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The survival amplitude for unstable particles at short times in relativistic quantum field theory

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ABSTRACT. We study in general the survival probability for unstable systems in relativistic quantum field theory (RQFT) and, then, in particular, the short-time behavior. Two specific models are investigated here: one superrenormalizable and the other simply renormalizable. In both models, we find that the said survival probability behaves like $P(t) = 1 - ct$ at very small t (c being a model-dependent constant). This —linear in t— behavior is essentially different from the quadratic one obtained in non-relativistic quantum mechanics also at very short times. The physical reason of this discrepancy is analyzed. Some related results in non-relativistic quantum field theory are also shown. Finally, some possibly relevant consequences of such a kind of short-time behavior of the survival probability for unstable relativistic particles are discussed.

RESUMEN. Estudiamos, en general, la amplitud de supervivencia de partículas inestables en Teoría Cuántica de Campos Relativistas, y, en particular, su comportamiento a tiempos cortos. Se analizan dos modelos específicos: uno superrenormalizable y otro simplemente renormalizable. En ambos modelos, concluimos que dicha amplitud de supervivencia se comporta como $P(t) = 1 - ct$ para tiempos t muy cortos (siendo c una constante dependiente del modelo). Este comportamiento lineal en t resulta ser esencialmente diferente del de tipo cuadrático obtenido en mecánica cuántica no relativista (también para tiempos cortos). Se analiza la razón física de dicha discrepancia. También se presentan y discuten otros resultados, relacionados con los anteriores, referentes a partículas inestables en el marco de la teoría cuántica de campos no relativistas. Finalmente, discutimos algunas posibles consecuencias del tipo de comportamiento a tiempo corto de la amplitud de supervivencia estudiado aquí.

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1. INTRODUCTION

The analysis of the finite-time evolution in relativistic quantum field theory (RQFT), and in particular the short-time behavior of the survival amplitude of unstable systems in RQFT, has not attracted a great deal of interest (a recent paper in which this question is dealt with will be commented at the end of the present work). Such a situation is probably due to the fact that most of the relevant (experimentally) properties of elementary particles can be appropriately described in terms of the S-matrix, which involves transition amplitudes just from $t = -\infty$ to $t = +\infty$, finite times thereby playing hardly a relevant role in such questions. Now one of the aims of this work is to study that finite-time evolution. This kind of analysis has some interest in connection with the short-time behavior of the non-decay probability of unstable systems in non-relativistic quantum mechanics (NRQM): we shall show that the existence of generic features related to the presence of infinite degrees of freedom in RQFT —in particular vacuum polarization and ultra- violet renormalization— gives rise to remarkable differences in the short-time behavior of the survival probability in both theories.

Now, as far as NRQM is concerned, the short-time behavior of the survival probability (SP) of unstable systems has been thoroughly studied (see, for instance, Ref. [1] and references therein) in particular in connection with the quantum violations of the "classical" exponential decay law. For the sake of completeness we shall present here the main result of those studies relevant to the purposes of this work.

Let $P(t)$ $(t > 0)$ be the SP for an unstable system which is represented in NRQM by some normalized state, $|\Psi\rangle$, and let H be the Hamiltonian governing the evolution of the quantum system. One has $P(t) = |\langle\Psi|\exp(-itH)|\Psi\rangle|^2$ (with $\hbar = 1$). If one assumes the finiteness of the energy dispersion in the initial (unstable) state

$$(\Delta E)^2 = \langle\Psi|H^2|\Psi\rangle - (\langle\Psi|H|\Psi\rangle)^2 < \infty, \tag{1}$$

as in fact happens in non-relativistic quantum systems, then it can be shown that at very short times

$$P(t) \simeq 1 - (\Delta E)^2 t^2, \qquad t \to 0^+. \tag{2}$$

Notice that Eq. (2) shows the violation of the exponential decay law at very short times whose consequences, such as the quantum Zeno effect, and possible experimental detection have been profusely discussed (see Ref. [13] for relevant experimental work concerning the quantum Zeno effect).

Another aim of the present work is to show that (2) does not hold in RQFT (at least in the two models here investigated) and that, instead, at very short times the PS evolves linearly in time at very short times $(t \to 0^+)$.

In Sect. 2, we carry out the analysis of the finite time evolution of the SP in a situation which, in some way, lies in between the NRQM and RQFT cases and that is helpful in order to get a proper understanding of the problems arising in the RQFT case: we are referring to the time evolution of the SP in some well-known non-relativistic quantum field models. General studies of finite time evolution and of the short-time one are carried out in Sect. 3, and the corresponding conclusions and open questions are discussed in Sect. 4.

2. SURVIVAL AMPLITUDE OF UNSTABLE PARTICLES IN SOME MODELS OF NON-RELATIVISTIC QUANTUM FIELD THEORY

2.1. Characterization of the models

Our purpose is to analyze the evolution for times $t > 0$, in particular for short times, of unstable particles, which have been formed or "prepared" previously, say, at $t \leq 0$. In this section, our study will be carried out in the framework of non-relativistic quantum field theory in three spatial dimensions.

Let $|i, \mathbf{p}_i\rangle$ be the initial state representing the unstable particle at $t = 0$, with three-momentum $\mathbf{p}_i$. Let H_0 be the field-theoretic hamiltonian describing that particle, regarded as a stable one, so that $|i, \mathbf{p}_i\rangle$ is an eigenstate of H_0 with eigenvalue $E_{0,i}$:

$$H_0|i, \mathbf{p}_i\rangle = E_{0,i}|i, \mathbf{p}_i\rangle. \tag{3}$$

We stress that H_0 also describes other particles besides the unstable one (say, its future decay products) and that the interaction giving rise to the decay is not included in H_0. We will suppose that for $t > 0$ there is an interaction represented by the interaction hamiltonian H_{I} that produces the decay. Then for $t > 0$, the total hamiltonian describing the time evolution of the unstable particle, its decay, and the dynamics of the resulting particles in the final state is

$$H = H_0 + H_{\mathrm{I}}. \tag{4}$$

None of the interactions considered in this section polarizes vacuum, by assumption, thereby restricting ourselves to non-relativistic field-theoretic situations.

To fix the ideas we consider: i) a non-relativistic quantum particle with (bare) mass m_0 and position and three-momentum operators $\mathbf{x}, \mathbf{p}$ $(= -i\nabla$, Planck's constant being set equal to one throughout this work), and for simplicity with zero spin; ii) a quantized spinless boson field described by the destruction and creation operators with momentum $\mathbf{k}$, $a(\mathbf{k})$, $a^+(\mathbf{k})$ $([a(\mathbf{k}), a^+(\mathbf{k}')] = \delta^{(3)}(\mathbf{k} - \mathbf{k}'))$. The energy of a boson with momentum $\mathbf{k}$ is $\omega(k)(k = |\mathbf{k}|)$, with $\omega(k) \geq \omega_0 > 0$ for any $\mathbf{k}$. Let $|0\rangle$ be the vacuum, *i.e.*, $a(\mathbf{k})|0\rangle = 0$ for any $\mathbf{k}$. For $t \leq 0$, the non-relativistic particle and the bosons evolve freely —there is no interaction among them. Then they are described by the hamiltonian

$$H_0 = \frac{p^2}{2m_0} + \int d^3\mathbf{k}\, \omega(\mathbf{k}) a^+(\mathbf{k}) a(\mathbf{k}). \tag{5}$$

For $t > 0$, we shall assume that an interaction described by

$$H_{\mathrm{I}} = \lambda \int d^3\mathbf{k} \left[\phi(\mathbf{k}) a(\mathbf{k}) \exp(i\mathbf{k}\mathbf{x}) + \phi^*(\mathbf{k}) a^+(\mathbf{k}) \exp(-i\mathbf{k}\mathbf{x})\right] \tag{6}$$

also acts. λ and $\phi(\mathbf{k})$ are a real coupling constant and a form factor, respectively. Notice that H_{I} does not polarize vacuum. The total hamiltonian H is given in Eq. (4), for $t > 0$. The non-relativistic particle, when it acquires a sufficient large three-momentum, will become unstable for $t > 0$, and the above bosons will appear in the final state after the decay, as we shall discuss later in more detail.

The total three-momentum operator is

$$\mathbf{P} = \mathbf{p} + \int d^3\mathbf{k}\, \mathbf{k} a^+(\mathbf{k}) a(\mathbf{k}). \tag{7}$$

One has

$$[H_0, \mathbf{P}] = [H, \mathbf{P}] = 0. \tag{8}$$

Let $\mathcal{H}(\mathbf{p}_0)$ be the subspace formed by all the eigenstates of $\mathbf{P}$ with eigenvalue P_0. Accordingly, the initial state $|i, \mathbf{p}_i\rangle$ representing the non-relativistic particle "prepared" at $t = 0$, which is a plane wave with three-momentum $\mathbf{p}_i$, belongs to $\mathcal{H}(\mathbf{p}_i)$. Notice that it satisfies (3), with $E_{0,i} = p_i^2/2m_i$ and that

$$\langle i, \mathbf{p} | i, \mathbf{p}' \rangle = \delta^{(3)}(\mathbf{p} - \mathbf{p}'). \tag{9}$$

Let $\sigma(H_0, \mathbf{p}_0)$ (resp., $\sigma(H, \mathbf{p}_0)$) be the set of all eigenvalues of H_0 (resp., H) when it acts only on states belonging to $H(\mathbf{p}_0)$. In general, both $\sigma(H_0, \mathbf{p}_0)$ and $\sigma(H, \mathbf{p}_0)$ will have discrete and continuous eigenvalues. In the present case, $E_{0,i} = p_i^2/2m_0$ is an isolated discrete eigenvalue of $\sigma(H_0, \mathbf{p}_i)$ for $E_{0,i} < \omega_0$.

We shall consider two models:

The polaron model. The non-relativistic particle and the boson field represent, respectively, an electron and the optical phonon field in an ionic crystal through which the electron moves. Furthermore, in this model one chooses $\omega(\mathbf{k}) = \omega_0 > 0$ for any $\mathbf{k}$ and $\phi(k) = i/k$; see Feynman [2] for further details.

The Gross-Nelson model. The non-relativistic particle and the boson field represent, respectively, a nucleon and the neutral pion field. One now chooses $\omega(\mathbf{k}) = (\mu^2 + k^2)^{1/2}$ and $\Phi(k) = [\omega(\mathbf{k})]^{-1/2}$, μ being the pion mass.

The behavior of this $\Phi(k)$ in the ultraviolet ($k \to \infty$) has posed several difficulties for a proper characterization of the hamiltonian H in the case of a stable nucleon. The difficulties have been solved through a suitable renormalization implemented by a dressing transformation; see Gross [3] and Nelson [4] for details.

2.2. Survival amplitude

In the polaron model, we shall suppose that the initially "prepared" state $|i, \mathbf{p}_i\rangle$ (fulfilling (3), (9)) has three-momentum $\mathbf{p}_i$ such that $p_i > (2m_0\omega_0)^{1/2}$. Then, the actual H_I [Eq. (6)] implies that this electron is unstable against a sort of "Cerenkov effect" in the ionic crystal, namely, the process electron ($\mathbf{p}_i$) $\to$ electron ($\mathbf{p}_i - \mathbf{k}$) + phonon ($\mathbf{k}$) occurs physically, the phonon momentum $\mathbf{k}$ fulfilling $p_i^2/2m_0 = (\mathbf{p}_i - \mathbf{k})^2/2m + \omega_0$. Then, the energy $E_{0,i} = p_i^2/2m_0$ of $|i, \mathbf{p}_i\rangle$ belongs to the continuum spectrum of $\sigma(H, \mathbf{p}_i)$.

Our main interest will be focussed on the non-decay (*i.e.*, survival) amplitude $A(t)$ and the associated probability $|A(t)|^2$ for $t > 0$. One has

$$A(t) = \frac{\langle i, \mathbf{p}_i | \exp(-itH) | i, \mathbf{p}_i \rangle}{\langle i, \mathbf{p}_i | i, \mathbf{p}_i \rangle}. \tag{10}$$

The infinite volume divergence of $\langle i, \mathbf{p}_i | i, \mathbf{p}_i \rangle$ [see Eq. (9)] cancels with a similar factor in the numerator in (10).

For short time $t > 0$, upon expanding $\exp(-itH)$ in (10), a formal manipulation easily yields

$$|A(t)|^2 = 1 - t^2(\Delta E^2), \tag{11}$$

$$(\Delta E)^2 = \frac{\langle i, \mathbf{p}_i | H^2 | i, \mathbf{p}_i \rangle}{\langle i, \mathbf{p}_i | i, \mathbf{p}_i \rangle} - \left(\frac{\langle i, \mathbf{p}_i | H | i, \mathbf{p}_i \rangle}{\langle i, \mathbf{p}_i | i, \mathbf{p}_i \rangle} \right)^2. \tag{12}$$

where ΔE is the energy uncertainty in the initial state $|i, \mathbf{p}_i\rangle$.

Some time ago, the decay of an unstable particle (proton decay, specifically, as conjectured by Grand Unified Theories) was studied in the framework of Lee-type models [5], which turn out to correspond to a simpler —and actually solvable— modified version of the class of models described by Eqs. (5), (6), also with a form factor $\phi(k)$. There [5], it was pointed out that the corresponding ΔE (also given by (12)) could be either finite or ultraviolet divergent, depending on the behavior of $\phi(k)$ at infinite momentum k —for previous work on unstable particles in Lee-type models, see Ref. [6].

Let us analyze in the polaron model the probability for the electron in the state $|i, \mathbf{p}\rangle$, with $p_i > (2m_0\omega_0)^{1/2}$, to survive at short time t, by using Eq. (11). For that purpose we need to evaluate $(\Delta E)^2$ (Eq. (12)). Some standard algebra yields

$$(\Delta E)^2 = \frac{\langle i, \mathbf{p}_i | H_{\mathrm{I}}^2 | i, \mathbf{p}_i \rangle}{\langle i, \mathbf{p}_i | i, \mathbf{p}_i \rangle} = \lambda^2 \int d^3\mathbf{k}\, |\phi(k)|^2, \tag{13}$$

which is linearly ultraviolet divergent for the polaron model. Consequently, (11) is no longer useful in the present frame-work to evaluate the survival probability at short times in the polaron model.

In spite of the failure of Eq. (11), the survival amplitude in the polaron model for any $t > 0$ can be studied through the following exact formula

$$A(t) = \int_{ic+\infty}^{ic-\infty} \frac{dE}{2\pi i} \frac{\exp(-iEt)}{E - E_{0,i} - R(E)}, \tag{14}$$

where $R(E)$ can be regarded as the self-energy of the unstable particle, and it is given by

$$R(E) = \frac{\langle i, \mathbf{p}_i | H_{\mathrm{I}} | \Psi \rangle}{\langle i, \mathbf{p}_i | i, \mathbf{p}_i \rangle}, \tag{15}$$

where in turn $|\Psi\rangle$ satisfies the integral equation

$$|\Psi\rangle = |i, \mathbf{p}_i\rangle + \frac{1}{E - H_0}\Big(I - |i, \mathbf{p}_i\rangle\langle i, \mathbf{p}_i|\Big)H_{\mathrm{I}}|\Psi\rangle, \tag{16}$$

I being the unit operator. In Eq. (14), c is a positive number, so that all singularities in the denominator lie below the path of integration. We refer to [7] for a thorough derivation and analysis of Eqs. (14)–(16); see also Ref. [8] for previous work.

Upon replacing $|\Psi\rangle$ in (15) by the series obtained by iterating (16) sucessivelly, one can express the self-energy $R(E)$ as a power series in H_{I}. The interesting point is that each term in that perturbation series is finite. This follows by using (5), (6), (3), (9) and the form factor of the polaron model and by performing some direct countings of powers of threemomenta in each term of that series. Since each $|\phi(k)|^2$ provides a k^{-2} and so does each $(E - H_0)^{-1}$, they are sufficient to overcome threemomentum space volume element $k^2\,dk$ as $k \to \infty$ and, hence, to render all integrals finite in the ultraviolet. Likewise, one can see that, in general, there are no infrared divergences. Then (14) shows that $A(t)$ does exist and is well defined for $t > 0$, in the polaron model. However, one should refrain from expanding naively the r.h.s. of (14) into a power series in t, as such a procedure could introduce ultraviolet divergences in the integration over E. Such divergences could be related to the one for ΔE commented above, which in fact came from a formal expansion of $\exp(-itH)$ into powers of t. We now turn to the Gross-Nelson model.

Let $|i, \mathbf{p}_i\rangle$ represent now a one nucleon state. It can also suffer a "Cerenkov effect" when $\mathbf{p}_i$ fulfills

$$E_{0,i} = \frac{p_i^2}{2m_0} = (\mu^2 + k^2)^{1/2} + \frac{(\mathbf{p}_i - \mathbf{k})^2}{2m_0}. \tag{17}$$

This equation has a non-trivial solution (when $\mathbf{p}_i$ and $\mathbf{k}$ are collinear, with $\mathbf{p}_i \cdot \mathbf{k} > 0$), if

$$p_i = k/2 + m_0(1 + \mu^2/k^2)^{1/2}, \quad k > k_0 > 0, \tag{18}$$

provided that k_0 fulfills

$$1 + \frac{\mu^2}{k_0^2} = \frac{4m_0^2\mu^4}{k_0^6}. \tag{19}$$

Then, the nucleon state $|i, \mathbf{p}_i\rangle$ "prepared" at $t = 0$, and with p_i fulfilling (17, 18) will become unstable when H_{I} be turned on for $t > 0$. In what follows, we shall be specifically interested in the survival amplitude for this unstable nucleon. To start with, a direct application of Eqs. (11, 12) for the short-time survival probability fails because the energy uncertainty ΔE, (13), for the unstable particle in the Gross-Nelson model turns out to be quadratically ultraviolet divergent. Moreover, a direct attempt to study the survival amplitude on the basis of (14) is also doomed at failure for the H characterizing the Gross-Nelson model. Consider the nucleon self-energy $R(E)$, as given formally through (15) and the series of iterations for (16). Now it turns out that each term in the series for $R(E)$

is ultraviolet divergent (since $|\phi(k)|^2$ now behaves as k^{-1} for $k \to \infty$). Consequently, (14) cannot be used. Fortunately, by applying for $t > 0$ the same unitary (dressing) transformation for the actual unstable nucleon case as Gross [3] and Nelson [4] did for the stable one, the above difficulties can be overcome and the survival amplitude becomes finite. Specifically, that unitary transformation is implemented by the unitary operator $\exp T$, where

$$T = \int d^3\mathbf{k}\left[\beta(\mathbf{k})\exp(i\mathbf{k}\mathbf{x})a(\mathbf{k}) - \beta^*(\mathbf{k})\exp(-i\mathbf{k}\mathbf{x})a^+(\mathbf{k})\right] = -\bar{T}^+ \tag{20}$$

$$\beta(\mathbf{k}) = -\frac{\lambda[1-\theta(\Lambda-k)]\phi(k)}{\omega(k)+(k^2/2m_0)}. \tag{21}$$

Λ is a non-negative arbitrary fixed constant and θ the usual step function. Then one finds that $[\exp T]\mathbf{P}[\exp(-T)] = \mathbf{P}$ and that, for $t > 0$, H transforms into the new hamiltonian H' as

$$[\exp T]H[\exp(-T)] = H' + E'; \tag{22}$$

$$E' = -\lambda^2\int d^3\mathbf{k}\,\frac{|\phi(k)|^2\,[1-\theta(\Lambda-k)]}{\omega(k)+k^2/2m_0}, \tag{23}$$

$$H' = H_0 + H'_{I,1} + H'_{I,2}; \tag{24}$$

$$H'_{I,1} = \lambda\int d^3\mathbf{k}\,\theta(\Lambda-k)\phi(k)\left[a(\mathbf{k})\exp(i\mathbf{k}\mathbf{x}) + a^+(\mathbf{k})\exp(-i\mathbf{k}\mathbf{x})\right], \tag{25}$$

$$H'_{I,2} = \frac{1}{2m}\left[\mathbf{A}^2 + (\mathbf{A}^+)^2 + 2\mathbf{A}^+\mathbf{A} + 2(\mathbf{p}\mathbf{A} + \mathbf{A}^+\mathbf{p})\right], \tag{26}$$

$$\mathbf{A} = -\int d^3\mathbf{k}\,\beta(\mathbf{k})\mathbf{k}\,a(\mathbf{k})\exp(i\mathbf{k}\mathbf{x}). \tag{27}$$

At this stage, we shall omit the logarihmic ultraviolet divergent constant self-energy E' in Eq. (23) and regard H', in (24), as the hamiltonian describing the unstable nucleon dynamics and decay for $t > 0$. Then, the corresponding survival amplitude is naturally given by Eq. (10) with the sole replacement of H by H'. The new energy uncertainty, as given by (12) with H replaced by H' as well, still turns out to be ultraviolet divergent as further computations show. Fortunately Eqs. (14)–(16), with H_{I} replaced by $H'_{I,1} + H'_{I,2}$ are finite and well-defined and characterize the nucleon survival amplitude in the (renormalized) Gross-Nelson model. In fact each term in the perturbative series for the actual $R(E)$ in powers of $H'_{I,1} + H'_{I,2}$ turns out to be finite in the ultraviolet (and in the infrared). This property follows from further countings of powers of three-momenta when use of the new interaction hamiltonian $H'_{I,1} + H'_{I,2}$ (see Eqs. 25)–(27) is made. Actually the finiteness of $R(E)$ in the present case is not surprising, but a direct consequence of the unitary transformation in Eq. (22), which removed E' (at the expense of introducing the more complicated interaction $H'_{I,1} + H'_{I,2}$), as in the stable nucleon case [3, 4]. Then, (14) can be employed to characterize the survival amplitude for any $t > 0$.

We will end this section by pointing out some curious —and perhaps somewhat paradoxical— property of the polaron model. As we will see, provided that certain dressing effects be properly accounted for in the initial state, a new survival amplitude can be introduced in such a way that the divergence in ΔE is cured and hence the corresponding survival probability at short times can be computed via (11). Specifically we will assume that the initially "prepared" unstable electron state is no longer the above $|i, \mathbf{p}_i\rangle$ but the following one, in which also $p_i > (2m_0\omega_0)^{1/2}$

$$|i, \mathbf{p}_i\rangle_{\mathrm{d}} = \exp(-T)|i, \mathbf{p}_i\rangle, \tag{28}$$

The operator T is given in (20), (21) with the form factor $\phi(k)$ characterizing the polaron model. $|i, \mathbf{p}_i\rangle_{\mathrm{d}}$ can be regarded as an electron state, partially dressed by a coherent phonon state, which becomes unstable for $t\rangle 0$, when H_{I} is turned on. Our choice for $|i, \mathbf{p}_i\rangle_{\mathrm{d}}$ is not accidental but inspired on the properties embodied in (22), (23), as the following analysis will show. Notice that the new initial state $|i, \mathbf{p}_i\rangle_{\mathrm{d}}$ is not an eigenstate of H_0 any longer —although it is still an eigenstate of $\mathbf{P}$.

The survival amplitude associated to the new initial state is

$$A(t)_{\mathrm{d}} = \frac{{}_{\mathrm{d}}\langle i, \mathbf{p}_i| \exp(-itH)|i, \mathbf{p}_i\rangle_{\mathrm{d}}}{{}_{\mathrm{d}}\langle i, \mathbf{p}_i|i, \mathbf{p}_i\rangle_{\mathrm{d}}} = \frac{\langle i, \mathbf{p}_i| \exp([-it(H' + E')|i, \mathbf{p}_i\rangle}{\langle i, \mathbf{p}_i|i, \mathbf{p}_i\rangle} \tag{29}$$

where (22) has been used. Notice that E' is finite for the actual polaron case.

The survival probability $|A(t)_{\mathrm{d}}|^2$ at short $t > 0$ is also given by (11, 12) with H replaced by $H' + E'$. By using (22)–(25) and $\langle i, \mathbf{p}_i|(H'_{I,1} + H'_{I,2})|i, \mathbf{p}_i\rangle = 0$, some easy power counting of three-momenta shows that $(\Delta E)^2 < \infty$ in the polaron model for the new initial dressed state. Then (11) can be used for the latter. Notice, however, that by dealing with $|i, \mathbf{p}_i\rangle_{\mathrm{d}}$, we are disregarding the following conceptual paradox: that state includes at $t = 0$ dressing effects associated to an interaction which will act for $t > 0$!

3. Unstable particles in RQFT: survival amplitude

3.1. *A general analysis*

We now turn to analyze the short time decay of unstable particles in local RQFT in 3 space dimensions. Now, one should be careful about both vacuum polarizing interactions and ultraviolet divergent renormalization.

For previous general analysis of unstable particles in RQFT we refer to [9]. More recently the short-time behavior has been treated in Ref. [10].

To begin with, let us state clearly the problem.

We consider —with respect to certain reference frame— some "free" hamiltonian H_0 which describes particles for $t \leq 0$ in local RQFT in the Schrödinger picture. Also, let $\mathbf{P}$ be the total three-momentum operator, with $[\mathbf{P}, H_0] = 0$, and let $|i, \mathbf{p}_i\rangle$ be the initially prepared state representing the unstable particle at $t = 0$. By assumption, $|i, \mathbf{p}_i\rangle$ is a common eigenstate of both H_0 and $\mathbf{P}$ with eigenvalues $E_{0,i}$ and $\mathbf{p_i}$, respectively, and the unstable particle is massive: $E_{0,i} > 0$ for any $\mathbf{p}_i$. In the cases to be analyzed below, the unstable particle will be either a neutral spinless boson or a neutral spin 1/2 fermion. Notice that upon "preparing" the unstable particle we are disregarding (or at least cavalier regarding) formal covariance since we are selecting one inertial frame, namely, that in which the initial state has been prepared at $t = 0$. As it seems a priori natural (and simplest) we shall select the inertial frame (S) in which the unstable particle is at rest at $t = 0$, *i.e.*, $\mathbf{p}_i = 0$.

Let $|0\rangle$ be the vacuum for H_0. Then, the initially prepared state can be written as

$$|i, \mathbf{p}_i = 0\rangle = |i\rangle = a^+(\mathbf{p}_i = 0)|0\rangle, \tag{30}$$

$a^+|\mathbf{p}_i\rangle$ being the associated creation operator for the unstable particle (in Schrödinger's picture). Possible spin dependence for it will be written later, when needed.

The interaction hamiltonian H_I, which acts only for $t > 0$ and its responsible for the decay, has the following properties (by assumption):

1. It commutes with $\mathbf{P}$.
2. It polarizes vacuum.
3. It gives rise to either a superrenormalizable local RQFT or to a renormalizable one and ,so, it includes the corresponding ultraviolet divergent renormalization counterterms. We shall start, for simplicity, with superrenormalizable interactions in the next subsection.
4. It involves a small coupling constant, in the sense that one can reliably apply perturbation theory. In this regard, a good example would be the interaction hamiltonian responsible for the conjectured proton decay in Grand Unified Theories (GUT's): in particular the order of magnitude of this H_I is much smaller than that of the strong (QCD) hamiltonian (which in turn would be the analogue of H_0). The total hamiltonian, H, for $t > 0$, is also given by the r.h.s. of (4) and it also commutes with $\mathbf{P}$.

It is worth noticing the following difference between the treatment in this section and that in the preceding one. On one side, in the non-relativistic field theoretic models of Sect. 2 the same hamiltonians H_0, H_I described both: i) particle structure and dressing (renormalization effects), ii) decay. On the other side, in the RQFT case of Sect. 3, one is distinguishing between the total hamiltonian for $t < 0$, H_0, describing the structure (and formation) of the initial state and the one for $t \geq 0$, $H_0 + H_1$, in which H_1 gives rise to the subsequent decay.

Notice that for a given initial state describing the unstable particle before decay, the resolution of the total hamiltonian for $t \geq 0$ into the sum of an unperturbed hamiltonian and a perturbation is not unique, in general. They are split up in this way as a practical consequence of our actual limitation to calculate using perturbation theory (should we have been able to solve the equations of motion exactly, then there would be no need to make such a split). In this setting, ambiguities of that sort appear to be unavoidable (and well known and understood).

An unstable particle could also be prepared in a state which is not an eigenstate of the chosen H_0. We recall, in this connection, that the lifetime of an unstable particle could depend on the way it has been prepared (see Schwinger's paper in Ref. [9]).

The survival amplitude $A(t)$, for the unstable particle at rest in S for $t > 0$ is still given by the r.h.s. of Eq. (10), with the actual H and $|i, \mathbf{p}_i\rangle$ being replaced by $|i\rangle = |i, \mathbf{p}_i = 0\rangle$.

Again, the volume divergence embodied in $\langle i|i\rangle = \delta^{(3)}(\mathbf{p}_i - \mathbf{p}_i)$ will cancel with a similar factor in the numerator of the counterpart of (10).

In order to study $A(t)$ in renormalized perturbation theory, we shall go over to the interaction picture (ip), for $t \geq 0$, in the standard way. In so doing, we shall give only the essential formulas, thereby omitting standard and easy calculations which are well documented [11,12]. By assumption, both Schrödinger's picture and ip coincide at $t = 0$. Let

$$U(t)_{\rm ip} = \exp(itH_0)\exp(-itH), \quad t \geq 0, \tag{31}$$

be the ip evolution operator. A generic operator B representing a dynamical variable in Schrödinger's picture becomes in the ip

$$B(t)_{\rm ip} = \exp(itH_0)B\exp(-itH_0). \tag{32}$$

Then, by using (31), (32), the survival amplitude for $t > 0$ can be easily cast as:

$$A(t) = \frac{\langle 0|a(\mathbf{p}_i = 0;\, t)_{\rm ip} U(t)_{\rm ip} a^+(\mathbf{p}_i = 0;\, t = 0)_{\rm ip}|0\rangle}{\langle i|i\rangle} \tag{33}$$

$a(\mathbf{p} = 0;\, t)_{\rm ip}$ being the ip destruction operator at $t > 0$. For later convenience, we have written $a^+(\mathbf{p}_i = 0;\, t = 0)_{\rm ip}$ instead of $a^+(\mathbf{p}_i = 0)$.

Equation (33) reminds the known representation for the one-particle Green's function for the decaying particle (as if it were stable) in a formally similar RQFT also in the ip (compare for instance with Eq. (35d), page 657, in Ref. [11]), although they turn out to be different physical and mathematical quantities.

One crucial difference between them is that the survival amplitude involves operators and states at finite non-negative times necessarily, whereas the one-particle Green's function requires them at all times, $-\infty < t < \infty$ (in the ip). Fortunately, in the present case we can also use the standard perturbative expansion for $U(t)_{\rm ip}$,

$$U(t)_{\rm ip} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^t dt_1 \ldots \int_0^t dt_n\, T\Big[H_{\rm I}(t_1)_{\rm ip} \ldots H_{\rm I}(t_n)_{\rm ip}\Big]. \tag{34}$$

The symbol T in the r.h.s. of (34) denotes the standard time-ordered product. Then, upon replacing Eq. (34) into (33), we get the basic formula which will enable us to calculate the survival amplitude in renormalized perturbation theory:

$$A(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^t dt_1 \ldots \int_0^t dt_n\, \frac{\langle 0|T[a(t)\, H_{\rm I}(t_1) \ldots H_{\rm I}(t_n)\, a^+(t = 0)]|0\rangle}{\langle i|i\rangle}, \tag{35}$$

(where, for simplicity, we have omitted ip in the operators, as well as $\mathbf{p}_i = 0$ in a and a^+). As $t \geq t_1, \ldots, t_n \geq 0$, it was allowed to write both $a(t)$ and $a^+(t = 0)$ inside each

T–product in (35). One can now evaluate $A(t)$ by using (35)and Wick's theorem in the standard way [11, 12].

Let $E_{0,i}(\mathbf{p}_i = 0) = m_i$ (> 0) be the renormalized rest mass of the unstable particle ($c = 1$ throughout this work). After applying Wick's theorem, the first stage in the computation consists in the cancellation of the infinite-volume divergence in $\langle i|i\rangle^{-1}$ with a similar factor which comes from the vacuum expectation value of the contractions in (35). Such straightforward calculations show, at the same time, that each perturbative contribution to $A(t)$ contains $\exp(-im_i t)$ as an overall factor. Consequently, after those cancellations, $A(t)$ bears the factorized form

$$A(t) = \exp(-im_i t)\tilde{A}(t), \tag{36}$$

where the "reduced" survival amplitude $\tilde{A}(t)$ tends to 1 for any $t > 0$ if $H_{\mathrm{I}} \to 0$. The structure of (36) can be inmediately checked from (33) when $H_{\mathrm{I}} = 0$. After having cancelled out the infinite-volume divergences contained in $\langle i|i\rangle^{-1}$ in each perturbative order, as implemented in Eq. (35), new infinite-volume divergences crop up in $A(t)$. In fact, since H_{I} polarizes the vacuum, upon applying Wick's theorem to each T–product one finds unavoidably the contraction of $a(t)$ with $a^+(t = 0)$. Such a special class of contractions give rise to new infinite-volume divergences generically known (and represented graphically) as disconnected vacuum contributions (dvc).

Let A'_{dvc} be the sum of all contributions associated to dvc, as generated by the application of Wick's theorem to the r.h.s. of (35), and let us write

$$A_{\mathrm{dvc}}(t) = 1 + A'_{\mathrm{dvc}}(t). \tag{37}$$

A_{dvc} embodies all the remaining infinite-volume divergences which are generated by (35) (those which do not cancel with $\langle i|i\rangle^{-1}$). It can be shown in general that, to all orders in perturbation theory, the "reduced" survival amplitude factorizes as

$$\tilde{A}(t) = A_{\mathrm{dvc}}(t)\, A_{\mathrm{ph}}(t). \tag{38}$$

$A_{\mathrm{ph}}(t)$, to be called hereafter the physical survival amplitude, is the sum of all perturbative contributions to $\tilde{A}(t)$ in each of which $a(t)$ is not contracted with $a^+(t = 0)$. Therefore, all the contributions to A_{ph} are free of infinite-volume divergences. Notice that $A_{\mathrm{ph}} \to 1$ as $H_{\mathrm{I}} \to 0$.

In local RQFT, ultraviolet divergences will generically appear in the perturbative expansion of $\tilde{A}(t)$. However, since H_{I} also includes, by assumption, all the necessary renormalization counterterms, it turns out that $A_{\mathrm{ph}}(t)$ is ultraviolet finite. In other words, all ultraviolet divergences which could appear in the contributions to $A_{\mathrm{ph}}(t)$ cancel out to all perturbative orders. The different key properties of $A(t)$, $\tilde{A}(t)$, $A_{\mathrm{dvc}}(t)$, and $A_{\mathrm{ph}}(t)$ will be checked below through several explicit computations in certain superrenormalizable and renormalizable models, to second-order perturbation theory. It is interesting to compare the structures in Eqs. (36), (38) with the analysis made in [10]: specifically, the squared modulus of Eqs. (36), (38) coincides with Eq. (15) in Ref. [10].

3.2. A superrenormalizable model

We shall consider an unstable relativistic neutral scalar massive boson (i) with mass $m_1 > 0$, which can decay into two different stable neutral scalar bossons (a, b) with masses m_a, m_b, so that $m_i > m_a + m_b$. All masses are the renormalized ones. Then, H_0 is the sum of three terms, each of which is similar to the second contribution in the r.h.s. of Eq. (5), with the corresponding energy and creation and annihilation operators (for the i, a, and b bosons).

The interaction hamiltonian for $t > 0$ in the Schrödinger picture is by assumption

$$H_{\mathrm{I}} = \lambda \int d^3\mathbf{x}\, \Phi_i(\mathbf{x})\, \Phi_a(\mathbf{x})\, \Phi_b(\mathbf{x}) - \delta m_i^2 \int d^3\mathbf{x}\, N[\Phi_i(\mathbf{x})]^2 + \text{o.c.t.} \tag{39}$$

$\Phi_i(\mathbf{x})$ is the standard free quantized Klein-Gordon field for the i boson: its well-known plane wave expansion in terms of the creation and annihilation operators (those appearing in (30)) will be omitted. Likewise for the field operators Φ_a and Φ_b. λ is a coupling constant with dimensions (mass)^2. δm_i^2 is the ultraviolet divergent mass renormalization counterterm for the unstable i boson: its explicit expression will be given below, as it will be needed later. The symbol N in Eq. (39) denotes normal ordering of field operators, while o.c.t. means other divergent renormalization counterterms, specifically, mass renormalization counterterms for the a and b bosons —they will not be written explicitly, as they are not needed in this work.

The RQFT model described by (39) is superrenormalizable. The mass renormalization for the i–boson is ($\epsilon \to 0^+$)

$$\delta m_i^2 = \frac{i\lambda^2}{(2\pi)^4} \int d^4k_1\, \frac{1}{(k_1^2 - m_a^2 + i\epsilon)(k_1^2 - m_b^2 + i\epsilon)}. \tag{40}$$

For this model, we shall give the contribution of order λ^2 to $A'_{\mathrm{dvc}}(t)$ to be denoted as $A'_{\mathrm{dvc}}(t)^2$. We apply Wick's theorem, we cancel out $\langle i|i\rangle^{-1}$ and express the non-vanishing contractions of boson field operators in terms of the corresponding free boson propagators; then we replace the latter by their Fourier integral representations and we carry out trivially all three-dimensional integrations over space coordinates as well as one integration over the three-momentum of the internal i boson. The integrations over times are not carried out explicitly at this stage. Thus, one arrives at ($k_j = (k_j^0, \mathbf{k}_j)$, $j = 1, 2$; $\epsilon \to 0^+$)

$$\begin{aligned} A'_{\mathrm{dvc}}(t) = &-\delta^{(3)}(\mathbf{0})\, \frac{i\lambda^2}{(2\pi)^6} \int_0^t dt_1 \int_0^t dt_2 \int d^4k_1\, d^4k_2 \int_{-\infty}^{\infty} dk_3^0 \\ &\times \exp\Big[-i(k_1^0 + k_2^0 + k_3^0)(t_1 - t_2)\Big] \\ &\times \frac{1}{(k_1^2 - m_a^2 + i\epsilon)(k_2^2 - m_b^2 + i\epsilon)\Big[(k_3^0)^2 - (\mathbf{k}_1 + \mathbf{k}_2)^2 - m_i^2 + i\epsilon\Big]}, \end{aligned} \tag{41}$$

which displays the infinite volume divergence $\delta^{(3)}(\mathbf{0})$ neatly.

We now turn to our main subject, *i.e.* the physical survival amplitude. Having dealt with the dvc, we shall work out the consequences of Wick's theorem for the remaining contributions, that are free of infinite-volume divergences. Some computations, patterned after those which have led to (41), yield a set of rules for writing the renormalized perturbative contributions to $A_{\rm ph}(t)$. The analogy, commented above, between Eq. (33) and the one particle Green's function, G, indicates that there is a one-to-one correspondence between the renormalized perturbative contributions (free of infinite volume divergences) for G and for $A_{\rm ph}(t)$ and, hence, between the rules yielding each of them separately.

In short one can say that the diagrams contributing to both G and $A_{\rm ph}(t)$ are formally identical, but that the computational rules yielding the contributions for G and $A_{\rm ph}(t)$ are not the same (exactly) but slightly different. Then, it is natural and simpler to formulate the rules for obtaining the renormalized perturbative contributions to $A_{\rm ph}(t)$ by starting from the ones for G in momentum space, which are well known (and need not be reproduced here) [11, 12].

We start by drawing a fully connected diagram D in four-momentum space, contributing to G to some perturbative order, n. Particular attention will be paid, in the sequel, to the two external vertices in which the two external lines for the i–boson (say, the incoming and outgoing ones) join the diagram. By convention, the incoming (outgoing) particle joins the diagram at vertex 2 (1). The two external vertices may coincide in some special case, as we shall see. In order to obtain the perturbative contribution of order n to $A_{\rm ph}(t)$ generated by D, we proceed as follows:

1) We attribute a time t_j to each vertex in D, including the two external ones, so that t_1 (t_2) corresponds to the external vertex 1 (2); in the special case in which both coincide we will write t_1, only.

2) For each internal line in D for the x–boson ($x = i,\ a,\ b$) carrying four-momentum $k = (k^0, \mathbf{k})$ and going from the vertex j towards the vertex j (which may be external or internal), we write the contribution ($\epsilon \to 0^+$)

$$\frac{1}{(2\pi)^4}\,\frac{\exp[-ik^0(t_j - t_s)]}{k^2 - m_x^2 + i\epsilon}.$$

Note that the external i–boson lines carry fourmomentum $(m_i, \mathbf{0})$.

3) For any three-boson vertex, we write $\lambda(2\pi)^3\delta^{(3)}(\mathbf{K})_{\rm tot})$, $\mathbf{K}_{\rm tot}$ being the sum of all three-momenta entering and leaving that vertex (with the same rules as those used when dealing with the diagram for G).

4) For any two-boson vertex at which only two lines with four-momenta k_1, k_2 join (which is necessarily generated by one of the mass renormalization counterterm appearing in (39), we simply write

$$i(2\pi)^3\delta^{(3)}(\mathbf{k}_1 - \mathbf{k}_2)\,\delta m_x^2 \quad (x = i, a, b).$$

5) We attribute no contribution to one of the two external vertices (irrespective of whether three bosons or two bosons meet in that vertex). In other words, for that vertex we do not make use of rules 3 and 4, for, otherwise, we would give rise to a volume divergence ($\delta^{(3)}(\mathbf{0})$): actually, it is the one which has already cancelled out with $\langle i|i\rangle^{-1}$.

6) We multiply by $(2m_i)^{-1}\exp[im_i(t_1-t_2)]$.

7) We integrate over all internal fourmomenta.

8) We integrate over all times t_j, for all vertices (internal and external) in $0 \le t_j \le t$.

9) Other factors, namely $(-i)^n/n!$, statistical factors (arising from different possibilities in performing volume-divergence-free contractions),..., are the same as in the standard rules yielding the renormalized perturbative contributions for G.

Using rules 1–9, the renormalized perturbative contribution to $A_{\rm ph}(t)$ of order λ^2, $A_{\rm ph}(t)^{(2)}$, can be inmediately written. After a few trivial simplifications, it reads

$$\begin{aligned} A_{\rm ph}(t) - 1 &\simeq A_{\rm ph}(t)^{(2)} \\ &= \frac{1}{2m_i}\left\{\frac{\lambda^2}{(2\pi)^5}\int d^4k_1\, d^4k_2\, \frac{\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2)}{k_1^2-m_a^2+i\epsilon)(k_2^2-m_b^2+i\epsilon)}\right. \\ &\quad \left.\times \int_0^t dt_1 \int_0^t dt_2\, \exp\left[i(m_i-k_1^0-k_2^0)(t_1-t_2)\right] + i\delta m_i^2 \int_0^t dt_1\right\} \end{aligned} \tag{42}$$

Notice that the contribution δm_i^2 in (42) is just associated to one diagram in which both external vertices coincide.

Upon performing the time integrations and trivial algebra, Eqs. (40) and (42) yield.

$$A_{\rm ph}(t)^{(2)} = \frac{\lambda^2}{2m_i}X. \tag{43}$$

X can be written as

$$X = X_1(X_1') + X_2, \tag{44}$$

where

$$X_1(X_1') = \int \frac{d^4k_i}{(2\pi)^4}\, \frac{1}{(k_1^2-m_a^2+i\epsilon)}X_1', \tag{45}$$

with

$$X_1' = \int_0^\infty \frac{dk_2^0}{2\pi}\, \frac{4\sin^2\left[\frac{t}{2}(m_i-k_1^0-k_2^0)\right]}{(m_i-k_1^0-k_2^0)^2}\, \frac{1}{(k_2^0)^2-(q^2+m_b^2)+i\epsilon}, \tag{46}$$

where $q = |\mathbf{k}_1|$, and (recalling (40))

$$X_2 = -\frac{t}{(2\pi)^4}\int \frac{d^4k_1}{(k_1^2-m_a^2+i\epsilon)(k_1^2-m_b^2+i\epsilon)}. \tag{47}$$

The integration in (46) requires some care. A convenient procedure to do it is differentiating the r.h.s., with respect to t, twice and then performing a residue integration over k_2^0. By using that procedure, and taking into account that

$$\left(\frac{dX_1'}{dt}\right)_{t=0^+} = X_1'(t=0^+) = 0,$$

we get

$$X_1' = Y_1' + Y_2', \tag{48}$$

where

$$Y_1' = \frac{i}{2(q^2+m_b^2)^{1/2}}\left[\frac{\exp[-it(k_1^0+E_b-m_i)]-1}{(k_1^0+E_b-m_i-i\epsilon)^2} + \frac{\exp[it(k_1^0-E_b-m_i)]-1}{(k_1^0-E_b-m_i+i\epsilon)^2}\right] \tag{49}$$

$$Y_2' = \frac{t}{(k_1^0-m_i)^2-E_b^2+i\epsilon} \tag{50}$$

(for convenience we have defined $E_b = (q^2+m_b^2)^{1/2}$)

The last expression for Y_2' is now introduced in (45) and the corresponding result added to X_2 (Eq. (47)), so obtaining

$$\begin{aligned} Y &= X_1(Y_2') + X_2 \\ &= \frac{it}{4\pi^2}\int_0^\infty dq\left[\frac{q^2}{E_a^2E_b+E_aE_b^2}\frac{m_i^2}{(E_a+E_b)^2-m_i^2-i\epsilon}\right], \end{aligned} \tag{51}$$

where, obviously, $E_a = (q^2+m_a^2)^{1/2}$. Notice that (51) is no longer ultraviolet divergent.

Now by separating out real and imaginary parts in (51) we get

$$Y = -\frac{Q}{8\pi m_i}t + itI, \tag{52}$$

where Q (CM momentum of the final particles) is given by

$$Q = \frac{1}{2m_i}\left[m_i^4 - 2m_i^2(m_a^2+m_b^2) + (m_a^2-m_b^2)^2\right]^{1/2} \tag{53}$$

and I, being essentially the principal part of the integral in (51), needs not be specified since itI is pure imaginary and its contribution to the survival probability is then of order λ^4.

We now turn to the remaining part of X_1', *i.e.* Y_1', given in Eq. (49). The latter once introduced in Eq. (45) and after integrating over k_2^0, yields the following result

$$\begin{aligned} Z &= X_1(Y_1') \\ &= \frac{1}{8\pi^2}\int_0^\infty dq\frac{q^2}{E_aE_b}\left[\frac{\exp[-it(E_a+E_b-m_i)]-1}{(E_a+E_b-m_i-i\epsilon)^2} + \frac{\exp[-it(E_a+E_b+m_i)]-1}{(E_a+E_b+m_i+i\epsilon)^2}\right] \end{aligned} \tag{54}$$

In (54), we shall use $(A - i\epsilon)^{-2} = PA^{-2} - i\pi\delta'(A)$ for any A (P is the principal part and δ' the derivative of the δ–function). The contributions to Z arising from $\delta'(A)$ and PA^{-2} are denoted by $Z(\delta')$ and $Z(RP)$, respectively. Some algebra yields, for any $t > 0$

$$Z(\delta) = \frac{Q}{8\pi m_i} t, \tag{55}$$

which exactly cancels out with the first term in the r.h.s. of (52). We now compute the remaining part of (54), $Z(RP)$. By expanding in power series in t and taking only into account linear and quadratic terms (as we are dealing with the short-time limit) we obtain after some calculation

$$Z(RP) = Z - Z(\delta) \simeq \frac{-t}{16\pi} + \frac{(m_a + m_b)t^2}{16\pi^2}. \tag{56}$$

Now, recalling the first equality in (42) and (43) we get for the short-time survival probability up to order t^2 ($t \geq 0$)

$$P(t) = |A_{\rm ph}(t)|^2 \simeq 1 - \frac{\lambda^2}{16\pi m_i} t + \frac{\lambda^2/\pi}{16\pi m_i}(m_a + m_b)t^2, \tag{57}$$

which is valid (as one realizes by analyzing the expansion in powers of t in (57)) for $0 \leq t < (m_a + m_b)^{-1}$ —a very short time indeed for "normal" decay processes.

In spite of not being directly concerned here with large time properties of decay, it seems worth studying briefly the survival amplitude as $t \to \infty$ (still at order λ^2), simply as some kind of consistency check of the present treatment. Thus, from (54) and recalling that

$$\lim_{t \to \infty} \frac{\sin(\omega t)}{\omega} = \pi\delta(\omega),$$

we easily obtain

$$\mathrm{Re}\, Z(RP) \underset{t \to \infty}{\longrightarrow} -\frac{Qt}{8\pi m_i}. \tag{58}$$

Now, as the imaginary part of $Z(RP)$ gives a contribution to $P(t)$ which is of order λ^4 and (see (55))

$$\mathrm{Re}\, Z(RP)_{t \to \infty} + Z(\delta) = 0,$$

we shall have, from (52) and (43)

$$P(t) \underset{t \to \infty}{\longrightarrow} 1 - \frac{\lambda^2 Q}{8\pi m_1^2} t. \tag{59}$$

Notice that, as expected, the above expression coincides with the one obtained by using Fermi's Golden Rule. We again point out that the main result of this subsection is Eq. (57), in which one sees that as $t \to 0^+$ the survival probability decreases linearly with t, instead of the quadratic behavior appearing in Non-relativistic Quantum Mechanics. A similar linear behavior will be present in the model we are going to study in the next subsection. The physical discussion of this result is left for the last Section of the paper.

3.3. A renormalizable model

We shall now consider an (unstable) relativistic neutral spin-1/2 fermion (i), with renormalized mass $m_i > 0$, which can decay into two relativistic stable neutral particles (a, b) with renormalized masses $m_a, m_b > 0$, so that: i) $m_i > m_a + m_b$, and ii) b is a spin-1/2 fermion while a is a scalar spinless boson.

Now, H_0 is the sum of three terms; the free hamiltonian corresponding to a is similar to the second term in the r.h.s. Eq. (5), while those corresponding to both i and b are the well-known ones for relativistic free Dirac fermions. For $t > 0$ the interaction hamiltonian in the Schrödinger picture is

$$H_\mathrm{I} = \lambda \int d^3\mathbf{x} \Big[\bar{\Psi}_b(\mathbf{x})\, \Psi_i(\mathbf{x})\, \Phi_a(\mathbf{x}) + \mathrm{h.c.} \Big] - \delta m_i \int d^3\mathbf{x} N \Big[\bar{\Psi}_i(\mathbf{x})\, \Psi_i(\mathbf{x}) \Big] + \mathrm{o.c.t.} \tag{60}$$

λ is a renormalized dimensionless coupling constant; Ψ_i and Ψ_b are the standard relativistic free Dirac operators. Thus, the plane wave expansion for Ψ_i contains the annihilation operator of the decaying i–particle, $a(\mathbf{p}_i, \sigma_i)$, which now depends on its helicity σ_i $(= \pm\frac{1}{2})$ as well.

The mass renormalization counterterm for the unstable i–fermion is to order λ^2 ($\not{k}_1 = k_1^0\gamma^0 - \mathbf{k}_1 \cdot \vec{\gamma}$)

$$\delta m_i = \frac{i\lambda^2}{(2\pi)^4} \int d^4k_1 \, \frac{(-\not{k}_1 + m_b)}{(k_1^2 - m_a^2 + i\epsilon)(k_1^2 - m_b^2 + i\epsilon)}, \tag{61}$$

which is ultraviolet divergent. Notice that this model is not superrenormalizable but simply renormalizable, a fact which will give rise to certain interesting differences in comparison with the model discussed in the previous subsection. As before, the symbols o.c.t. in Eq. (60) denote the remaining necessary ultraviolet divergent renormalization counterterms (for masses, wave functions and coupling constants). A crucial difference between the actual renormalizable case and the previous superrenormalizable one is that o.c.t. now contains, among other contributions (irrelevant for our purposes) that corresponding to the renormalization of the wave function for the unstable fermion. The latter contribution comes from a term

$$(Z_{i,2} - 1)\Psi_i \Big[\tfrac{i}{2}\, \gamma^\mu \partial_\mu - m_i \Big] \Psi_i$$

in the corresponding lagrangian density, $Z_{i,2}$ being the (ultraviolet divergent) associated wave function renormalization constant for the unstable fermion —it will play an interesting role in the subsequent computation of the survival amplitude.

The dvc and the rules for obtaining the physical survival amplitude $A_\mathrm{ph}(t)$ in renormalized perturbation theory for the present renormalized model are similar to those for the superrenormalizable one, except for some simple modifications. For the case of $A_\mathrm{ph}(t)$, the latter can be simply worked out just by appealing to the corresponding analogy with the Feynman rules for the one-particle's Green function of the i–fermion in momentum space. For the actual $A_\mathrm{ph}(t)$, the modifications of the previous rules 1–9 of the superrenormalizable case (besides the trivial one amounting to replace the three-boson vertices by new vertices with one boson and two fermions) are:

2′) For each internal fermion line with four-momentum k, we replace $(k^2 - m_x^2 + i\epsilon)^{-1}$ in rule 2 by the fermion propagator $(\not{k} - m_x + i\epsilon)^{-1}$ $(x = i, b)$ —other factors remaining unchanged.

4′) For any two fermion vertex, at which only two lines with fourmomenta k_1, k_2 join (necessarily generated by the mass and wave function renormalization counterterms of the unstable fermion), we write

$$i(2\pi)^3\delta^{(3)}(\mathbf{k}_1 - \mathbf{k}_2)\left\{\delta m_i + \left[\gamma^0 \tfrac{1}{2}(k_1^0 + k_2^0) - \vec{\gamma}\cdot\mathbf{k}_1 - m_i\right]\left(Z_{i,2} - 1\right)\right\}$$

6′) We multiply by $\bar{u}(\mathbf{p}_i = 0, \sigma_i)\exp(im_i t_1)$ at left and by $u(\mathbf{p}_i = 0, \sigma_i)\exp(-im_i t_2)$ at right (as we consider normalized Dirac spinors). $u(\mathbf{p}_i = 0, \sigma_i)$ is a Dirac spinor for the i–fermion at rest with spin projection σ_i along some given axis $(\bar{u}u = 1)$.

We shall concentrate in discussing briefly $A_{\rm ph}(t)^{(2)}$, the renormalized contribution to $A_{\rm ph}(t)$ of second order in λ. Using the rules one gets for $t > 0$

$$A_{\rm ph}(t) - 1 \simeq A_{\rm ph}(t)^{(2)} = \bar{u}(\mathbf{p}_i = 0, \sigma_i)Ku(\mathbf{p}_i = 0\sigma_i). \tag{62}$$

On the other hand, the matrix K is given by the r.h.s. of (42) with the simple substitutions: i) $(2m_i)^{-1}$ by unity, ii) $(k_2^2 - m_b^2 + i\epsilon)^{-1}$ by $(k_2 - m_b + i\epsilon)^{-1}$, iii) δm_i^2 by $\delta m_i + (\gamma^0 m_i - m_i)(Z_{i,2} - 1)$. In turn, we shall use Eq. (61) and approximate $Z_{i,2} - 1$ by its standard (ultraviolet divergent) approximation to order λ^2 which for brevity will not be written here. The inclusion of both δm and $Z_{i,2} - 1$, as indicated above, will imply that K is ultraviolet finite.

After some calculations, similar to those in the previous subsection (integrations over k_2^0 and k_1^0, cancellation of ultra-violet divergences, a cancellation similar to that between (55) and the first term in the r.h.s. of (52), etc.) one finds

$$A_{\rm ph}(t)^{(2)} \simeq -\frac{\lambda^2}{16\pi}m_b t + i\lambda^2[\ldots] \tag{63}$$

where the quantity in $[\ldots]$ is real, so that the corresponding contribution to the survival probability is of order λ^4 and, hence, will not be considered here. From (63) we have

$$P(t) = |A_{\rm ph}(t)|^2 \simeq \left|1 + A_{\rm ph}(t)^{(2)}\right|^2 \simeq 1 - \frac{\lambda^2}{8\pi}m_b t. \tag{64}$$

In this case the quadratic term in t is much more complicated than in the previous, superrenormalizable one, and we do not think its expression to be especially relevant. The main point to notice in the above equation is that, again, the survival probability for $t \to 0$ decreases —from its value 1 at $t = 0$— in a linear way in t, although the corresponding coefficient is different from the one in the previous case, as now it depends on the mass of the final fermion, b, whereas (compare (57)) the linear term in the previous case depends just on the mass of the initial unstable particle.

4. Conclusions and outlook

In this paper we have dealt with the problem of the finite time evolution of field-theoretic quantum systems, and, in particular, with the behavior of the survival probability of unstable particles in RQFT. We have analyzed two cases: one of a (superrenormalizable) three-boson interaction and the other of a two fermion-one boson (renormalizable) interaction. In both cases, we have found that the SP decreases at very short times linearly with the time, instead of quadratically as predicted by NRQM. The origin of the different behavior in NRQM and RQFT seems to be due to the fact that in the latter there are vacuum polarization and ultraviolet divergences which made the energy dispersion in the initial (unstable) state to be infinite, then invalidating the main assumption of NRQM, as already commented in the Introduction.

In fact certain difference between NRQM and QFT is already present even in the non-relativistic version of the latter, as we have tried to show through the study of two "classic" models: the polaron model and the Gross-Nelson model. Such a study has been carried out with the aim (mainly) of showing some special features —regarding the question of the short-time behavior of the survival probability for unstable particles— which properly belong to systems with infinite degrees of freedom. In the case of RQFT —the chief subject of this work— and as already pointed out, the most dramatic difference with NRQM is the linear —in t— behavior of the survival amplitude at small t.

However, such a behavior is valid only for very small times ($t \leq (m_a + m_b)^{-1}$). For t–values sufficiently higher one recovers the "normal" exponential decay law. In fact, it is enormously difficult to carry out an experiment to clearly establish that non-exponential behavior (due to the smallness of the relevant times). For instance, assuming that the results of this paper could be naively extrapolated to the case of neutron β–decay we obtain that the times in which the linear (non-exponential) law would apply are those smaller than $t \leq (m_p + m_e)^{-1} \simeq 10^{-23}$ s, which are by now impossible to measure.

Now besides the difficult experimental question of finding some kind of evidence for such a behavior, two other points are worth discussing. The first has to do with the extension of the results obtained here for two rather simple RQFT models to more realistic cases, in particular to Gauge-invariant theories as QED, Weinberg-Salam, QCD, etc. Of course, no conclusions can be rigorously drawn before making an appropriate treatment, but as the origin of the odd linear behavior at very small t seems very "profound" (in the sense of being actually related to the fluctuations of the quantum vacuum) one could reasonably guess that such a behavior will still appear in those, more realistic, cases. The second point concerns the covariance of the method presented here: as commented in Sect. 3, formal covariance is destroyed because a particular inertial frame (the one in which the unstable particle is at rest) has in fact been selected. It is true that one could try to do a formally covariant treatment by using the Tomonaga-Schwinger equation and appropriate spacelike surfaces; however, besides technical difficulties, it is hard to envisage the possible advantages of that treatment, as the very definition of the survival probability (or of the decay probability) entails fixing a frame in which the unstable system is "prepared" ("observed",...) at $t = 0$.

In this paper we have considered RQFT models just at zero temperature. The case of finite temperature can be, however, of some interest, not only by its own, but also because it could be important in the quantum evolution of the early universe, as the times involved in the relevant process at such early stages (we are thinking for instance in times of $\sim 10^{-12}$ s after the Big Bang) are comparable to the times discussed here. However, one should note that finite temperatures actually involve a new time-scale (associated precisely with the temperature) and, therefore, this case would require a careful study.

Finally we will comment upon another alternative treatment of the survival amplitude for an unstable particle in RQFT at very short times. In a recent paper [10], the unrenormalized survival amplitude is singular as $t \to 0^+$ and, in order to regularize it, the authors have introduced a characteristic time, related to the time resolution of the measuring apparatus. As we hopefully have shown here, such a "time resolution" regularization procedure can be replaced alternatively by a more conventional renormalization procedure, which makes the survival amplitude to exist and be finite for all times.

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