

# The theory of multifractal elasticity: basic laws and constitutive equations

ALEXANDER S. BALANKIN

*Instituto Tecnológico y de Estudios Superiores de Monterrey  
Campus Estado de México, México 52629*

Recibido el 18 de agosto de 1995; aceptado el 29 de enero de 1996

**ABSTRACT.** Two basic laws of multifractal elasticity are formulated in general form, leading to a closed system of constitutive equations. This approach allows to explain the elasticity of materials governed by multifractal and hookean mechanisms. It is also noted that for materials of pure multifractal origin their elastic behavior can be used to estimate three first generalized dimensions (metric, information, and correlation) of random (multifractal) microstructure. With the introduced new framework, classical formulas for rubber-like and spring-like elasticity are derived as well as constitutive equations for superelastic materials. The theoretical results reasonably agree with experimental data and computer simulations reported in the literature.

**RESUMEN.** Se formulan en forma general dos leyes básicas de la elasticidad multifractal, que conducen a un sistema de ecuaciones constitutivas. Este método permite explicar el comportamiento elástico de materiales gobernados por mecanismos multifractales y *hookeanos*. Se discute que para materiales de origen puramente multifractal, su comportamiento elástico puede ser empleado para estimar las dimensiones métrica, de información y de correlación de microestructuras estocásticas (multifractales). En este nuevo marco de referencia, se derivan las fórmulas clásicas de elasticidad *rubber-like* y *spring-like*, así como las ecuaciones constitutivas para materiales superelásticos. Los resultados teóricos obtenidos coinciden razonablemente con los datos experimentales y con las simulaciones computacionales reportadas en la literatura.

PACS: 03.40.Dz; 62.20.Dc; 64.30.+t

In the last ten years the theory of materials with random (fractal or multifractal) microstructure has become an attractive topic in mechanics and physics of solids [1-6]. Statistical properties of a multifractal are characterized by a spectrum of generalized dimensions  $d_q$  [1]. Generally, the generalized dimension  $d_q = 0$  is equal to the metric (fractal) dimension evaluated by means of the box-counting algorithm, *i.e.*,  $d_0 = d_F$ , which for multifractal structure is greater than its topological dimension  $d_T$ , but smaller than, or equal to, the topological dimension  $d$  of the enveloping Euclidean space, *i.e.*,  $d_T < d_F \leq d$ ; the generalized dimension of order  $q = 1$  is equal to the information dimension  $d_1 = d_I$ ; and the dimension  $d_q = 2$  is equal to the correlation integral exponent  $d_C = d_2$ , also called *correlation dimension*. Aerogel, colloidal aggregates, polymers, some types of composite materials, porous media, etc. have a multifractal structure in a wide range of spatial scales  $L_0 < L < L_M$  [1-3].

In practice, four different types of elasticity [4]<sup>1</sup> are broadly used for modeling elastic behavior of materials with multifractal microstructure: 1) elasticity of energetic nature (elastic continuum, crystals, polycrystals, etc.), based on the generalized Hooke's law; 2) entropy elasticity (elastomers), based on Gaussian statistics of elements (micro) structure; 3) hyper- and superelasticity, which are described by using phenomenological models for elastic potential; and 4) spring-like elasticity (long polymer chains, foams, some structural composites, etc.) based on the empirical formula for spring.

In the series of our works [7–9] a new type of elasticity was suggested that is the multifractal elasticity (see also reviews [5, 6]). Some models of elastic behavior for materials with fractal and multifractal microstructures have been developed in the works [10–14] on the basis of two postulated phenomenological laws of reversible deformations of multifractal structures.

In this article we give a more general formulation for the basic laws of multifractal elasticity and derive the closed system of constitutive equations. It will also be shown that the entropic rubber-like elasticity, super- and hyperelasticity, and spring-like elasticity are the specific cases of the multifractal elasticity. The generalization of the developed theory which combines the multifractal and hookean models of elastic response is also discussed.

Basic laws of reversible deformations of an elastic multifractal can be formulated in the general form as follows.

1) “When the external force  $F$  is applied to an elastic isotropic multifractal object, deformations occur mostly on the length scale beyond certain characteristic length, which depends of  $F$ . Thus, the presence of an external stress leads to the appearance of the unique new characteristic scaling length  $L_F$ ”.<sup>2</sup>

If this law is valid, from the second law of thermodynamics it follows that the force obeys relation

$$F = \left( \frac{\partial \Psi}{\partial L_F} \right)_T = \left( \frac{\partial U}{\partial L_F} \right)_T - T \left( \frac{\partial S}{\partial L_F} \right)_T. \quad (1)$$

The first term on the right of Eq. (1) is evidently the energy component of internal forces, and the second terms is the entropic component.

2) *The reversible deformations of multifractals are homeomorphic and affine.*

It is clear that a homeomorphic deformation (one-to-one and onto transformation) does not change the metric (fractal) dimension of a deformed multifractal. Moreover in Ref. [15] it was shown that there are no changes in the information and correlation dimensions after affine transformation (deformation) of multifractal structure, while the limiting values

<sup>1</sup> Theories of these types of elastic behavior are based on different phenomenological laws (experimental facts) and lead to different types of constitutive equations.

<sup>2</sup> The initial morphology of a multifractal can either be characterized by one or more length scale parameters  $L_i$  or does not have them at all. In the first case the  $L_F$  may have physical significance such as characteristic dimension of blobs in a polymer, characteristic size of cells, or mean distance between inclusions in a composite material, characteristic radius of correlations in a random network and aerogel, etc. As it follows from the law postulated above, only one of the set of scale parameters  $L_i$  (or their invariant combination) depends on the external forces. If the initial multifractal structure has no scale parameters then the physical meaning of  $L_F$  is the characteristic length above which deformations occur.

of generalized dimension spectrum, *e.g.*,  $d_\infty$  and  $d_{-\infty}$  after affine transformation in general differ from those for initial multifractal structure. Therefore, if the second postulate (law) is valid the metric, information, and correlation dimensions of deformed multifractal structure are constant.

The mass density  $\rho$  of a material with multifractal microstructure generally depends on the length scale  $L$  [3]. Dimensional analysis implies the following general functional relation:

$$\rho = \rho_0 \Psi \left( \frac{\xi_c}{L}, \frac{L_F}{L} \right) = \rho_0 \Psi(\lambda_c, \lambda_f), \quad (2)$$

where  $\Psi$  is a dimensionless function of its dimensionless arguments  $\lambda_c = \xi_c/L$  and  $\lambda_f = L_F/L$ ;  $\xi_c$  is the correlation length of multifractal (micro)structure. It is well known that within the bounded interval

$$L_F \ll L \ll \xi_c \quad (3)$$

the mass density of multifractal structure  $\rho(L)$  obeys a power law behavior [3]. According to inequality (3) the second argument of the dimensionless function  $\Psi(\lambda_c, \lambda_f)$  is small ( $\lambda_f \ll 1$ ); so that at the asymptotic self-similar state we can apply to the function  $\Psi$  a scaling (incomplete self-similarity) representation [16]

$$\rho = \rho_p \lambda_f^{-\alpha} \psi(\lambda_c), \quad \alpha = d - d_F, \quad (4)$$

where  $d_F$  is the fractal (metric, box-counting) dimension of the multifractal and  $\psi$  is a dimensionless function of  $\lambda_c$ . Notice that  $\rho$  possess scaling behavior (4) in the initial state of the multifractal structure as well as after any deformation of this structure, but  $\xi_c$  and  $d_F$ , generally speaking, may be different before and after the deformation.

However, according to the first law of reversible deformations, only the characteristic length  $L_F$  changes after elastic (reversible) deformation of elastically isotropic multifractal, so that  $\xi_c$  must be constant. Furthermore, elastic deformations are homeomorphic by definition and thus do not change the metric dimension  $d_F$  of the deformed multifractal, so that the scaling exponent  $\alpha$  is also constant.

By this means, the mass density of elastic multifractal structure after its reversible (homeomorphic) deformation caused by different external forces  $F_1$  and  $F_2$  is equal to  $\rho(F_1) = \rho_0 \lambda_{f1} \psi(\lambda_c)$  and  $\rho(F_2) = \rho_0 \lambda_{f2} \psi(\lambda_c)$ , respectively. Hence the ratio  $\rho(F_1)/\rho(F_2)$  is not dependent on neither the variable length scale  $L$  nor the correlation length  $\xi_c$  and scales as

$$\frac{\rho(F_1)}{\rho(F_2)} = \left( \frac{\lambda_{f1}}{\lambda_{f2}} \right)^{-\alpha} = \left( \frac{L_{F1}}{L_{F2}} \right)_{T=\text{const}}^{-\alpha} = \lambda_F^{-\alpha}, \quad \alpha = d - d_F, \quad (5)$$

*i.e.* the relation governing the change in the mass density  $\rho$  of elastically deformed multifractals is similar to the relation that governs the change in the mass density because

of the geometric change in the dimensionalities of (multi)fractal structure<sup>3</sup> (within the interval (3)!).

It immediately follows from Eq. (5) that under uniaxial tension (compression) the change in the dimensionality of multifractal structure in the direction of external force  $F_x$ , which is  $\lambda_x = L_x/l_x$ , is accompanied by the change in the lateral dimensionalities of deformed multifractal structure in orthogonal directions of the surrounding  $d$ -space with  $\lambda_i = L_i/l_i$ , where  $i = 1, 2, \dots, d - 1$ . The lateral deformations  $\lambda_i$  are related to  $\lambda_x = \lambda_F$  as

$$\lambda_i = \lambda_{\perp} = \lambda_x^{-\nu_F} = \lambda_F^{-\nu_F}, \quad i = 2, 3, \dots, d, \tag{6}$$

where  $\nu_F$  is the transverse deformation exponent.<sup>4</sup>

Substituting (6) into Eq. (5) we obtain  $\alpha = 1 - (d - 1)\nu_F$ , so that

$$\nu_F = -\frac{\ln \lambda_{\perp}}{\ln \lambda_F} = \frac{d_F}{d - 1} - 1. \tag{7}$$

Therefore, if the first postulate (law) is valid, the transverse deformation exponent  $\nu_F$  of an elastically isotropic multifractal is defined uniquely by its metric dimension.<sup>5</sup>

The data of Table I demonstrate that the results of calculations by the analytical formula (7) agree well with results of computer simulations of the elastic properties of two-dimensional percolation networks near a percolation threshold (see also the Appendix), as well as with the experimental data for aerogel SiO<sub>2</sub> (which were obtained in the studies of the longitudinal and transverse elastic waves propagation), rubber, and a strongly twisted nondeformable polymer filaments.

At the same time, it must be emphasized that Eqs. (5) and (7) are not valid for materials obeying conventional Poisson's effect, or, what is equivalent, the generalized Hooke's law. For such materials the correlation length  $\xi_c$  is equal to the sample size in the direction of applied external force, *i.e.*,  $\xi_c \equiv L_x$ ; so that  $\xi_c$  and

$$\lambda_C = \frac{\lambda_{c1}}{\lambda_{c2}} = \frac{\xi_{c1}}{\xi_{c2}} \tag{8}$$

must change after deformation. At the same time, for an elastic continuum  $\lambda_f = 0$  and  $\lambda_F \equiv 1$ , while for a regular elastic lattice  $\lambda_f = a/a_0$  ( $a$  and  $a_0$  are the interatomic distances before and after elastic deformation, respectively) and  $\lambda_F = \lambda_C$ . Hence, for materials obeying Poisson's effect our first postulate is not valid.

<sup>3</sup> Notice that formerly this fact was postulated as the second law (see, for example reviews [5, 6]). In fact, as is shown above, Eq. (5) is a direct consequence of the first law and of the homeomorphism of reversible deformations.

<sup>4</sup> Notice that the exponent of lateral deformations  $\nu_F = -\ln \lambda_{\perp}/\ln \lambda_F$  is equal to the Poisson's ratio  $\nu = -\sqrt{|\lambda_{22} - 1|/|\lambda_{11} - 1|}$  only in the limit of infinitely small strains  $\epsilon_{ii} = \sqrt{|\lambda_i^2 - 1|} \ll 1$ .

<sup>5</sup> At first glance, it is surprising that lateral deformations are independent of the detailed geometry of deformed multifractal. Notice, however, that similar situations (power law distributions of stresses and strains with exponents which are a function only of Poisson's ratio) are common within singular problems in the classical theory of the elastic continuum obeying the intermediate asymptotic behavior (see, for example [17, 18]).

TABLE I. Comparison of Poisson's ratio  $\nu$ , calculated using analytical relationship (7), with the computed values, based on two-dimensional elastic random networks, and with experimental data for aerogel SiO<sub>2</sub>, strongly twisted nondeformable polymer filament, and rubber.

Properties	Two-dimensional random network of size $L \times L$ near the percolation threshold ( $\xi_C$ — correlation length)		Aerogel SiO <sub>2</sub>	Strongly twisted nondeformable filament	Rubber
	$L/\xi_C \rightarrow \infty$	$L/\xi_C \rightarrow 0$			
Connectedness of random network	Elasticity of network is determined by dangling bonds	Bonds, determining elasticity of network are multiduplicated	Fractal cluster	Monomer	Polymer network
Fractal dimension of elastic backbone $d_F$	Dimension of geodesic line, $d_F = d_{\min}$ , $d_{\min} = 1.1 \pm 0.02$ [29]	Dimension of red bonds, $d_F = d_{rb}$ , $d_{rb} = 3/4$ [29]	$d_F$ , measured by small-angle neutron scattering and molecular adsorption [31]: $2.3 \pm 0.1$	Dimension of self-avoiding random walk: $d_{s-a} = 2$ [29]	$d_F = 3.00 \pm 0.04$ [25, 26]
$\nu_F$ , Eq. (7)	$0.1 \pm 0.01$	$-1/3$	$0.15 \pm 0.05$	0	$0.50 \pm 0.02$
Poisson's ratio (results of numerical simulation and experimental data)	$0.08 \pm 0.04$ [28]	$-0.33 \pm 0.01$ [28]	$0.12 \pm 0.08$ [30]	0 [27]	$0.50 \pm 0.01$ [22]

Furthermore, the metric dimension of an elastic continuum as well as a regular elastic lattice ( $d_F \equiv d_q$  for all  $q$ ), which can be considered as a limiting cases for multifractal structures, is equal to the topological dimension of structure  $d_T$ ; so that

$$d_F = \begin{cases} d, & \alpha = \begin{cases} 0, & d_T = d \\ 1, & d_T = d - 1. \end{cases} \end{cases} \quad (9)$$

Now, it is easy to understand that in the case of an euclidean elastic structure obeying generalized Hooke's law the dimensionless function  $\psi_P(\lambda_C) = \psi(\lambda_{c1})/\psi(\lambda_{c2})$  scales (in the limit of infinitely small strains!) as

$$\psi_P = \left( \frac{\xi_{c1}}{\xi_{c2}} \right)^{-\beta} \equiv \left( \frac{L_x(F_1)}{L_x(F_2)} \right)^{-\beta} = \lambda_C^{-\beta}, \quad (10)$$

where  $\beta = 1 - (d_T - 1)\nu$ . The relation for the conventional Poisson's effect  $\epsilon_{jj} = \nu\epsilon_{11}$  (see footnote 4) may be derived by substitution of Eqs. (10) and (6) with  $d_T = d$  in the general relation (4).

Therefore, in the general case of an elastic multifractal structure, the lateral deformations of which are governed by the combination of lateral deformations of the multifractal nature with those associated with the conventional Poisson's effect, the scaling relation (5)

can be generalized into

$$\frac{\rho(F_1)}{\rho(F_2)} = \lambda_F^{-(d-d_F)} \lambda_C^{(d_F-1)\nu-1}. \tag{11}$$

It should be emphasized, that relation (11) is valid only in the limit of infinitesimally small strains  $\varepsilon_{ii} = \sqrt{|\lambda_i^2 - 1|} \ll 1!$

Now, if a multifractal structure possesses Poisson’s effect with  $\lambda_C = \lambda_F$ , the transverse deformation exponent is equal to

$$\nu_F = \nu - \frac{d - d_F}{d - 1}. \tag{12}$$

This relation governs the lateral deformation of the multifractal as well as the Poisson’s nature.

Below we will consider only the elastic multifractals’ reversible behavior, which is governed by the two laws postulated above. The lateral deformations of such a structure have a pure metric origin ( $\lambda_C \equiv 1$ )<sup>6</sup> and are governed by the metric dimension of the structure (see Eq. (7)).

Looking back at the two proposed laws of reversible deformations for multifractal structures, we see that we need to know the changes in the entropy and internal energy as functions of deformations before the closed system of constitutive equations can be obtained.

Using the definitions of the information and correlation dimensions and their scaling properties [1, 2] with relations (1), (5) valid, it is easy to show that the changes in the thermodynamic entropy  $\Delta S(\lambda_i)$  and in the internal energy  $U(\lambda_i)$  during the reversible deformation of an elastic multifractal in  $d$ -dimensional space can be represented in the form [9]

$$T \Delta S = -C_2 \left( \sum_{i=1}^d \lambda_i^{d_i} - d \right), \tag{13}$$

and

$$\Delta U = -C_1(\lambda_F^{\alpha_C} - 1), \quad \alpha_C = d - d_C, \tag{14}$$

respectively. Here  $C_1$  and  $C_2$  are constants (notice that parameters  $C_1$  and  $C_2$  can also be determined for any detailed model of the structure [19]).

Substituting (13) and (14) into (1), and using relations (5)–(7) we can derive the relationships between external force  $F_i$  and relative deformations  $\lambda_i$  of multifractal structure. For example, in the case of uniaxial deformation (tension or compression) we obtain

$$F_1 = C_2 \left\{ d_1 \lambda_1^{d_1-1} - d_1 [d_F - (d - 1)] \lambda_1^{-d_1 [d_F / (d-1) - 1] - 1} - \frac{C_1}{C_2} (d - d_C) \lambda_1^{d-d_C-1} \right\}. \tag{15}$$

---

<sup>6</sup> It is pertinent to note that in the case  $\lambda_C = \lambda_F$  and  $\nu = 1/(d-1)$  Eqs. (11), (12) are equivalent to Eqs. (5), (7), so that all results considered below are also valid in this more general case.

According to the obvious condition  $F(\lambda_i = 1) = 0$ , it follows from Eq. (15) that

$$\frac{C_1}{C_2} = d_1 \frac{d - d_F}{d - d_C} \leq d_F, \tag{16}$$

so that<sup>7</sup>

$$F_1 = C_2 d_1 \left\{ \lambda_1^{d_1-1} - [d_F - (d - 1)] \lambda_1^{-d_1[d_F/(d-1)-1]-1} - (d - d_F) \lambda_1^{d-d_C-1} \right\}. \tag{17}$$

Thus behavior patterns of elastic multifractals (for which two laws postulated above are valid!) are completely determined by the metric, information, and correlation dimensions.

Moreover, it is easy to see that in the limit of infinitely small strains, Eqs. (17) and (7) can be generalized by using Eqs. (11) and (12).

In the case of monofractal structure all generalized dimensions are equal to the metric (fractal) dimension, *i.e.*,  $d_q \equiv d_F = d_1 = d_C$  [1], and Eq. (16) results in the equality

$$\frac{C_1}{C_2} = d_F. \tag{18}$$

So that, Eq. (15) may be rewritten in the form

$$F_1 = C_1 \left\{ \lambda_1^{d_F-1} - [d_F - (d - 1)] \lambda_1^{-\nu_F d_F-1} - (d - d_F) \lambda_1^{\alpha-1} \right\}. \tag{19}$$

The stress  $\sigma_{11}$  is related to the force  $F_1(\lambda_1)$  by obvious equation  $\sigma_{11} = F_1 \lambda_1^{1-\alpha}$ , which by using Eqs. (7), (19) may be written in the form

$$\sigma_{11} = \frac{E}{1 + 6\nu_F + 4\nu_F^3} \left[ (\lambda_1^{1+4\nu_F} - 1) - 2\nu_F (\lambda_1^{-1-2\nu_F^2} - 1) \right], \tag{20}$$

where  $E = (\partial\sigma_{11}/\partial\varepsilon_{11})_T$  is the Young modulus.

It is easy to see that, within the limit of infinitesimally small strains,  $|\varepsilon_{11}| = \sqrt{|\lambda_1^2 - 1|} \ll 1$ , Eq. (20) leads to its classic counterpart for elastic continuum. Thus, for monofractal structure we have  $C_1 = 2(1 + \nu_F)C_2 = (1 + 6\nu_F + 4\nu_F^3)E$ . Similarly, we can derive the relations  $\sigma_{ij}(\lambda_k)$  for  $n$ -axial deformation of an elastically isotropic monofractal in the  $d$  space. The pure shear is essentially a biaxial loading under the stresses  $\sigma_{11}$  and  $\sigma_{22}$  such that there is no change in length along second direction, *i.e.*,  $\lambda_2 = 1$  (see Ref. [20]).

The relationships between the elastic moduli, *i.e.*, Young's modulus  $E$ , shear modulus  $G$ , and balk modulus  $B$ , and Lamé coefficients  $\lambda, \mu$  of elastically isotropic (mono!)fractal,

---

<sup>7</sup> In Refs. [5, 6] we have conjectured that for an elastic multifractal  $d_1(d - d_F) = d_C(d - d_C)$ . This conjecture was made in the spirit of the Dirac's manifest that "a physical law must possess mathematical beauty". However, there is no experimental evidence in support of this conjecture. In this sense the Eqs. (16) and (17) are more correct than the related equations in Refs. [5, 6].

which are derived by analogy to the derivation of the corresponding relationships in classical theory of elastic continuum [20], are the following:

$$G = \frac{E(d-1)}{2d_F}, \quad B = \frac{E}{d(d-d_F)}, \quad B = \lambda + \frac{2}{d}\mu. \quad (21)$$

Notice that these relations differ from those which were conjectured for elastic fractals by Bergman and Kantor [21]. On the other hand, substituting Eq. (7) in the Eqs. (21) we obtain expressions that for  $d = 2$  and  $d = 3$  are identical to those for two- and three-dimensional elastically isotropic continuum!

Classical formulas for rubber-like elasticity [22]

$$F = \frac{E}{3}(\lambda_1 - \lambda_1^{-2}), \quad \lambda_2 = \lambda_3 = \lambda_1^{-1/2}(\rho = \text{const}), \quad (22)$$

may be derived within a framework of Eqs. (17) and (5)–(7) for multifractals obeying

$$d_F = d = 3, \quad \text{and} \quad d_I = 2.$$

Notice that the condition  $d_I = 2$  is associated with the Gaussian statistics [7, 23], which is used in the classical models of rubber-like elasticity [22], while the equality  $d_F = d = 3$  is the condition of incompressibility (see Eq. (5)), which is also assumed in the classical theory [22].

In fact, however, calculations based on Eq. (22), with the value of  $E$  adjusted by fitting, are in reasonable agreement with the experiments only in the range of relatively small strains ( $\lambda_i < 1.2$ ) [22]. Traditionally, the refinement of the relation (22) is made by phenomenological modifications of the entropic theory, or by using empirical models for the elastic potential [22, 24]. At the same time, elastomers are known to have fractal or multifractal microstructure [3, 25, 26]. Therefore, it is natural to describe rubber elasticity of polymers by using the results of the concept of multifractal elasticity discussed above.

Generally the generalized dimensions of polymer networks swelled in a good solvent are within the range  $2 < d_q \leq 3$  [3, 25, 26]. Assuming in the first approximation that  $d_F = d_I = d_C$  and substituting Eq. (7) into Eq. (19), we obtain the relationship between the nominal stress  $F_1$  and the strain factor  $\lambda_1$  in the case of uniaxial tension (compression) of an elastomer in the following form:

$$F_1 = \frac{E}{1 + 6\nu_F + 4\nu_F^3} \left\{ \lambda_1^{1+2\nu_F} - 2\nu_F \lambda_1^{-1-2\nu_F(1+\nu_F)} - (1 - 2\nu_F) \lambda_1^{-2\nu_F} \right\}, \quad (23)$$

which was first derived in our work [7] by other means.<sup>8</sup> Notice, that behavior (23) differs from (22) even in the limit of incompressibly deformed material, when Eq. (23) reduces to the formula

$$F_2 = \frac{E}{4.5}(\lambda_1^2 - \lambda_1^{-2.5}), \quad (24)$$

---

<sup>8</sup> We emphasize that Eq. (23) is a limiting case of the general equation (19) which gives a richer set of various behavior patterns.



obeying the experimentally established asymptotic (see Ref. [22])

$$F_1 \propto \lambda_1^2, \quad \text{when } \lambda_1 \gg 1.$$

It was shown in Refs. [5–11] that the calculations based on Eq. (23) agree well with experimental data for rubbers without any adjustment of parameters (except  $E$ ) right up to  $\lambda_1 = 7$ .

The nonlinear stress-elongation asymptotic for superelastic networks [27],

$$\sigma_{11} \propto \lambda_1^{1/3}, \quad (25)$$

is a special case of constitutive equations (17) and (7), which is valid for multifractal structures possessing

$$d_F + d_I - d = \frac{1}{3}, \quad (26)$$

for example  $d_F = 2$ ,  $d = 3$ ,  $d_I = 4/3$ .

The basic relation of spring-like elasticity [22]

$$F = E(\lambda - 1) \quad (27)$$

may be also derived from (17) and (5)–(7) in the case of multifractal structures for which  $d_F = d_I = 2$  ( $d = 3$ ).

Furthermore, the pseudo-elastic, super-elastic, and rubber-like behavior of alloys with thermoelastic martensitic transformation [3] can be also evaluated within a framework of the multifractal theory of elasticity.<sup>9</sup>

Hence the proper regard for the real morphology allows an adequate description for the behavior of a reversible deformed material with fractal or multifractal microstructure. Considered examples revealed that two laws of reversible deformations of multifractal structure, which are postulated in present work, are valid at least for some classes of materials with (multi)fractal (micro)structure.

We hope this work will stimulate experimental research along this line. We expect that these investigations will support our concept of multifractal elasticity. If so, experimental data on elastic behavior can be used to estimate the metric, information, and correlation dimensions of the (multi)fractal microstructure.

#### ACKNOWLEDGMENTS

The author is grateful to G. Cherepanov for useful discussions. The courtesy and support of the staff and faculty of Materials Research Center at Monterrey University of Technology, and especially, A. Bravo, are gratefully acknowledged.

---

<sup>9</sup> This topic will be considered in more details in a forthcoming paper.

## APPENDIX

The percolating network is a fundamental model for describing geometrical features of random systems [2, 29]. There are two main kinds of percolating networks: “site” and “bond”. To create a site-percolating network, each intersection (site) of an initially prepared  $d$ -dimensional lattice is occupied at random with probability  $p$ . Sites are connected if they are adjacent along a principal direction. In a bond-percolation network, all sites are initially occupied and bonds are occupied randomly with probability  $p$ . At a critical (different) concentration  $p = p_C$  ( $p_C$  also depends on the dimension and symmetry of original lattice), both site and bond percolation exhibit a single, infinite cluster spanning all space.

When the occupation probability  $p$  is less than  $p_C$ , there are only finite clusters, whereas if  $p > p_C$  an infinite cluster is present as well as finite clusters. The cluster both at  $p_C$  and away from  $p_C$  are characterized by the number of sites,  $s$ , in the cluster and by the radius of gyration,  $R_g(s)$ , of the cluster. Over what distances are occupied sites connected? The connectedness length  $\xi_C$  is defined as the average root mean square distance between occupied sites that belong to the same and *finite* cluster. This connectedness length is also called the correlation length. The lower cutoff scale characterizing the percolation cluster is the length  $a$  that forms the lattice spacing of the original lattice.

For the problem of the electrical conductivity of a random resistor network and transport dynamic properties, another object is relevant: the “backbone” of an infinite percolation cluster [29]. The backbone is defined as the network of unblocked connected bonds, through which one can go to infinity by at least two nonintersecting paths. In other words, the backbone is a set of bonds, through which electric current would flow were a voltage applied to the cluster at infinitely remote electrodes. The rest of the cluster is referred to as a collection of “dead” or “dangling ends”. A dangling end can be disconnected from the cluster by cutting a single bond. In random resistor network near the percolation threshold, some bonds carry the highest current. These bonds form the set of single connected bonds of the backbone. This set of bonds were called “red bonds” (when one red bond is cut, the current flow stops). The rest of bonds of the backbone is the “blue bonds” (blue bonds carry current, but when a blue bond is cut, the resistance of the system only slowly increases). Another important concept is the minimal or “chemical” path between two points of a backbone, which is the shortest path between the two points. The shortest path between the sites  $i$  and  $j$  realizes along the “geodesic line” and can be defined as the minimal number of steps by which we can reach  $j$  from  $i$ , with restriction to existing paths between connected sites. This is termed the chemical distance  $L_{\text{chem}}$  (notice that it is not the same as the linear length measured between the two points and referred to as the “Euclidean distance”).

For any length scale  $L \gg \xi_C$ , a percolating system is macroscopically homogeneous. But for  $a \ll L \ll \xi_C$ , the system possesses statistical scale invariance and may be referred to as a fractal or multifractal object, all properties of which obey a power law dependence on its characteristic linear size  $L$  [29]. For example, the total number of bonds (or sites) of the cluster scales as  $N_{\text{cl}} \sim L^{d_{\text{cl}}}$ , where  $d_{\text{cl}}$  is the fractal dimension of the cluster; the total number of backbone bonds scales as  $N_{\text{bb}} \sim L^{d_{\text{bb}}}$ ; the number of red bonds  $N_{\text{red}}$  scales with  $L$  as  $N_{\text{red}} \sim L^{D_{\text{rb}}}$ . The chemical dimension  $d_{\text{chem}}$  is defined from the relation

$N_{\text{chem}} \sim L_{\text{chem}}^{d_{\text{chem}}}$ . The chemical length scales with  $L$  as  $L_{\text{chem}} \sim L^{d_{\text{min}}}$ , where the fractal dimension of geodesic line  $d_{\text{min}}$  is equal to the ratio  $d_{\text{cl}}/d_{\text{chem}}$ .

Recently, there has been a considerable interest in the elastic properties of percolating random networks [21, 28]. It was found that for elastic problem the critical value of occupation probability  $p_{\text{C el}}$  differs from  $p_{\text{C}}$  [32]. For example, for percolation on the two-dimensional triangular lattice  $p_{\text{C}} = 0.3473$ , whereas  $p_{\text{C el}} = 0.58$  [32]. Hence the elastic backbone which governs elastic behavior of percolating network also differs from backbone governed electrical resistance. Among the surprises that were uncovered was the phenomenon of a negative Poisson's ratio  $\nu$  of percolation network of size  $L \ll \xi_{\text{C}}$  near the percolation threshold ( $L \gg a$ ) [28]. It was also found that the Poisson's ratio is sensitive to the precise value of  $L/\xi_{\text{C}}$ , changing from  $-0.33$  when  $L/\xi_{\text{C}} \rightarrow 0$  to about  $+0.08$  when  $L/\xi_{\text{C}} \rightarrow \infty$  (a zero Poisson's ratio was found for  $L/\xi_{\text{C}} \simeq 0.2$ ) [28]. The most important result of the referred works is that the limiting values of  $\nu$  (namely  $\nu(0) = -0.33$  and  $\nu(\infty) = 0.08$ ) are independent of the elementary (microscopic) force constants of the bonds. This give the rise to describe the elastic behavior of such networks within a framework of the theory of multifractal elasticity (see Table I).

## REFERENCES

1. T. Vicsek, *Fractal Growth Phenomena*, London, World Scientific (1992).
2. A.S. Balankin, *Synergetics of Deformed Solids*, Moscow, Department of Defense USSR press (1991).
3. V.S. Ivanova, A.S. Balankin, I.J. Bunin, and A.A. Oksogoev, *Synergetics and Fractals in Material Science*, Nauka, Moscow (1994).
4. G.P. Cherepanov, A.S. Balankin and V.S. Ivanova, in *A Topical Encyclopedia of Current Knowledge Dedicated Alan Arnold Griffith*, Edited by G.P. Cherepanov, Krieger, Melbourne, U.S.A. (1996).
5. A.S. Balankin and P. Tamayo, *Rev. Mex. Fís.* **40** (1994) 506.
6. G.P. Cherepanov, A.S. Balankin and V.S. Ivanova, *Eng. Fracture Mech.* **51** (1995) 997.
7. A.S. Balankin, *Sov. Phys. Tech. Lett.* **17** (1991) 632.
8. A.S. Balankin, *Dokl. Russian Academy of Sci.* **319** (1991) 1098.
9. A.S. Balankin, *Sov. Phys. Dokl.* **37** (1992) 379.
10. A.S. Balankin and A.L. Bugrimov, *Polymer Sci.* **34** (1992) 889.
11. A.S. Balankin and A.L. Bugrimov, in *Abstr. XXVIII Int. Congr. Theor. and Appl. Mech. Haifa, Israel, August 22-28, 1992*, Int. Union Theor. and Appl. Mech. Press, Haifa (1992) 17.
12. A.S. Balankin, *Phys. Lett.* **A210** (1996) 51.
13. A.S. Balankin, *Sov. Phys. Solid State* **34** (1992) 658.
14. A.S. Balankin, *Phys. Rev.* **B 53** (1996) 5438.
15. A.S. Balankin, *J. Phys.* **A** (in press).
16. G.I. Barenblatt, *Similarity, self-similarity and intermediate asymptotics*, New York, Consultant Bureau (1979).
17. G.P. Cherepanov, *Mechanics of Composite Materials*, Moscow, Nauka (1983).
18. A.S. Balankin, *Sov. Tech. Phys. Lett.* **17** (1991) 229; 391.
19. S.V. Panyukov, *Sov. Phys. JETP* **71** (1990) 372.
20. A.E.H. Love, *A Treatise on the Mathematical Theory of Elasticity*, Dover, New York (1944).
21. D.J. Bergman and Y. Kantor, *Phys. Rev. Lett.* **53** (1984) 511.
22. L. Treloar, *The Physics of Rubber Elasticity*, Clarendon Press, Oxford (1975).
23. C.E. Puente, *Phys. Lett.* **A 161** (1992) 441.

24. S. Edwards and T. Bilgis, *Rev. Progr. Phys.* **51** (1988) 243.
25. A.M. Aharoni, in *Synthesis, Characterization, and Theory of Polymer Networks and Gels*, edited by S.M. Aharoni, Plenum Press, New York (1992).
26. M. Kallala, C. Sanches and B. Cabane, *Phys. Rev.* **E 48** (1993) 3692.
27. S.P. Obukhov, M. Rubinstein and R.H. Colby, *Macromolecules* **27** (1994) 3191.
28. E. Duering and D.J. Bergman, *Phys. Rev.* **B 37** (1988) 9460.
29. M.B. Isichenko, *Rev. Modern. Phys.* **64** (1992) 961.
30. M. Gronauer, A. Kadur and J. Fricke, in *Proc. 1st Symp. Aerogels, Wurzburg, Germany, 1985*, Springer-Verlag, Berlin (1986).
31. B.M. Smirnov, *Physics of Fractal Clusters*, Nauka, Moscow (1991).
32. S. Feng and P.N. Sen, *Phys. Rev. Lett.* **52** (1984) 216.