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An application of group theory to the solution of the N–level atom. A geometric representation

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ABSTRACT. Electric and magnetic interactions are considered throughout the formalism of group theory. We claim here that a general theoretical prescription may be established for the exact solution of N–level atoms. However the achievement of a geometrical representation is hevealy based on spin vector behavior of electric dipolar interaction. As an example an explicit geometrical representation for 3 level atoms is presented using this pseudospin vector analogy.

RESUMEN. Las interacciones eléctricas y magnéticas son consideradas a través del formalismo de la teoría de grupos. Afirmamos que se puede resolver exactamente el átomo de N niveles mediante el uso de una prescripción teórica general de teoría de grupos. Sin embargo, una representación geométrica sólo puede lograrse a través del *comportamiento* espinorial de la interacción dieléctrica. Como un ejemplo de aplicación del método, resolvemos el átomo de 3 niveles utilizando esta analogía seudoespinorial.

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1. INTRODUCTION

In this paper we discuss electric dipole interactions (EDI) and magnetic dipole interactions (MDI). At the classical level, the difference between EDI and MDI is based on the fact that the electric dipole moment is a *polar* vector, whereas the magnetic dipole moment is an *axial* vector. In the quantum domain, instead, the difference between EDI and MDI tends to disappear, and we can analyse them in equivalent forms.

We know from elementary quantum mechanics that the theory of angular momentum allows a geometrical representation; we also know that the theory of angular momentum is fundamental in our understanding of MDI and multilevel systems. The natural conclusion is to look for a geometrical representation of MDI; in fact, the firs attempt in this direction was done , in a very extensive way, by R. Gilmore [1], upon consideration of symmetrized state. This geometric representation of MDI provided a method to penetrate areas like nuclear magnetic resonance and nuclear quadrupole resonance (NQR). We then inquire wether a similar approach is possible for EDI; we think that a geometrical representation of EDI could, in principle, provide a method to explore areas like laser cooling of atoms, both theoretically and experimentally [2]; in this specific area, for example, a geometrical representation of EDI would provide a method to obtain analytic, exact solutions.

The idea of a geometric representation of EDI was first proposed by Feynman, Vernon and Hellwarth [3] in 1957, for two-level atoms; the extension to N–level atoms was later pursued by several authors, from different points of view [4–10]. We want to mention in particular a paper by Cook and Shore [11], in which the authors present a nice analogy that leads them to the original geometrical representation of Ref. [3].

This paper is organized as follows: In Sect. 2 we discuss briefly the comparison between EDI and MDI, both classically and quantically. In Sect. 3 we describe the general formalism for N–level atoms using SU(N). In Sect. 4 we study in particular the well-known case of 3-level atoms. In Sect. 5 we present EDI in an explicit form.

2. COMPARISON BETWEEN EDI AND MDI

In classical mechanics the time evolution of angular momentum $\mathbf{L}$ is given by

$$\frac{d\mathbf{L}}{\mathrm{dt}} = \boldsymbol{\tau}_{\mathrm{ext}}, \tag{1}$$

where $\boldsymbol{\tau}$ is the torque exerted by external forces; now, if $\boldsymbol{\tau}_{\mathrm{ext}}$ may be expressed by

$$\boldsymbol{\tau}_{\mathrm{ext}} = \mathbf{L} \times \boldsymbol{\Omega}, \tag{2}$$

means that $\mathbf{L}$ precesses with angular velocity $\boldsymbol{\Omega}$. Thus, in an external magnetic field $\mathbf{H}$, the magnetic dipole moment $\mathbf{M}$ precesses according to the equation

$$\frac{d\mathbf{M}}{\mathrm{dt}} = \mathbf{M} \times (\gamma \mathbf{H}), \tag{3}$$

where $\mathbf{M} = \gamma \mathbf{H}$ is an axial vector. It is not possible to have an equivalent equation for the electric dipole moment $\mathbf{p}$, although is known that in an external electric field the torque experimented by the dipole is given by

$$\boldsymbol{\tau}_{\mathrm{ext}} = \mathbf{r} \times [\mathbf{p} \cdot \nabla \mathbf{E}] + \mathbf{p} \times \mathbf{E}.$$

If the field is homogeneous, then $\nabla \mathbf{E} = 0$ and we obtain

$$\boldsymbol{\tau}_{\mathrm{ext}} = \mathbf{p} \times \mathbf{E}. \tag{4}$$

However, a direct relationship between $\mathbf{p}$ and $\mathbf{L}$ does not exist and therefore it is not possible to have an equation like (3), we find, though, that (2) and (4) have a suggestive similarity.

2.1. Quantum similarities

The hamiltonian for a magnetic dipole moment **M** in a magnetic field **H** is

$$H_m = -\mathbf{M} \cdot \mathbf{H} = \gamma \hbar H \hat{I}_z, \tag{5}$$

where we have introduced the dimensionless angular momentum $\hat{I}$, and assumed **H** parallel the z–axis; as consequence, eigenvalues of energy are integer multiples of $\gamma\hbar H$, *i.e.*,

$$E = -m(\gamma \hbar H), \qquad m = I, I-1, \ldots, -I; \tag{6}$$

I corresponds to eigenvalues of $\hat{I}$.

In order to study transitions among levels is usual to introduce [12] an oscillating perturbative field perpendicular to the static magnetic field **H**. The corresponding Hamiltonian is then

$$H_{\mathrm{p}} = -\gamma \hbar H_x^0 \hat{I}_x \cos wt. \tag{7}$$

Now, $\hat{I}_x$ has matrix elements of the form $\langle m'|I_x|m\rangle$ with $m' = m \pm 1$. Thus the allowed transitions through the coupling with H_x take place among adjacent levels (for a given **L**). This is the quantum form to describe a vector operator associated to a classical motion equation given by (3). For electric interaction the Hamiltonian, in the case of two levels may be written as [13]

$$\mathbf{p} \cdot \mathbf{E} = (p_{12} \cdot E^{\dagger})\sigma + \sigma^{\dagger}(p_{21} \cdot E)$$

E and **p** have been expanded as

$$\begin{aligned} \mathbf{E}(r,t) &= E(r)e^{-iwt} + E^{\dagger}e^{iwt}, \\ \mathbf{p} &= p_{12}\sigma e^{-iwt} + p_{21}\sigma^{\dagger}e^{iwt}, \end{aligned}$$

where σ and $\sigma^{\dagger}$ are projection operators, and

$$\sigma_{ij}\sigma_{k\ell} = \sigma_{i\ell}\,\delta_{jk}.$$

So that σ are ladder operators and transitions among no adjacent levels are allowed.

3. GENERAL FORMALISM FOR N–LEVEL ATOMS USING SU(N)

Let us consider an N–level atom. The state vector $|\psi\rangle$ may be written

$$|\psi\rangle = \sum_{i=1}^{N} a_i |\psi_i\rangle, \tag{8}$$

where a_i are probability amplitudes of occupation of eigenstates $|\psi_i\rangle$.

We may consider coefficients a_i as elements of a N–dimensional vector basis, that will be assumed as that of definitorial representation of the special unitary group SU(N):

$$\mathbf{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_N \end{bmatrix}.$$

The number of generators G_i is $N^2 - 1$ and, in the definitorial representation they correspond to $N \times N$ matrices. Now, let us propose the basis associated to the regular representation whose dimension is $N^2 - 1$. Elements c_i in this new representations are calculated according to the rule

$$c_i = a^\dagger G_i a, \tag{9}$$

where $i = 1 \ldots N^2 - 1$, and G_i are expressed in the definitorial representation. As an example, in SU(2), G_i correspond to the Pauli matrices $(\sigma_x, \sigma_y, \sigma_z)$, and c_i are given by

$$c_x = a^\dagger \sigma_x a, \quad c_y = a^\dagger \sigma_y a, \quad c_z = a^\dagger \sigma_z a \tag{10}$$

This case corresponds to two-level atoms; a geometrical representation was proposed in 1957 introducing (10) as an ad-hoc definition. Thus

$$\begin{aligned} c_x &= \begin{bmatrix} a_1^* & a_2^* \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \\ &= a_2^* a_1 + a_1^* a_2. \end{aligned} \tag{11}$$

Analogously, using

$$\sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

we have

$$\begin{aligned} c_y &= i(a_1 a_2^* - a_1^* a_2), \\ c_z &= |a_1|^2 - |a_2|^2. \end{aligned} \tag{12}$$

Then, the equation of motion for $\mathbf{c}$ is

$$\dot{\mathbf{c}} = -\frac{2}{\hbar}\mathbf{c} \times \boldsymbol{\alpha} = \mathbf{w} \times \mathbf{c}, \tag{13}$$

describing the precession of $\mathbf{c}$ with frequency $\mathbf{w}$. This geometrical representation allows to obtain the observables of the system using the values of $\mathbf{c}$, and this simplifies the calculations.

Now, is it possible to get equations like (13) for the atoms with higher number of levels? or does an $N^2 - 1$ dimensional vector $\mathbf{c}$ associated to the regular representation of SU(N) show precession? We may answer this question by deriving (9) with respect to time:

$$\dot{\mathbf{c}} = \frac{d}{dt}(a^\dagger \mathbf{G} a) = \dot{a}^\dagger \mathbf{G} a + a^\dagger \mathbf{G} \dot{a}. \tag{14}$$

Using Schrödinger equation

$$i\hbar \dot{a} = \hat{H} a, \tag{15}$$

we have

$$\begin{aligned} \dot{\mathbf{c}} &= -\frac{1}{i\hbar}[(\hat{H}a)^\dagger \mathbf{G} a - a^\dagger \mathbf{G}(\hat{H}a)] \\ &= -\frac{1}{i\hbar}[a^\dagger \hat{H}^\dagger \mathbf{G}\mathbf{a} - \mathbf{a}^\dagger \mathbf{G}(\hat{\mathbf{H}}\mathbf{a})] \\ &= -\frac{1}{i\hbar}\{a^\dagger [\hat{H}, G] a\}, \end{aligned} \tag{16}$$

where we have used the hermiticity of H.

We observe that Eq. (16) is valid in general form, as for any value of N. According Eq. (9) the number of generators $\mathbf{G}$ fixes the dimension of vector space.

If $\mathbf{G}$ belongs to SU(N), $\mathbf{c}$ belong to R^{N^2-1}; thus according to group theory, for arbitrary N, we may expand $\mathbf{H}$ as

$$\hat{H} = \alpha_i G_i + \beta I. \tag{17}$$

I is the identity, α, β are complex constants and we use the sum convention for repeated index, with $i, j, k, \ell, \ldots = 1, \ldots, N^2 - 1$. Each component c_ℓ of $\mathbf{c}$ may be written as

$$\dot{\mathbf{c}}_\ell = -\frac{1}{i\hbar}\left\{a^\dagger [\hat{H}, G_\ell] a\right\} \tag{18}$$

$$= -\frac{1}{i\hbar}\left\{a^\dagger [(\alpha_k G_k + \beta I), G_\ell] a\right\}. \tag{19}$$

By using the conmutation rule for generators in SU(N),

$$[G_k, G_\ell] = i f_{k\ell m} G_m,$$

Eq. (19) takes the form

$$\begin{aligned}\dot{\mathbf{c}}_\ell &= -\frac{1}{i\hbar}\Big\{a^\dagger[(\alpha_k G_k), G_\ell]a\Big\}\\ &= -\frac{1}{i\hbar}\Big\{a^\dagger\alpha_k[G_k, G_\ell]a\Big\}\\ &= -\frac{1}{i\hbar}\Big\{a^\dagger(\alpha_k i f_{k\ell m}G_m)a\Big\},\end{aligned}$$

where $f_{k\ell m}$ are the structure constants of the group. Finally,

$$\begin{aligned}\dot{\mathbf{c}}_\ell &= -\frac{1}{i\hbar}\Big\{\alpha_k\, i f_{k\ell m}(a^\dagger G_m a)\Big\}\\ &= -\frac{1}{i\hbar}\Big\{\alpha_k\, i f_{k\ell m}c_m\Big\}\\ &= -\frac{1}{\hbar}\Big\{\alpha_k\, f_{k\ell m}c_m\Big\} \qquad (20)\end{aligned}$$

or

$$\dot{\mathbf{c}}_\ell = -\frac{1}{\hbar}\Big\{f_{k\ell m}\alpha_k c_m\Big\}. \qquad (21)$$

As the Levi-Civita symbol, the constants $f_{k\ell m}$ are antisymmetrical. Equation (21) is analogous form to the vector product of $\boldsymbol{\alpha}$ and $\mathbf{c}$ in R^3 between $\boldsymbol{\alpha}$ and $\mathbf{c}$. However, our space R has in general a higher dimension. For SU(3), the vector $\mathbf{c}$ has 3 components and processes in ordinary space. Thus we may consider (21) as the form of a vector product in the space of the regular representation defined by (9). In fact, Elguin [14] and later Hioe and Eberly [15], have worked 3–level atoms by considering the time evolution equation as a rotation of $\mathbf{c}$ in an 8–dimensional space. In this case $\mathbf{G}$ in (17) corresponds to 8 Gell-Mann matrices.

Equation (21) will be useful if we may give it a geometrical representation in the 3–dimensional ordinary space. In fact this is impossible for a vector with N^2-1 components; all we may state is that Eq. (21) has the form of a vector product or (21) defines the vector product among vectors in the regular representation, and describes precession of $\mathbf{c}$ in a N^2-1 dimensional space.

4. 3-LEVEL ATOMS AND SU(3)

In this case vectors c_ℓ are calculated according Eq. (9):

$$\begin{aligned}
c_1 &= \tfrac{1}{2}(a_1a_2^* + a_1^*a_2) = \mathbf{U}_x, & c_5 &= \tfrac{i}{2}(a_1a_3^* - a_1^*a_3) = \mathbf{V}_y,\\
c_2 &= \tfrac{i}{2}(a_2^*a_1 - a_1^*a_2) = \mathbf{U}_y, & c_6 &= \tfrac{1}{2}(a_2a_3^* + a_2^*a_3) = \mathbf{W}_x,\\
c_3 &= \tfrac{1}{2}(|a_1|^2 - |a_2|^2) = \mathbf{U}_z, & c_7 &= \tfrac{i}{2}(a_2a_3^* - a_2^*a_3) = \mathbf{W}_y,\\
c_4 &= \tfrac{1}{2}(a_1a_3^* + a_3a_1^*) = \mathbf{V}_x, & c_8 &= \tfrac{1}{2\sqrt{3}}\Big(|a_1|^2 + |a_2|^2 - 2|a_3|^2\Big)\\
& & &= \tfrac{1}{2\sqrt{3}}\mathbf{V}_z + \tfrac{1}{2\sqrt{3}}\mathbf{W}_z.
\end{aligned} \qquad (22)$$

Vectors c_1, c_2, c_3, denominated as block **U**, are equivalent to those describing transitions in two level atoms, *i.e.*, correspond to components c_x, c_y, c_z, whereas c_4, c_5, correspond to vectors c_x, c_y, of another block **V**, describing transitions between levels 1 and 3. c_6, c_7, describe transitions between levels 2 and 3. c_8 contains components c_z of the last two blocks; in fact we may write

$$c_8 \equiv \tfrac{1}{2\sqrt{3}}\left(|a_1|^2 - |a_3|^2\right) + \tfrac{1}{2\sqrt{3}}\left(|a_2|^2 - |a_3|^2\right).$$

In order to separate these three blocks **U**, **V**, **W**, let us introduce 9 matrices instead of the 8 ones of Gell-Mann, the ninth being a lineal combination of the other eight matrices. In practice this is equivalent to a change of representation, so that it is necessary to calculate again the constants $f_{\ell nm}$ (see Appendix). Thus , instead of the matrix

$$G_8 = \frac{1}{2\sqrt{3}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix},$$

we propose

$$G'_5 = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad G'_8 = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

so that we obtain the new components

$$c'_5 = \mathbf{V}_z = \tfrac{1}{2}\left(|a_1|^2 - |a_3|^2\right), \qquad c'_8 = \mathbf{W}_z = \tfrac{1}{2}\left(|a_2|^2 - |a_3|^2\right).$$

In this way we propose the blocks (or vectors)

$$(\mathbf{U}_x, \mathbf{U}_y, \mathbf{U}_z), \qquad (\mathbf{V}_x, \mathbf{V}_y, \mathbf{V}_z), \quad \text{and} \quad (\mathbf{W}_x, \mathbf{W}_y, \mathbf{W}_z).$$

Now we want to present a geometrical representation of Eq. (21) for $N = 3$. In this case we introduce the following convention for indices:

$$\begin{aligned} \ell, m, n &\longrightarrow 1 \ldots 8, \\ i, j, k &\longrightarrow 1, 2, 3 \longrightarrow \mathbf{U}, \\ a, b, c &\longrightarrow 4, 5, 6 \longrightarrow \mathbf{V}, \\ r, s, t &\longrightarrow 7, 8, 9 \longrightarrow \mathbf{W}; \end{aligned} \tag{23}$$

then, from $\dot{c}_\ell = \frac{f_{\ell nm}\alpha_n c_m}{\hbar}$,

$$\dot{c}_\ell = f_{\ell im}\, w_i c_m + f_{\ell am}\, w_a c_m + f_{\ell rm}\, w_r c_m, \tag{24}$$

where $w_n = \alpha_n/\hbar$. Expanding the m index

$$\begin{aligned}\dot{c}_\ell = {} & f_{\ell ij}\, w_i c_j + f_{\ell ia} w_i c_a + f_{\ell ir}\, w_i c_r \\ & + f_{\ell ai}\, w_a c_i + f_{\ell ab}\, w_a c_b + f_{\ell ar}\, w_a c_r \\ & + f_{\ell rj}\, w_r c_j + f_{\ell rb}\, w_r c_b + f_{\ell rs}\, w_r c_s,\end{aligned} \tag{25}$$

we have

$$\begin{aligned}\dot{c}_\ell = {} & f_{\ell ij}\, w_i c_j + f_{\ell ab}\, w_a c_b + f_{\ell rs}\, w_r c_s \\ & + f_{\ell ia}\,(w_i c_a - w_a c_i) + f_{\ell ir}\,(w_i c_r - w_r c_i) + f_{\ell ar}\,(w_a c_r - w_r c_a),\end{aligned} \tag{26}$$

where we have defined $w_n = \alpha_n/\hbar$ and took into account the antisymmetry of constants $f_{\ell nm}$. Let us note with $\ell = 1, 2, 3$, that the first term on the right hand side is an ordinary vector product $(\mathbf{W}_u \times \mathbf{U})$; with $\ell = 4, 5, 6$, the second term on the right hand side is $(\mathbf{W}_v \times \mathbf{V})$; and with $\ell = 7, 8, 9$, we get $(\mathbf{W}_\omega \times \mathbf{W})$ for the third term on right hand side, where

$$\begin{aligned}\mathbf{W}_u &= \text{precession frequency of } \mathbf{U}, \\ \mathbf{W}_v &= \text{precession frequency of } \mathbf{V}, \\ \mathbf{W}_\omega &= \text{precession frequency of } \mathbf{W}.\end{aligned}$$

In order to represent geometrically Eq. (26) it is necessary to give the structure constants and to write the 8 components in terms of components of blocks (vectors) **U**, **V** and **W**. As this equation does not restrict the Hamiltonian, the dynamics of the problem may correspond to electric or magnetic interaction or both.

Now we write the nine components of vector **c**:

$$\begin{aligned}\dot{c}_1 = \dot{u}_x = {} & (w_2 c_3 - w_3 c_2) + \tfrac{w_2}{2}(c_6 - c_9) + \tfrac{1}{2} c_2 (w_9 - w_6) \\ & + \tfrac{1}{2}(w_4 c_8 - w_8 c_4) - \tfrac{1}{2}(w_5 c_7 - w_7 c_5),\end{aligned} \tag{27}$$

$$\begin{aligned}\dot{c}_2 = \dot{u}_y = {} & (w_3 c_1 - w_1 c_3) + \tfrac{w_1}{2}(c_9 - c_6) + \tfrac{c_1}{2}(w_6 - w_9) \\ & + \tfrac{1}{2}(w_4 c_7 - w_7 c_4) + \tfrac{1}{2}(w_5 c_8 - w_8 c_5),\end{aligned} \tag{28}$$

$$\dot{c}_3 = \dot{u}_z = (w_1 c_2 - w_2 c_1) + (w_4 c_5 - w_5 c_4) + (w_8 c_7 - w_7 c_8). \tag{29}$$

The time evolution of vector **U** is described by

$$\dot{\mathbf{U}} = \dot{\mathbf{U}}_x + \dot{\mathbf{U}}_y + \dot{\mathbf{U}}_z;$$

in other words, as it is seen throughout Eqs. (27)–(29), $\dot{\mathbf{U}}$ may be described as a cross product between frequency $\mathbf{W}_u$; and the vector **U**, plus the remaining non rotating terms we named generically $\boldsymbol{\Gamma}_u$,

$$\dot{\mathbf{U}} = \mathbf{W}_u \times \mathbf{U} + \boldsymbol{\Gamma}_u, \tag{30}$$

$$\begin{aligned}\boldsymbol{\Gamma}_u = {} & \tfrac{w_2}{2}(c_6 - c_4) + \tfrac{1}{2}c_2(w_9 - w_6) + \tfrac{1}{2}(w_4c_8 - w_8c_4) - \tfrac{1}{2}(w_5c_7 - w_7c_5) \\ & + \tfrac{w_1}{2}(c_9 - c_6) + \tfrac{c_1}{2}(w_6 - w_4) + \tfrac{1}{2}(w_4c_7 - w_7c_4) + \tfrac{1}{2}(w_5c_8 - w_8c_5) \\ & + (w_4c_5 - w_5c_4) + (w_8c_7 - w_7c_8).\end{aligned}$$

Analogously,

$$\begin{aligned}\dot{c}_4 = \dot{v}_x = {} & (w_5c_6 - w_6c_5) - \tfrac{1}{2}(w_3c_5 - w_5c_3) - \tfrac{1}{2}(w_1c_8 - w_8c_1) \\ & - \tfrac{1}{2}(w_2c_7 - w_7c_2) + \tfrac{1}{2}(w_5c_9 - w_9c_5), \qquad (31)\end{aligned}$$

$$\dot{c}_5 = \dot{v}_y = (w_6c_4 - w_4c_6) + \tfrac{1}{2}(w_3c_4 - w_4c_3) + \tfrac{1}{2}(w_1c_7 - w_7c_1), \qquad (32)$$

$$\dot{c}_6 = \dot{v}_z = (w_4c_5 - w_5c_4) - \tfrac{1}{2}(w_2c_1 - w_1c_2) + (w_7c_8 - w_8c_7); \qquad (33)$$

and

$$\dot{\mathbf{V}} = \mathbf{W}_v \times \mathbf{V} + \boldsymbol{\Gamma}_v, \qquad (34)$$

$$\begin{aligned}\boldsymbol{\Gamma}_v = {} & -\tfrac{1}{2}(w_3c_5 - w_5c_3) - \tfrac{1}{2}(w_1c_8 - w_8c_1) - \tfrac{1}{2}(w_2c_7 - w_7c_2) \\ & + \tfrac{1}{2}(w_5c_9 - w_9c_5) + \tfrac{1}{2}(w_3c_4 - w_4c_3) + \tfrac{1}{2}(w_1c_7 - w_7c_1) \\ & - \tfrac{1}{2}(w_2c_1 - w_1c_2) + (w_7c_8 - w_8c_7).\end{aligned}$$

Finally,

$$\begin{aligned}\dot{c}_7 = \dot{\omega}_x = {} & (w_8c_9 - w_9c_8) + \tfrac{1}{2}(w_2c_4 - w_4c_2) - \tfrac{1}{2}(w_1c_5 - w_5c_1) \\ & + \tfrac{1}{2}(w_3c_8 - w_8c_3) + \tfrac{1}{2}(w_6c_8 - w_8c_6), \qquad (35)\end{aligned}$$

$$\begin{aligned}\dot{c}_8 = \omega_y = {} & (w_9c_7 - w_7c_9) + \tfrac{1}{2}(w_1c_4 - w_4c_1) + \tfrac{1}{2}(w_2c_5 - w_5c_2) \\ & - \tfrac{1}{2}(w_3c_7 - w_7c_3) - \tfrac{1}{2}(w_6c_7 - w_7c_6), \qquad (36)\end{aligned}$$

$$\dot{c}_9 = \dot{\omega}_z = (w_7c_8 - w_8c_7) + (w_4c_5 - w_5c_4) + (w_2c_1 - w_1c_2); \qquad (37)$$

and

$$\dot{\mathbf{W}} = \mathbf{W}_\omega \times \mathbf{W} + \boldsymbol{\Gamma}_\omega, \qquad (38)$$

again $\boldsymbol{\Gamma}_w$ corresponds to the remaining non-rotating terms in the time evolution equation of $\mathbf{W}$, then

$$\begin{aligned}\boldsymbol{\Gamma}_w = {} & \tfrac{1}{2}(w_2c_4 - w_4c_2) - \tfrac{1}{2}(w_1c_5 - w_5c_1) + \tfrac{1}{2}(w_3c_8 - w_8c_3) \\ & + \tfrac{1}{2}(w_6c_8 - w_8c_6) + \tfrac{1}{2}(w_1c_4 - w_4c_1) + \tfrac{1}{2}(w_2c_5 - w_5c_2) \\ & - \tfrac{1}{2}(w_3c_7 - w_7c_3) - \tfrac{1}{2}(w_6c_7 - w_7c_6) + (w_4c_5 - w_5c_4) + (w_2c_1 - w_1c_2).\end{aligned}$$

The complete set of previous equations is not necessary for describing the radiation-matter interaction. As an example, components u_z, v_z, w_z, correspond to atomic population difference between levels (1–2), (1–3) and (2–3) respectively and their time evolution equation might be our particular interest (population inversion in lasers, by example). In contrast, in problems involving a deep knowledge of dipole transition (as luminescence, super-radiance, etc.) a solution to equations containing components x, y of vectors $\mathbf{c}, \mathbf{v}, \mathbf{w}$ will be necessary.

5. EDI IN AN EXPLICIT FORM

In the following lines we consider an atom in an electric field in the long-wave approximation.

The hamiltonian is

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \hat{\mathcal{H}}_{\mathrm{p}} = \hat{\mathcal{H}}_0 - \mathbf{p} \cdot \mathbf{E}. \tag{39}$$

$\hat{\mathcal{H}}_0$ is the non-perturbed part of Hamiltonian and $\hat{\mathcal{H}}_{\mathrm{p}} = -\mathbf{p} \cdot \mathbf{E}$ is the electric perturbation. The state vector satisfies the Schrödinger equation:

$$i\hbar|\dot{\psi}\rangle = \hat{\mathcal{H}}|\psi\rangle$$

each component $|\psi_n\rangle$ satisfying in this approximation

$$i\hbar|\dot{\psi}_n\rangle = \hat{\mathcal{H}}|\psi_n\rangle. \tag{40}$$

Eigenvectors $|\psi_n\rangle$ are eigenfunctions of $\mathcal{H}_0$, in the long-wave approximation [16], when $[\mathcal{H}, \mathcal{H}_0] \cong 0$.

Now we consider a specific interaction where the electric field is a plane wave with elliptic polarization defined as

$$E = (\hat{\epsilon}_x E_x^0 + i\hat{\epsilon}_y E_y^0) e^{-iw_L t} = \hat{\epsilon} E^0(r)\, e^{-iw_L t}. \tag{41}$$

We must take into account that a radiation field contains in general positive and negative frequencies. By using a field as (41) we may introduce a RWA (rotating wave approximation), where there is not negligible optical terms.

Turning to Eq. (41)

$$\begin{aligned} i\hbar|\dot{\psi}_n\rangle &= (\mathcal{H}_0 - \mathbf{P} \cdot \mathbf{E})|\psi_n\rangle \\ &= (\mathcal{H}_0 - \mathbf{P} \cdot \hat{\epsilon} E_0^{\dagger} e^{-iw_L t})|\psi_n\rangle, \end{aligned} \tag{42}$$

where we consider a monochromatic plane wave as that generated by a laser. This equation may be written in a coordinate system rotating with the same angular frequency as the field, by introducing the transformation

$$|\psi_n\rangle = e^{-iw_L t}|\psi_n'\rangle. \tag{43}$$

Eq. (42) is then

$$i|\dot{\psi}'_n\rangle = [\Delta n - \Omega']|\psi'_n\rangle, \tag{44}$$

where as definitions

$$\Delta n \equiv (w_n - w_L), \qquad \Omega' \equiv \Omega e^{-iw_L t}, \qquad \Omega \equiv \frac{\mathbf{P}\cdot\mathbf{E}}{\hbar}; \tag{45}$$

Δn is known as detuning and Ω is the Rabi transition frequency in the rotating frame.

By introducing $|\psi_m\rangle\langle\psi_m|$ in the second term on right hand side of (44), we write in matrix form

$$i\begin{bmatrix}|\dot{\psi}_1\rangle\\ |\dot{\psi}_2\rangle\\ |\dot{\psi}_3\rangle\end{bmatrix} = \begin{bmatrix}\Delta_{11} & -\Omega^{\dagger}_{12} & -\Omega^{\dagger}_{13}\\ -\Omega^{\dagger}_{21} & \Delta_{22} & -\Omega^{\dagger}_{23}\\ -\Omega^{\dagger}_{31} & -\Omega^{\dagger}_{32} & \Delta_{33}\end{bmatrix}\begin{bmatrix}|\psi_1\rangle\\ |\psi_2\rangle\\ |\psi_3\rangle\end{bmatrix}, \tag{46}$$

where we have explicitly put $\Delta_{11} = w_{11} - w_L$, as detuning for the first active level, similarly $\Delta_{22} = w_{22} - w_L$ as detuning for the second active level and $\Delta_{33} = w_{33} - w_L$ as detuning for the third active level. On the other hand Ω_{ij} $(i,j = 1,2,3)$ is the Rabi transition frequency between levels i and j, which take account of the strength of the coupling for this (ij) particular coupling.

Using Eq. (46) and appendix A, we may calculate frequencies to propose a geometrical solution for 3–level atoms. The vality of this solution is guaranteed for the followings reasons:

1. Equation (21) stablishes that **c** process in an 8 (extended to 9) dimensional space. So that the behavior of **c** is like an *axial* vector in 8 dimensions. However it is not clear that this means a rotation in the ordinary 3–space.
2. Definition (9) and Eqs. (22) suggest that the pseudovector **c** describes properly transitions in 3–level atoms; block **U** for example corresponds to the known description for 2–levels. Analogously with block **V** and **W**. The general time evolution of the system takes into account **U**, **V** and **W**.
3. Finally, Eqs. (29), (33) and (37) say that the components $\mathbf{U}_z$, $\mathbf{V}_z$ and $\mathbf{W}_z$ are mixed in a simple form for $t > 0$.

We now assume that initially $(t = 0)$ **U**, **V** and **W** coincide with the z axis and are now indistinguishable. The splitting of the system in **U**, **V** and **W** will happen for $t > 0$.

We have drawn independently each one of the vectors. For any instant t, **U**, **V** and **W** are located in a non-diagonal coordinate system (Fig. 1) but we may through a rotation of coordinates [17] (Euler angles) get the coincidence of the z axis with these three vectors. We write any vector in a non diagonal system as

$$J = bJ_x + cJ_y + aJ_z;$$

the same vector in a diagonal system is

$$J = \left(a^2 + b^2 + c^2\right)^{\frac{1}{2}} J'_z,$$

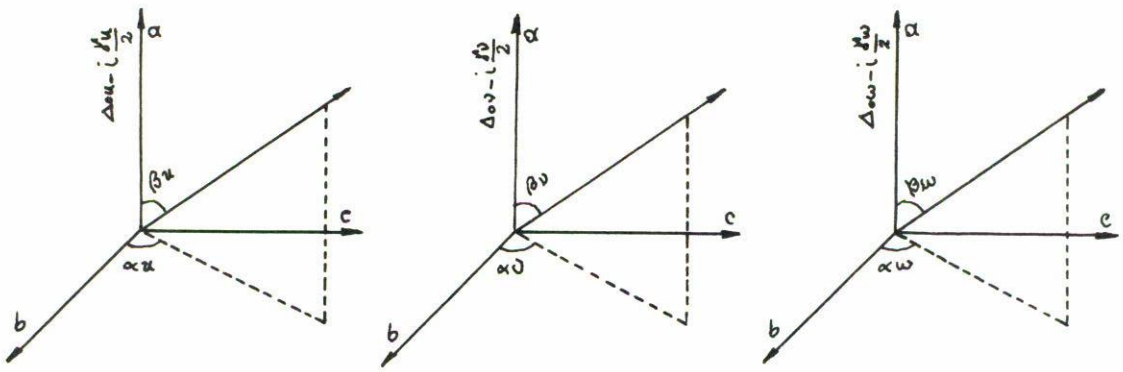

FIGURE 1. The vectors **U**, **V** and **W** for $t > 0$.

where $(a^2 + b^2 + c^2)^{\frac{1}{2}}$ is the value of the angular momentum for $m = 1$. The angles α, β in Fig. 1 for each case are given by

$$\tan \alpha_u = \frac{c_u}{b_u}, \qquad \tan \beta_v = \frac{c_v}{b_v}, \qquad \tan \gamma_w = \frac{c_w}{b_w}; \tag{47}$$

or in general

$$\tan \alpha_i = \frac{\operatorname{Im}(\Omega_0)_i}{\operatorname{Re}(\Omega_0)_i} \qquad (i = u, v, w).$$

Analogously for β

$$\tan \beta_i = \frac{\sqrt{b_i^2 + c_i^2}}{a_i} = \frac{|\Omega_0|_i}{\left(\Delta_{0i} - \frac{\operatorname{Im}(\gamma_{0i})}{2}\right)}; \tag{48}$$

$\boldsymbol{\Omega}_0$ is the Rabi transition frequency associated with the basic energy levels, in the diagonal system after RWA, and formally follows definition (45), in other words

$$\boldsymbol{\Omega}_0 = \frac{\mathbf{P} \cdot \mathbf{E}_0}{\hbar},$$

where $\mathbf{E}_0$ is the electric field amplitude, $\mathbf{P}$ the dipolar electric moment corresponding to the grown state transition and γ_{0i} is a parameter including losses due to collisions or ionizations.

In the new system, vectors are diagonal and eigenvalues of **U**, **V** and **W** are given as

$$\lambda_u = m\lambda_u^0, \qquad \lambda_v = m\lambda_v^0, \qquad \lambda_w = m\lambda_w^0.$$

This vector is expressed in the original system through the Euler rotation matrices $D^j_{MM'}$. These matrices will be expressed as in Ref. [11], *i.e.*, by identifying an energy level n with a specific magnetic sublevel, thus

$$N = 2J + 1 \longrightarrow: J = \tfrac{1}{2}(N - 1), \qquad n = M + J + 1 \longrightarrow: M = n - \tfrac{1}{2}(N - 1).$$

In our case ($N = 3$) we have $J = 1$ or $M = +1, 0, -1$ corresponding respectively to the levels $n = 1, 2, 3$. The components of vectors **U**, **V** and **W**, in the non diagonal system are

$$U_{MK} = \sum_{M'=1}^{3} D^3_{M'M}(\alpha_u, \beta_u, 0)\, U'_{M'K}, \tag{49}$$

$$V_{MK} = \sum_{M'=1}^{3} D^3_{M'M}(\alpha_v, \beta_v, 0)\, V'_{M'K}, \tag{50}$$

$$W_{MK} = \sum_{M'=1}^{3} D^3_{M'M}(\alpha_w, \beta_w, 0)\, W'_{M'K}. \tag{51}$$

The time evolution equation giving information on the dynamics of 3–level atoms mix all the components of the vectors in the original system. These components have been defined for the new system in Eq. (21); however with that definition they do not give information in presence of electric fields neither revels the mixing of fields. We may say, then, that the solution for z is given by

$$\begin{aligned} U_{ZK} = {} & \sum D^3_{MM'}(\alpha_u, \beta_u, 0)\, U'_{M'K} + \sum D^3_{MM'}(\alpha_v, \beta_v, 0)\, V'_{M'K} \\ & + \sum D^3_{MM'}(\alpha_w, \beta_w, 0)\, W'_{M'K}. \end{aligned} \tag{52}$$

From now on we have $J = 1$ (3 levels) so that

$$D^3_{MM'}(\alpha_u, \beta_u, 0) = e^{iM'\alpha_u}\, d_{MM'}(\beta),$$

and the same for vectors **V**, **W**.

The final expression for U_z is

$$\begin{aligned} U_z = {} & e^{-i\alpha_u}\, d_{31}(\beta_u)\, U'_{1K} + e^{-i2\alpha_u}\, d_{32}(\beta_u)\, U'_{2K} + e^{-i3\alpha_u}\, d_{33}(\beta_u)\, U'_{3K} \\ & + e^{-i\alpha_v} d_{31}(\beta_v)\, V'_{1K} + e^{-i2\alpha_v}\, d_{32}(\beta_v)\, V'_{2K} + e^{-i3\alpha_v}\, d_{33}(\beta_v)\, V'_{3K} \\ & + e^{-i\alpha_w}\, d_{31}(\beta_w)\, W'_{1K} + e^{-i2\alpha_w}\, d_{32}(\beta_w)\, W'_{2K} + e^{-i3\alpha_w}\, d_{33}(\beta_w)\, W'_{3K}, \end{aligned} \tag{53}$$

and equivalent equations for V_z, W_z, or any other component. Equation (53) give us all the information concerning to atomic population inversion. In other words, the real part of the first line of (53) (that associated to **U**′), expresses the population difference between levels 1 and 2; whereas the second line does the same with levels 1 and 3. The real part of the first line of (53) is shown in Figs. 2 to 5 for 4 different values of laser radiation w_L. In the resonant situation (zero detuning), we get a similar result to Sargent III y Horowitz [18], although they work with two coincident Rabi frequencies. In practice it is equivalent to working with just 2 active levels.

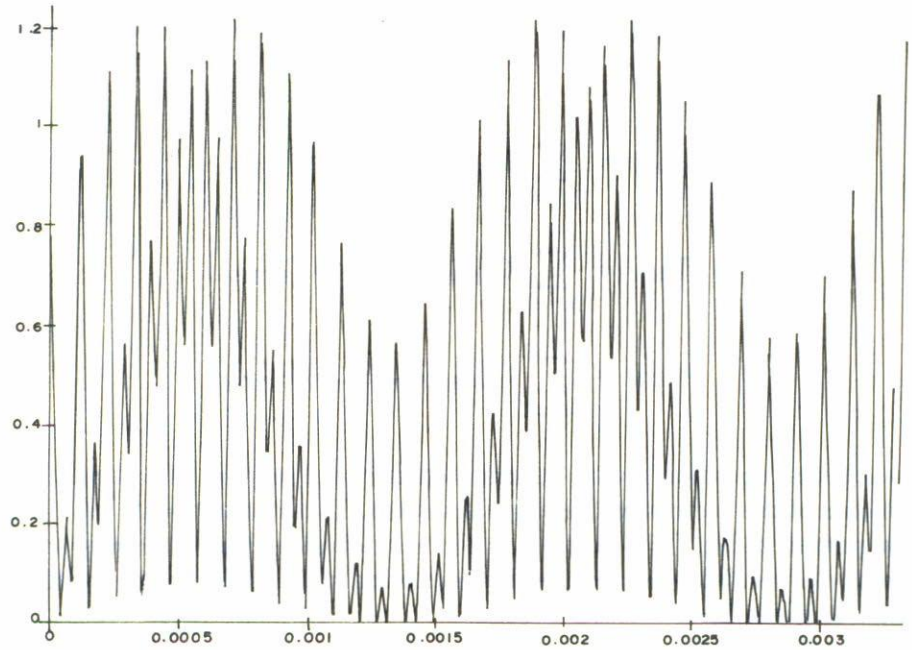

FIGURE 2. Square module of the first line of $U_z(t/\tau)$, $t_{\max} = 10$ periods of the laser ($\omega_L = 10000$ Å).

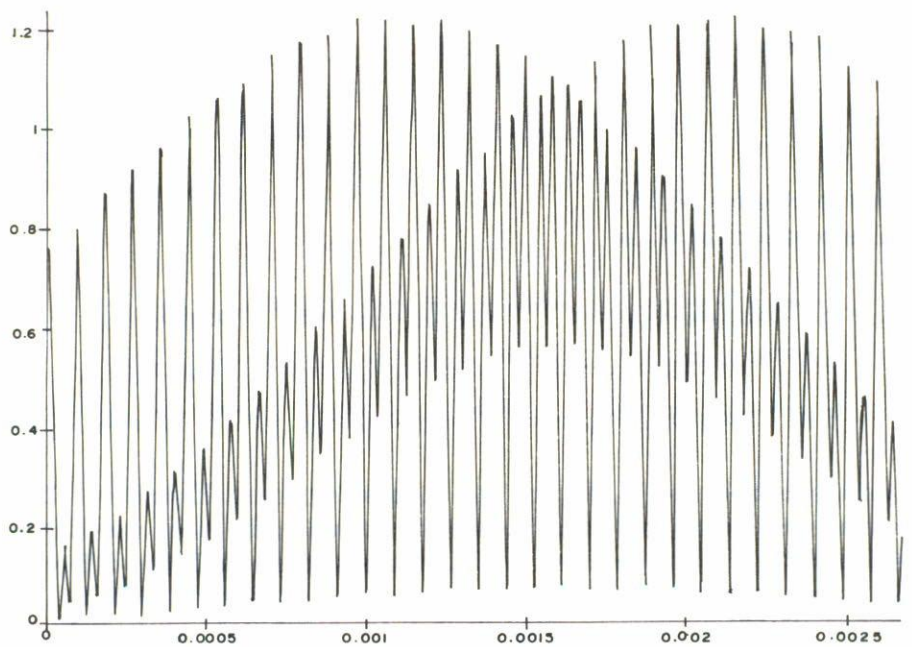

FIGURE 3. Square module of the first line of $U_z(t/\tau)$, $t_{\max} = 10$ periods of the laser ($\omega_L = 8000$ Å).

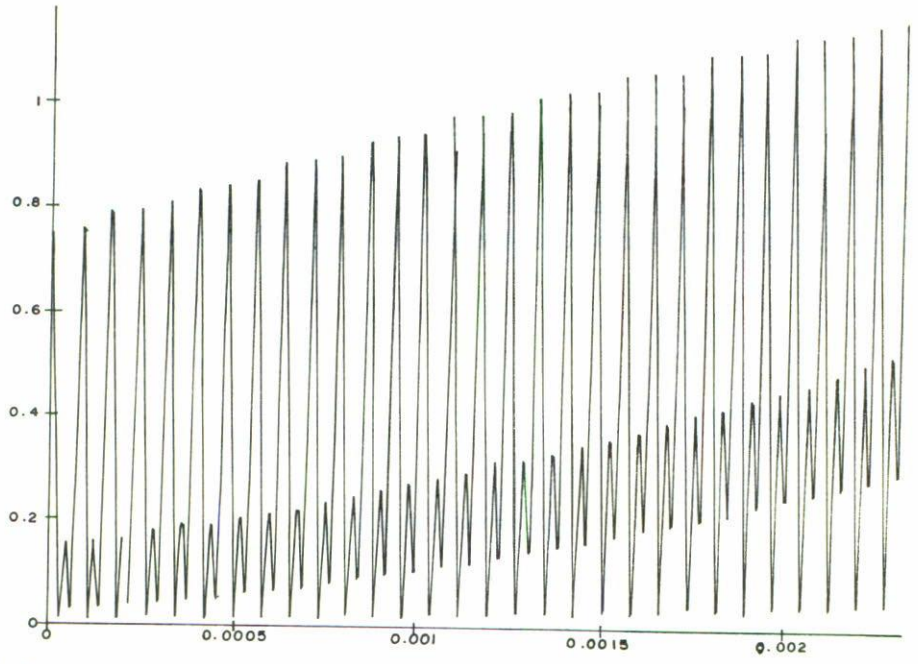

FIGURE 4. Square module of the first line of $U_z(t/\tau)$, $t_{\max} = 10$ periods of the laser ($\omega_L = 7000$ Å).

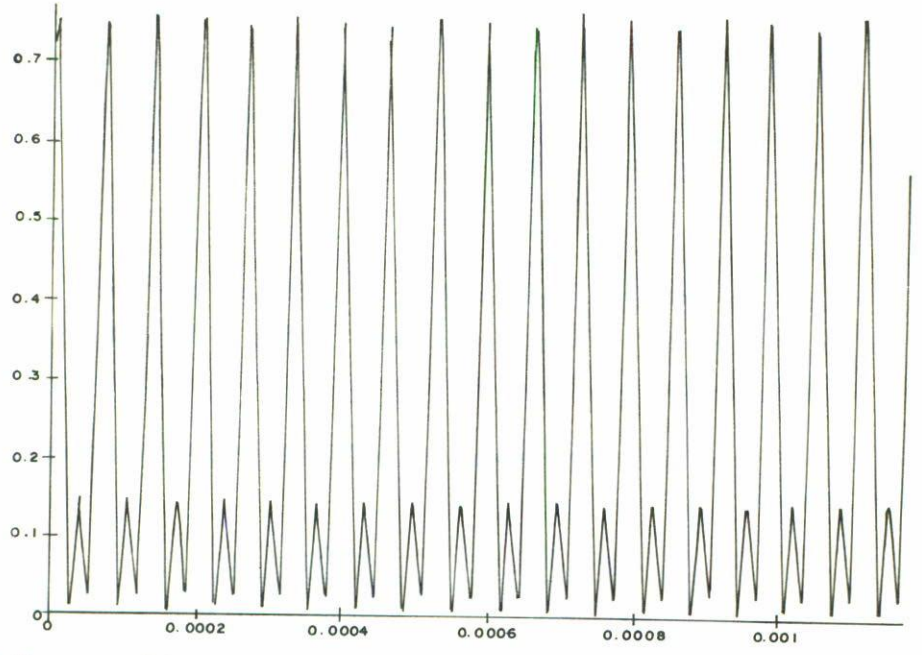

FIGURE 5. Square module of the first line of $U_z(t/\tau)$, $t_{\max} = 6$ periods of the laser ($\omega_L = 5889.9$ Å).

6. CONCLUSIONS

It has been demonstrated that EDI may be treated as MDI in a vector space defined by Eq. (9). This is a general theory whose dynamics is determined by Eq. (21). As will be noted is a matrix development worked for 3 levels by Aravind [19] and refined by Dattoli and A. Torre [20]; the same that we have generalized to any number N of levels. The purpose of this technique, introducing pseudo-spin is to get a geometrical representation in ordinary 3–dimensional space in such a way that time evolution may be easily followed.

APPENDIX

A. FREQUENCY CALCULATION

$$H = \alpha_i G_i + \beta I \tag{A1}$$

$$\begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix} = \begin{bmatrix} 0 & \alpha_1 & 0 \\ \alpha_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -i\alpha_2 & 0 \\ i\alpha_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \alpha_3 & 0 & 0 \\ 0 & -\alpha_3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & 0 & \alpha_4 \\ 0 & 0 & 0 \\ \alpha_4 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -i\alpha_5 \\ 0 & 0 & 0 \\ i\alpha_5 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \alpha_6 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\alpha_6 \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha_7 \\ 0 & \alpha_7 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i\alpha_8 \\ 0 & i\alpha_8 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \alpha_9 & 0 \\ 0 & 0 & -\alpha_9 \end{bmatrix},$$

from here

$$\begin{aligned} \mathbf{H}_{11} &= \boldsymbol{\alpha}_3 + \boldsymbol{\alpha}_6, & H_{21} &= \alpha_1 + i\alpha_2, & H_{31} &= \alpha_4 + i\alpha_5, \\ H_{12} &= \alpha_1 - i\alpha_2, & \mathbf{H}_{22} &= -\boldsymbol{\alpha}_3 + \boldsymbol{\alpha}_9, & H_{32} &= \alpha_7 + i\alpha_8, \\ H_{13} &= \alpha_4 - i\alpha_5, & H_{23} &= \alpha_7 - i\alpha_8, & \mathbf{H}_{33} &= -\boldsymbol{\alpha}_6 - \boldsymbol{\alpha}_9. \end{aligned}$$

Let us note that this equations could depend on the fact that the Hamiltonian has or not zeros on the diagonal

$$\begin{aligned} \alpha_1 &= \frac{H_{12} + H_{21}}{2}, & \alpha_2 &= \frac{i(H_{12} - H_{21})}{2}, \\ \alpha_4 &= \frac{H_{13} + H_{31}}{2}, & \alpha_5 &= \frac{i(H_{13} - H_{31})}{2} \\ \alpha_7 &= \frac{H_{23} + H_{32}}{2}, & \alpha_8 &= \frac{i(H_{23} - H_{32})}{2}. \end{aligned} \tag{A2}$$

The diagonal elements have the equations:

$$H_{11} = \alpha_3 + \alpha_6, \qquad H_{22} = -\alpha_3 + \alpha_9 \qquad H_{33} = -\alpha_6 - \alpha_9. \tag{A3}$$

The Gell-Mann Matrices are

$$G_1 = \tfrac{1}{2}\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}, \quad G_2 = \tfrac{1}{2}\begin{bmatrix}0 & -i & 0\\ i & 0 & 0\\ 0 & 0 & 0\end{bmatrix}, \quad G_3 = \tfrac{1}{2}\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{bmatrix},$$

$$G_4 = \tfrac{1}{2}\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{bmatrix}, \quad G_5 = \tfrac{1}{2}\begin{bmatrix}0 & 0 & -i\\ 0 & 0 & 0\\ i & 0 & 0\end{bmatrix}, \quad G_6 = \tfrac{1}{2}\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}, \tag{A4}$$

$$G_7 = \tfrac{1}{2}\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & -i\\ 0 & i & 0\end{bmatrix}, \quad G_8 = \tfrac{1}{2\sqrt{3}}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -2\end{bmatrix}.$$

The structure constants for this particular representation are:

$k\ell m$	$f_{k\ell m}$
123	ϵ_{123}
147	1/2
156	−1/2
246	1/2
257	1/2
345	1/2
367	−1/2
458	$1/2\sqrt{3}$
678	$1/2\sqrt{3}$

The Structure Constants for the Proposition Gell-Mann + 1 with

$$G'_5 = \frac{1}{2}\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{bmatrix}, \qquad G'_8 = \frac{1}{2}\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}, \tag{A5}$$

are

$$\begin{aligned}
&f_{148} = \tfrac{1}{2}, && f_{582} = \tfrac{1}{2}, && f_{378} = -\tfrac{1}{2},\\
&f_{157} = -\tfrac{1}{2}, && f_{594} = \tfrac{1}{2}, && f_{472} = \tfrac{1}{2},\\
&f_{162} = -\tfrac{1}{2}, && f_{678} = \tfrac{1}{2}, && f_{123} = \epsilon_{123},\\
&f_{175} = \tfrac{1}{2}, && f_{247} = \tfrac{1}{2}, && f_{456} = \epsilon_{456},\\
&f_{184} = -\tfrac{1}{2}, && f_{274} = -\tfrac{1}{2}, && f_{789} = \epsilon_{789}.\\
&f_{192} = \tfrac{1}{2}, && f_{345} = \tfrac{1}{2},
\end{aligned} \tag{A6}$$

B.

The intensity of any atomic transition is given as

$$f_{ka} = \frac{2m}{3\hbar} w_{ka} |e r_{ka}|^2. \tag{B7}$$

The subindices a and k in f_{ka} include all the quantum numbers of the final and initial states. In particular f_{ka} depend on the magnetic quantum numbers. It is convenient to define an average transition intensity for $|n\ell\rangle \longrightarrow |n'\ell'\rangle$, which is independent of the magnetic quantum numbers and therefore of the polarization radiation:

$$f_{(\underbrace{n',\ell'}_{k}:\underbrace{n,\ell}_{a})} = \frac{1}{2\ell+1} \sum_{m'=-\ell'}^{\ell'} \sum_{m=-\ell}^{\ell} f_{n'\ell'm',n\ell m}; \tag{B8}$$

the average it is running on the initial states. As an example we have taken in the present work these transitions for the sodium atom:

$$3\left(^2S_{\frac{1}{2}}\right) \longrightarrow \begin{cases} 3\left(^2P_{\frac{1}{2}}\right) & 5889.9 \text{ Å} \rightarrow f_{ka} = 0.324 \\ 3\left(^2P_{\frac{3}{2}}\right) & 5895.9 \text{ Å} \rightarrow f_{ka} = 0.648 \\ 4\left(^2P_{\frac{3}{2}}\right) & 3302.9 \text{ Å} \rightarrow f_{ka} = 0.102 \\ 4\left(^2P_{\frac{1}{2}}\right) & 303.9 \text{ Å} \rightarrow f_{ka} = 0.051 \end{cases}$$

$$3\left(^2P_{\frac{1}{2}}\right) \longrightarrow 4\left(^2S_{\frac{1}{2}}\right) \rightarrow 22084 \text{ Å} \rightarrow f_{ka} = 0.167$$

$$3\left(^2P_{\frac{3}{2}}\right) \longrightarrow 4\left(^2S_{\frac{1}{2}}\right) \rightarrow 22057 \text{ Å} \rightarrow f_{ka} = 0.335$$

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