Revista Mexicana de Física **42**, *No. 3 (1996) 498–504*

Gravitomagnetic mass in the linearized Einstein theory

G.F. TORRES DEL CASTILLO
Departamento de Física Matemática, Instituto de Ciencias
Universidad Autónoma de Puebla, 72000 Puebla, Pue., México

AND

J. HERNÁNDEZ SÁNCHEZ
Facultad de Ciencias Físico Matemáticas
Universidad Autónoma de Puebla
Apartado postal 1152, 72001 Puebla, Pue., México

Recibido el 6 de julio de 1995; aceptado el 7 de febrero de 1996

ABSTRACT. The gravitational field produced by a "gravitomagnetic monopole" in the linearized Einstein theory is found and a quantization condition for the ordinary mass is obtained by considering the motion of a particle in the field of a gravitomagnetic monopole.

RESUMEN. Se halla el campo gravitacional producido por un "monopolo gravitomagnético" en la teoría de Einstein linealizada y se obtiene una condición de cuantización para la masa ordinaria considerando el movimiento de una partícula en el campo de un monopolo gravitomagnético.

PACS: 04.20.–q; 04.25.–g

1. INTRODUCTION

As is well known, assuming the existence of magnetic charges, Maxwell's equations can be modified in such a way that these equations show more symmetry between the electric and magnetic fields. Even though it is a straightforward matter to propose the equations

$$\nabla \cdot \mathbf{B} = 4\pi\rho_{\mathrm{m}}, \quad \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} - \frac{4\pi}{c}\mathbf{J}_{\mathrm{m}}, \tag{1}$$

by analogy with

$$\nabla \cdot \mathbf{E} = 4\pi\rho_{\mathrm{e}}, \quad \nabla \times \mathbf{B} = -\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{J}_{\mathrm{e}}, \tag{2}$$

where ρ_{e}, ρ_{m} are the electric and magnetic charge densities and $\mathbf{J}_{\mathrm{e}}$, $\mathbf{J}_{\mathrm{m}}$ are the electric and magnetic current densities, respectively, in order to maintain the relation between the electric and magnetic fields with the usual potentials,

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, \tag{3}$$

it is necessary to assume that the potentials have singularities. (Alternatively, it can be assumed that the potentials correspond to a connection in a non-trivial principal bundle.) By demanding that the singularities be unobservable in any quantum process, Dirac [1,2] arrived at the conclusion that the existence of a magnetic monopole implies the quantization of the electric charge (see, *e.g.*, Ref. [3] and the references cited therein).

The equations of general relativity allow us to consider sources of the gravitational field analogous to the magnetic charges; in fact, there exist known exact solutions of the Einstein field equations characterized by a "magnetic mass" parameter (also called "dual mass", "gravitomagnetic mass", and "NUT parameter") in addition to the ordinary mass parameter (see, *e.g.*, Refs. [4, 5]). However, in order to study some of the features of the gravitational field produced by a gravitomagnetic mass, it is convenient to consider the linearized Einstein theory, where the analysis is simpler. (A detailed analysis of the geodesic equations for the Taub-NUT metric, which represents the gravitational field of a spherically symmetric source with ordinary and gravitomagnetic mass, is given in Ref. [6].)

In this paper we obtain a solution of the linearized Einstein field equations that represents the gravitational field of a gravitomagnetic monopole, making use of the gauge-invariant description of the gravitational field given by the curvature tensor. We show that, in the limit of small velocities, the motion of a particle in the field of a gravitomagnetic monopole is identical to that of a charged particle in the field of a magnetic monopole (see also Ref. [6]) and, therefore, we obtain a quantization condition analogous to Dirac's relation (see also Refs. [7–9]). In Sect. 2 we summarize the basic equations of the linearized Einstein theory and we find, in this approximation, the metric corresponding to the gravitational field of a gravitomagnetic monopole. In Sect. 3 we show that in the limit of small velocities the geodesic equation for this metric is identical to the equation of motion of a charged particle in the field of a magnetic monopole and we obtain a quantization condition relating the ordinary and the gravitomagnetic masses. Greek indices run from 0 to 3 and Latin indices $i, j, \ldots$, from 1 to 3. Indices are raised and lowered by means of the Minkowski metric.

2. The field of a gravitomagnetic monopole in the linearized Einstein theory

The Einstein field equations linearized about the Minkowski metric show several analogies with the equations for the electromagnetic field. By expressing the metric of the space-time in the form

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}, \tag{4}$$

where $(\eta_{\alpha\beta}) = \mathrm{diag}\,(-1, 1, 1, 1,)$, one finds that the curvature tensor to first order in $h_{\alpha\beta}$ is

$$K_{\alpha\beta\gamma\delta} = \tfrac{1}{2}\{\partial_\alpha\partial_\gamma h_{\beta\delta} - \partial_\beta\partial_\gamma h_{\alpha\delta} + \partial_\beta\partial_\delta h_{\alpha\gamma} - \partial_\alpha\partial_\delta h_{\beta\gamma}\}, \tag{5}$$

with $\partial_\alpha \equiv \partial/\partial x^\alpha$, and from the linearized Einstein vacuum field equations, $K^\alpha{}_{\beta\alpha\gamma} = 0$, it follows that the tensor fields

$$E_{ij} \equiv K_{0i0j}, \quad B_{ij} \equiv -\tfrac{1}{2}\epsilon_{ikl}K_{kl0j} \tag{6}$$

are symmetric and trace-free and satisfy (see, *e.g.*, Refs. [10, 11])

$$\partial_i B_{ij} = 0, \quad \epsilon_{ijk}\partial_j E_{kl} = -\frac{1}{c}\frac{\partial}{\partial t}B_{il}, \tag{7}$$

$$\partial_i E_{ij} = 0, \quad \epsilon_{ijk}\partial_j B_{kl} = \frac{1}{c}\frac{\partial}{\partial t}E_{il}, \tag{8}$$

which are analogous to the source-free Maxwell equations. The curvature tensor (5) is invariant under the gauge transformations

$$h_{\alpha\beta} \to h_{\alpha\beta} - \partial_\alpha\xi_\beta - \partial_\beta\xi_\alpha, \tag{9}$$

where ξ_α is an arbitrary four-vector field. By combining Eqs. (5) and (6) one obtains the expressions

$$B_{ij} = \epsilon_{ilk}\partial_l\tfrac{1}{2}(\partial_j h_{0k} - \partial_0 h_{kj}), \qquad E_{ij} = -\partial_i\tfrac{1}{2}(\partial_0 h_{0j} - \partial_j h_{00}) - \partial_0\tfrac{1}{2}(\partial_j h_{0i} - \partial_0 h_{ij}), \tag{10}$$

which are analogous to Eqs. (3).

The Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2Gm}{c^2 r}\right)c^2\,dt^2 + \left(1 - \frac{2Gm}{c^2 r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\varphi^2),$$

which corresponds to the gravitational field of a point mass m, to first order in the mass parameter m can be represented in the form (4) with

$$h_{00} = \frac{2Gm}{c^2 r}, \qquad h_{ij} = \frac{2Gm}{c^2 r^3}x_i x_j, \qquad h_{0i} = 0. \tag{11}$$

Then, according to Eqs. (10) the "electric" and "magnetic" parts of the curvature are

$$E_{ij} = \frac{Gm}{c^2}\left(\frac{3x_i x_j}{r^5} - \frac{\delta_{ij}}{r^3}\right), \qquad B_{ij} = 0. \tag{12}$$

By analogy with Eq. (12), the gravitational field of a gravitomagnetic monopole is determined by

$$E_{ij} = 0, \qquad B_{ij} = \frac{Gn}{c^2}\left(\frac{3x_i x_j}{r^5} - \frac{\delta_{ij}}{r^3}\right), \tag{13}$$

where n is the "gravitomagnetic mass" of the monopole. It can be readily seen that the metric perturbation

$$h_{00} = 0 = h_{ij}, \qquad h_{0i} = \frac{2Gn}{c^2} \frac{(-y, x, 0)}{r(\pm r - z)}, \tag{14}$$

is a solution of the linearized Einstein vacuum field equations (*i.e.*, $\partial^\alpha \partial_\alpha h_{\beta\gamma} - \partial_\beta \partial^\alpha h_{\alpha\gamma} - \partial_\gamma \partial^\alpha h_{\alpha\beta} + \partial_\beta \partial_\gamma h^\alpha{}_\alpha = 0$), for $r \neq 0$, and that the independent components of the curvature (5) are given by Eqs. (13). Thus, in the linearized Einstein theory, the gravitational field of a gravitomagnetic monopole is given by

$$\begin{aligned} ds^2 &= -c^2\, dt^2 + dx^2 + dy^2 + dz^2 + \frac{4Gn}{c} \frac{(x\, dy - y\, dx)\, dt}{r(\pm r - z)} \\ &= -c^2\, dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) + \frac{4Gn}{c}(\pm 1 + \cos\theta)\, d\varphi\, dt, \end{aligned} \tag{15}$$

modulo the gauge transformations (9). (Note that the two metric perturbations given by Eqs. (14) are related by a gauge transformation with $\xi_0 = -4Gn/c^2 \arctan(y/x)$, $\xi_i = 0$.)

It may be noticed that the vector field

$$\gamma_i \equiv h_{0i} = \frac{2Gn}{c^2} \frac{(-y, x, 0)}{r(\pm r - z)}, \tag{16}$$

is the vector potential for the field of a magnetic monopole of magnetic charge $-2Gn/c^2$, *i.e.*,

$$\nabla \times \gamma = -\frac{2Gn}{c^2} \frac{\mathbf{r}}{r^3}. \tag{17}$$

As in the case of the vector potential (16), the metric perturbation (14) is singular on the positive or negative z-axis, according to whether one takes the plus or minus sign in Eq. (14).

It can be readily seen that the metric (15) is equivalent, up to the gauge transformations (9), to the Taub-NUT solution (see, *e.g.*, Ref. [12])

$$ds^2 = -U^{-1}\, dr^2 + (2l)^2 U (d\psi + \cos\theta\, d\varphi)^2 + (r^2 + l^2)\, (d\theta^2 + \sin^2\theta\, d\varphi^2),$$

where

$$U \equiv -1 + \frac{2(mr + l^2)}{r^2 + l^2},$$

to first order in l, when $m = 0$, making the identifications $c\, dt = 2l\, d\psi$ and $l = -Gn/c^2$.

3. A QUANTIZATION CONDITION

In general relativity, any test particle subject only to the gravitational force moves along a geodesic of the space-time. The spatial components of the geodesic equation for a particle with non-zero rest mass are

$$\frac{d^2x^i}{d\tau^2} = -\Gamma^i_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}, \tag{18}$$

where τ is the proper time of the particle and the $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols corresponding to the space-time metric $g_{\alpha\beta}$. In the limit $v \ll c$, where v is the speed of the particle, Eq. (18) reduces to

$$\frac{d^2x^i}{dt^2} = -c^2\Gamma^i_{00} - 2c\Gamma^i_{0j}\frac{dx^j}{dt}. \tag{19}$$

Using the expression for the Christoffel symbols to first order in $h_{\alpha\beta}$ and taking into account Eqs. (14), one finds that $\Gamma^i_{00} = 0$ and $\Gamma^i_{0j} = (\partial_j h_{0i} - \partial_i h_{0j})/2$; therefore, under the present assumptions, the equation of motion for a particle of mass m in the gravitational field of a gravitomagnetic monopole is

$$\begin{aligned} m\frac{d\mathbf{v}}{dt} &= mc\mathbf{v} \times (\nabla \times \boldsymbol{\gamma}) \\ &= -2Gmn\frac{\mathbf{v}}{c} \times \frac{\mathbf{r}}{r^3} \end{aligned} \tag{20}$$

[see Eqs. (16) and (17)]. This last equation coincides with the equation of motion of a particle with electric charge e in the field of a magnetic monopole of magnetic charge g if we make the identification

$$-2Gmn = eg \tag{21}$$

(see also Ref. [6]). Hence, following the same steps as in Refs. [13, 3], by considering the deflection at large impact parameters of a particle of mass m by the gravitational field of a gravitomagnetic monopole, and assuming that the change of the component of the angular momentum along the incident direction must be a multiple of $\hbar$, one obtains the Dirac quantization condition with $-2Gmn$ in place of eg, that is

$$\frac{2Gmn}{\hbar c} = \frac{N}{2}, \qquad (N = 0, \pm 1, \pm 2, \ldots). \tag{22}$$

A similar result was obtained in Ref. [8] by a different procedure, making use of the analogy of the Maxwell equations with

$$\begin{aligned} &\nabla \cdot \mathbf{H} = 0, & &\nabla \times \mathbf{g} = 0, \\ &\nabla \cdot \mathbf{g} = -\frac{4\pi G}{c^2}\mu, & &\nabla \times \mathbf{H} = \frac{4}{c}\left(\frac{\partial \mathbf{g}}{\partial t} - \frac{4\pi G}{c^2}\mu\mathbf{v}\right), \end{aligned} \tag{23}$$

where $H_i \equiv \epsilon_{ijk}\partial_j h_{0k}$, $g_i \equiv -\partial_i h_{00}/2$ and μ and $\mathbf{v}$ are the mass density and the velocity of the matter, respectively, which follow from the linearized Einstein field equations, provided that $v \ll c$ (see, *e.g.*, Ref. [14]). However, the derivations based on the analogy of Eqs. (23) with Maxwell's equations are not very reliable since, even in the case of vacuum, the invoked analogy is not complete and the fields $\mathbf{g}$ and $\mathbf{H}$ are not invariant under the gauge transformations (9). (It may be noticed that the "gravitoelectric" and "gravitomagnetic" fields $\mathbf{g}$ and $\mathbf{H}$ corresponding to the metric perturbations (11) and (14) are $\mathbf{g} = (Gm/c^2)\mathbf{r}/r^3$ and $\mathbf{H} = (-2Gn/c^2)\mathbf{r}/r^3$, respectively, which are not simply related by the replacement of m by n. Compare with Eqs. (12) and (13).)

As is well known, an equation of motion of the form

$$m\frac{d\mathbf{v}}{dt} = k\mathbf{v} \times \frac{\mathbf{r}}{r^3}, \tag{24}$$

where k is a constant, implies that

$$\mathbf{M} \equiv \mathbf{r} \times (m\mathbf{v}) - k\frac{\mathbf{r}}{r} \tag{25}$$

is a constant of motion (see, *e.g.*, Ref. [15] and the references cited therein). In the case of a charged particle in the field of a magnetic monopole, where $k = eg/c$, $\mathbf{M}$ represents the *total* angular momentum ($-(eg/c)\mathbf{r}/r$ is the angular momentum of the corresponding electromagnetic field) (see, *e.g.*, Ref. [3]); however, in the case of the interaction with a gravitomagnetic monopole, the meaning of the term $(2Gmn/c)\mathbf{r}/r$ is not clear.

Choosing the z–axis along the direction of $\mathbf{M}$, from Eq. (25) one finds that $\mathbf{M}\cdot\mathbf{r} = -kr$; on the other hand, $\mathbf{M}\cdot\mathbf{r} = Mr\cos\theta$, where M is the magnitude of $\mathbf{M}$, therefore,

$$\cos\theta = -\frac{k}{M} = \text{const.}, \tag{26}$$

which means that the trajectory of a particle governed by Eq. (24) lies on a cone with its vertex at the origin. Furthermore, Eq. (24) implies that v is a constant and that the acceleration is normal to the cone (26). Hence, using the fact that a curve on a surface is a geodesic (with respect to the intrinsic geometry of the surface) if the acceleration of the curve is normal to the surface, one concludes that the trajectory is such that it becomes a straight line when the cone is unfolded. The same conclusion can be obtained by expressing Eq. (24) in spherical coordinates. Then, making use of Eq. (26), one finds that

$$\begin{aligned} r^2\frac{d\varphi}{dt} &= \text{const.}\,(\equiv l), \\ \frac{d^2r}{dt^2} &= r\sin^2\theta\left(\frac{d\varphi}{dt}\right)^2. \end{aligned} \tag{27}$$

Hence, $d/dt = (l/r^2)d/d\varphi$, and from the second equation in (27) it follows that $d^2u/d\varphi^2 = -(\sin^2\theta)u$, where $u \equiv 1/r$. Thus, $1/r = A\sin(\varphi\sin\theta) + B\cos(\varphi\sin\theta)$, where A and B are arbitrary constants, which represents a straight line when the cone is unfolded.

4. Concluding remarks

According to Eq. (22), mass is quantized in units of $\hbar c/4Gn$. Since there is no experimental evidence of such a quantization, $\hbar c/4Gn$ would have to be very small, which yields a lower bound for the gravitomagnetic mass of a gravitomagnetic monopole (see also Ref. [9]). If one assumes that the gravitational interaction satisfies Newton's third law, then the force on a gravitomagnetic monopole in the gravitational field of a mass m would be minus the right-hand side of Eq. (20), which is different from the force on a particle with ordinary mass in the same gravitational field, which would mean that the gravitomagnetic monopoles do not obey the equivalence principle.

Acknowledgment

The authors are grateful to one of the referees for bringing to their attention Refs. [6,9]. This work was supported in part by CONACYT.

References

1. P.A.M. Dirac, *Proc. R. Soc. London* **A 133** (1931) 60.
2. P.A.M. Dirac, *Phys. Rev.* **74** (1948) 817.
3. J.D. Jackson, *Classical Electrodynamics*, 2nd ed., Wiley, New York (1975), Chap. 6.
4. E.T. Newman, L. Tamburino and T. Unti, *J. Math. Phys.* **4** (1963) 915.
5. M. Demiański and E.T. Newman, *Bull. Acad. Pol. Sci. Ser. Math. Phys.* **14** (1966) 653.
6. R.L. Zimmerman and B.Y. Shahir, *Gen. Rel. Grav.* **21** (1989) 821.
7. A. Magnon and H.-Y. Guo, *Gen. Rel. Grav.* **19** (1987) 809.
8. A. Zee, *Phys. Rev. Lett.* **55** (1985) 2379.
9. M. Mueller and M.J. Perry, *Class. Quantum Grav.* **3** (1986) 65.
10. W.B. Campbell and T. Morgan, *Physica* **53** (1971) 264.
11. G.F. Torres del Castillo, *Rev. Mex. Fís.* **37** (1991) 443.
12. S.W. Hawking and G.F.R. Ellis, *The Large Scale Structure of Space-Time*, Cambridge University Press, Cambridge (1973).
13. A.S. Goldhaber, *Phys. Rev.* **140** (1965) 1407.
14. V.B. Braginsky, C.M. Caves and K.S. Thorne, *Phys. Rev.* **D15** (1977) 2047.
15. E. Piña, T. Ortiz and A. Muñoz, *Rev. Mex. Fís.* **33** (1988) 554. (In Spanish.)