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The electrical resistance of a solid body

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ABSTRACT. Two operational ways to define the electrical resistance of a homogeneous, isotropic solid body is presented. In the first one the incoming and the outgoing stationary current densities through part of the surface of the body are supposed to be given. In the second one, it is the value of the electrostatic potential that is assumed to be fixed on part of the surface of the body. It is shown that a piece of material, of the same conductivity, added in "parallel" produces, in both cases, a decreasing of the electrical resistance of the body. Finally an application to a two-dimensional body with a non-conventional shape is presented.

RESUMEN. Se presentan dos formas operacionales de definir la resistencia eléctrica de un cuerpo sólido homogéneo e isotrópico. En la primera, se da la densidad de corriente estacionaria que ingresa y que sale del cuerpo a través de su superficie. En la segunda, se especifica el potencial electrostático sobre regiones de la superficie del cuerpo. Se muestra además que si un trozo de material de la misma conductividad es agregado "en paralelo" al cuerpo, su resistencia eléctrica disminuye. Finalmente, se presenta una aplicación a un sólido bidimensional con una geometría no convencional.

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1. INTRODUCTION

The concept of electrical resistance is usually defined for cylindrical bodies in which the current density is assumed to be uniform and along the axis of the cylinder; bodies with different shapes are very seldom considered [1–3]. In this note we present two operational ways to define the electrical resistance of a homogeneous, isotropic solid body:

i) the current density is specified on the surfaces S_i and S_o (they are the regions of the boundary of the body through which the electrical current comes in and goes out respectively), see Fig. 1;

ii) the electrostatic potential is fixed on the surfaces S_i and S_0.

These two cases do not exhaust all the possibilities, there are situations where the current density is given in a region and the potential in another one. Such boundary conditions are found in stationary problems of heat conduction, where the role of the potential is played by the temperature of the conducting body.

To obtain the electrical resistance R of the body one has to solve the Laplace equation with given boundary conditions. The value of R depends on the shape of the body, the intrinsic conductivity σ of the material, the location of the surfaces S_i and S_o and, on the form of the incoming and outgoing currents in case i), and on the functional form of the fixed potential in case ii).

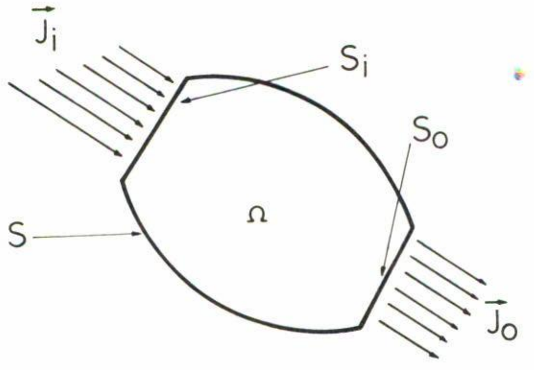

FIGURE 1. Solid body with incoming $\vec{J}_\mathrm{i}$ and outgoing $\vec{J}_\mathrm{o}$ current densities.

It is also shown here that the resistance of the body decreases when a piece of material, with the same conductivity, is added in "parallel" to the original body. The proof of this fact, so obvious for a conventional electric circuit, has not been found in the examined literature for a body of arbitrary shape. In these proofs we assume, in case ii), that the electrostatic potential is not modified when we add the piece in "parallel", this is equivalent to assume that a tension source, as the one used in circuit theory, is connected to the body. For the case i) one assumes that the current densities on S_i and S_o are fixed, that means that we have, in this case, a current source connected to the body. Certainly ideal tension and current sources do not exist but from a practical point of view it is known that it is possible to construct devices whose behavior, under certain conditions, are very close to the ideal ones.

Finally we present a numerical calculation of the resistance of a two-dimensional body with a non-conventional form with fixed current density. We add then a piece of material of the same resistivity in "parallel" and compare the resistance with the one already calculated. In this case one is faced with a Neumann type of problem. These calculations are performed using a numerical procedure developed by W. Lamberti and the author to solve the Laplace equation with given boundary conditions [4].

2. BOUNDARY CONDITIONS

2.1. *Fixed current density*

Figure 1 shows a homogeneous and isotropic solid body of conductivity σ and whose electrical resistance we want to find. Let us assume that a steady current I enters through the surface S_i and the same current comes out through S_o. Considering that the material satisfies Ohm's law and neglecting magnetic effects we have

$$\vec{J}(\vec{r}) = \sigma \vec{E}(\vec{r}), \quad \vec{r} \in \Omega, \tag{1}$$

where $\vec{J}(\vec{r})$ is the current density, $\vec{E}(\vec{r})$ the electrostatic field and Ω is the volume of the body.

Due to the stationary character of the problem we have

$$\nabla \cdot \vec{J} = \nabla \cdot (\sigma \vec{E}) = -\sigma \nabla \cdot \nabla \phi = -\sigma \, \Delta \phi(\vec{r}) = 0, \quad \vec{r} \in \Omega, \tag{2}$$

with $\phi(\vec{r})$ as the electrostatic potential, with the following boundary conditions:

$$\begin{aligned} \hat{n} \cdot \nabla \phi|_S &= 0, \\ \sigma \hat{n} \cdot \nabla \phi|_{S_\mathrm{i}} &= -\hat{n} \cdot \vec{J}_\mathrm{i}, \\ \sigma \hat{n} \cdot \nabla \phi|_{S_\mathrm{o}} &= -\hat{n} \cdot \vec{J}_\mathrm{o}, \end{aligned} \tag{3}$$

with the constraint

$$-\int_{S_\mathrm{i}} \vec{J}_\mathrm{i} \cdot d\vec{a} = \int_{S_\mathrm{o}} \vec{J}_\mathrm{o} \cdot d\vec{a} = I, \tag{4}$$

where $d\vec{a}$ is the surface element in the direction of the outward normal $\hat{n}$ to the volume Ω.

The Joule power dissipated in the body is

$$\begin{aligned} \int_\Omega \vec{E} \cdot \vec{J} \, d^3r &= \sigma \int_\Omega \nabla \phi \cdot \nabla \phi \, d^3r = \sigma \oint_{\partial \Omega} \phi \nabla \phi \cdot d\vec{a} \\ &= -\int_{S_\mathrm{o}} \phi \vec{J}_\mathrm{o} \cdot d\vec{a} - \int_{S_\mathrm{i}} \phi \vec{J}_\mathrm{i} \cdot d\vec{a} \equiv RI^2. \end{aligned} \tag{5}$$

Here we have used the boundary conditions given in Eq. (3). From Eq. (5) we obtain the electrical resistance R:

$$R = -\frac{1}{I^2} \left[\int_{S_\mathrm{o}} \phi \vec{J}_\mathrm{o} \cdot d\vec{a} + \int_{S_\mathrm{i}} \phi \vec{J}_\mathrm{i} \cdot d\vec{a} \right]. \tag{6}$$

In the particular case that $\vec{J}_\mathrm{o} \cdot \hat{n}$ and $\vec{J}_\mathrm{i} \cdot \hat{n}$ are constant on the surfaces S_o and S_i we have

$$\begin{aligned} R &= -\frac{1}{I^2} \left[\hat{n} \cdot \vec{J}_\mathrm{o} \int_{S_\mathrm{o}} \phi \, da + \hat{n} \cdot \vec{J}_\mathrm{i} \int_{S_\mathrm{i}} \phi \, da \right] \\ &= \frac{1}{I} \left[\frac{1}{A_\mathrm{i}} \int_{S_\mathrm{i}} \phi \, da - \frac{1}{A_\mathrm{o}} \int_{S_\mathrm{o}} \phi \, da \right] \equiv \frac{V}{I}. \end{aligned} \tag{7}$$

V is the difference between the averaged electrostatic potential over the surfaces S_i and S_o. Equation (6) provides us with an expression for the resistance of a homogeneous isotropic solid body of any shape as a function of the incoming $\vec{J}_\mathrm{i}(\vec{r})$ and outgoing $\vec{J}_\mathrm{o}(\vec{r})$ current densities.

We prove now that the electrical resistance of the body decreases if we add a piece of material Π, with the same conductivity σ, in parallel to the original body. Figure 2 shows

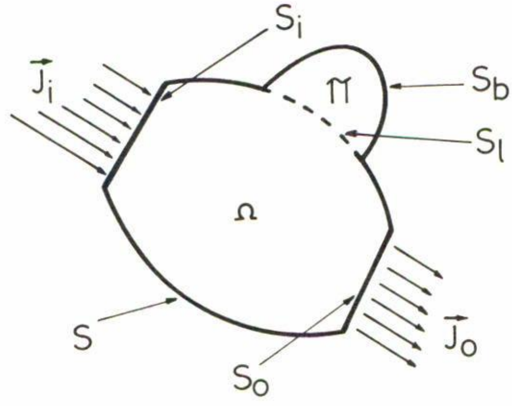

FIGURE 2. The solid body of Fig. 1 with a piece of material Π added in "parallel".

the original body of volume Ω with the piece Π added. We will calculate the electrical resistance of the original and of the enlarged bodies and we will compare them.

Let $\phi(\vec{r})$ and $\phi_1(\vec{r})$ be the electrostatic potentials of the original body and of the enlarged one, assuming that the incoming $\vec{J}_\mathrm{i}$ and the outgoing $\vec{J}_\mathrm{o}$ current densities are the same in both cases. The potentials ϕ and ϕ_1 satisfy

$$\begin{aligned} \Delta\phi(\vec{r}) &= 0, \quad \vec{r} \in \Omega; \\ \Delta\phi_1(\vec{r}) &= 0, \quad \vec{r} \in (\Omega + \Pi). \end{aligned} \tag{8}$$

The Joule power dissipated in both cases are

$$W = \int_\Omega \vec{E} \cdot \vec{J}\, d^3r, \quad W_1 = \int_{\Omega+\Pi} \vec{E}_1 \cdot \vec{J}_1\, d^3r. \tag{9}$$

We consider now the integral

$$\begin{aligned} \int_\Omega (\vec{E} - \vec{E}_1) \cdot (\vec{J} + \vec{J}_1)\, d^3r = &\int_\Omega (\vec{E} - \vec{E}_1) \cdot (\vec{J} - \vec{J}_1)\, d^3r \\ &+ 2\int_\Omega \vec{J}_1 \cdot (\vec{E} - \vec{E}_1)\, d^3r \equiv A + Q, \end{aligned} \tag{10}$$

with

$$\begin{aligned} A &= \int_\Omega (\vec{E} - \vec{E}_1) \cdot (\vec{J} - \vec{J}_1)\, d^3r \geq 0, \\ Q &= 2\int_\Omega \vec{J}_1 \cdot (\vec{E} - \vec{E}_1)\, d^3r \end{aligned} \tag{11}$$

Equation (10) can also be written in the form

$$\begin{aligned} \int_\Omega (\vec{E} - \vec{E}_1) \cdot (\vec{J} + \vec{J}_1)\, d^3r &= \int_\Omega (\vec{E} \cdot \vec{J} - \vec{E}_1 \cdot \vec{J}_1)\, d^3r \\ &= W - \int_{\Omega+\Pi} \vec{E}_1 \cdot \vec{J}_1\, d^3r + \int_\Pi \vec{E}_1 \cdot \vec{J}_1\, d^3r. \end{aligned} \tag{12}$$

We have then

$$W - W_1 = A + Q - \int_\Pi (\vec{E}_1 \cdot \vec{J}_1)\, d^3r. \tag{13}$$

Consider now

$$\begin{aligned} Q &= 2\int_\Omega \sigma\, \nabla\phi_1 \cdot \nabla(\phi - \phi_1)\, d^3r = 2\sigma \oint_{\partial\Omega} \phi_1 \nabla(\phi - \phi_1) \cdot d\vec{a} \\ &= 2\sigma \int_{S_l} \phi_1 \nabla(\phi - \phi_1) \cdot d\vec{a} = -2\sigma \int_{S_l} \phi_1 \nabla\phi_1 \cdot d\vec{a}, \end{aligned} \tag{14}$$

where S_l is the common boundary of Ω and Π.

Since the normal derivative of $\phi_1$vanishes on the surface S_b we can write Q in the form

$$\begin{aligned} Q &= -2\sigma \int_{S_l} \phi_1 \nabla\phi_1 \cdot d\vec{a} + 2\sigma \int_{S_\mathrm{b}} \phi_1 \nabla\phi_1 \cdot d\vec{a} \\ &= 2\sigma \oint_{\partial\Pi} \phi_1 \nabla\phi_1 \cdot d\vec{a} = 2\int_\Pi (\vec{E}_1 \cdot \vec{J}_1)\, d^3r. \end{aligned} \tag{15}$$

Equations (13) and (15) give

$$W - W_1 = (R - R_1)I^2 = A + \int_\Pi \vec{E}_1 \cdot \vec{J}_1\, d^3r \geq 0 \Rightarrow R \geq R_1, \tag{16}$$

as we wanted to prove.

2.2. Fixed electrostatic potential

Consider now the solid body where the potential has been fixed on the surfaces S_i and S_o. The boundary conditions in this case are

$$\hat{n} \cdot \nabla\phi\Big|_S = 0, \qquad \phi\Big|_{S_\mathrm{i}} = \phi_\mathrm{i}(\vec{r}), \qquad \phi\Big|_{S_\mathrm{o}} = \phi_\mathrm{o}(\vec{r}). \tag{17}$$

The Joule power dissipated in the body is

$$\begin{aligned} \int_\Omega \vec{E} \cdot \vec{J}\, d^3r &= \sigma \oint_{\partial\Omega} \phi\, \nabla\phi \cdot d\vec{a} \\ &= \sigma \int_{S_\mathrm{i}} \phi\, \nabla\phi \cdot d\vec{a} + \sigma \int_{S_\mathrm{o}} \phi\, \nabla\phi \cdot d\vec{a} = \frac{V^2}{R} \end{aligned} \tag{18}$$

where we have defined V by

$$V \equiv \frac{1}{A_\mathrm{i}} \int_{S_\mathrm{i}} \phi\, da - \frac{1}{A_\mathrm{o}} \int_{S_\mathrm{o}} \phi\, da, \tag{19}$$

i.e. the difference between the averaged potential on the surfaces S_i and S_o. With this definition we have the following expression for the resistance R

$$R = V^2 \left[\sigma \int_{S_\mathrm{i}} \phi \nabla\phi \cdot d\vec{a} + \sigma \int_{S_\mathrm{o}} \phi \nabla\phi \cdot d\vec{a} \right]^{-1} . \tag{20}$$

In particular when the potential is constant on the surfaces S_i and S_o, with values V_i and V_o respectively, we get for R:

$$R = \frac{V_\mathrm{i} - V_\mathrm{o}}{I}, \tag{21}$$

with I given in Eq. (4). We pass now to prove that the electrical resistance decreases when we add a piece Π, of the same conductivity σ, in "parallel" to the original solid body. Let W and W_1 be the power dissipated in the original body and in the enlarged one. Since the potential are kept fixed the value of V is the same for the two bodies. Consider now the difference

$$\begin{aligned} W_1 - W &= \int_{\Omega+\Pi} \vec{E}_1 \cdot \vec{J}_1 \, d^3r - \int_\Omega \vec{E} \cdot \vec{J} \, d^3r \\ &= \int_\Omega (\vec{E}_1 \cdot \vec{J}_1 - \vec{E} \cdot \vec{J}) \, d^3r + \int_\Pi \vec{E}_1 \cdot \vec{J}_1 \, d^3r. \end{aligned} \tag{22}$$

The second term of the right hand side of Eq. (22) is greater than or equal to zero. The first term can be written in the form

$$\begin{aligned} \int_\Omega (\vec{E}_1 \cdot \vec{J}_1 - \vec{E} \cdot \vec{J}) \, d^3r &= \int_\Omega \left[(\vec{E}_1 - \vec{E}) \cdot (\vec{J}_1 - \vec{J}) + 2\vec{E} \cdot (\vec{J}_1 - \vec{J}) \right] d^3r \\ &\equiv F + 2 \int_\Omega \vec{E} \cdot (\vec{J}_1 - \vec{J}) \, d^3r \end{aligned} \tag{23}$$

with $F \geq 0$. The last term of Eq. (23) is

$$\begin{aligned} 2 \int_\Omega \vec{E} \cdot (\vec{J}_1 - \vec{J}) \, d^3r &= 2\sigma \int_\Omega \nabla\phi \cdot \nabla(\phi_1 - \phi) \, d^3r = 2\sigma \oint_{\partial\Omega} (\phi_1 - \phi) \nabla\phi \cdot d\vec{a} \\ &= 2\sigma \int_{S_\mathrm{i}} (\phi_1 - \phi) \nabla\phi \cdot d\vec{a} + 2\sigma \int_{S_\mathrm{o}} (\phi_1 - \phi) \nabla\phi \cdot d\vec{a} = 0 \end{aligned} \tag{24}$$

since $\nabla\phi \cdot d\vec{a}$ vanishes on S and ϕ_1 and ϕ are equal on S_i and on S_o. We are then left with the inequality

$$W_1 - W = V^2 (\frac{1}{R_1} - \frac{1}{R}) \geq 0 \Rightarrow R_1 \leq R \tag{25}$$

as we wanted to prove.

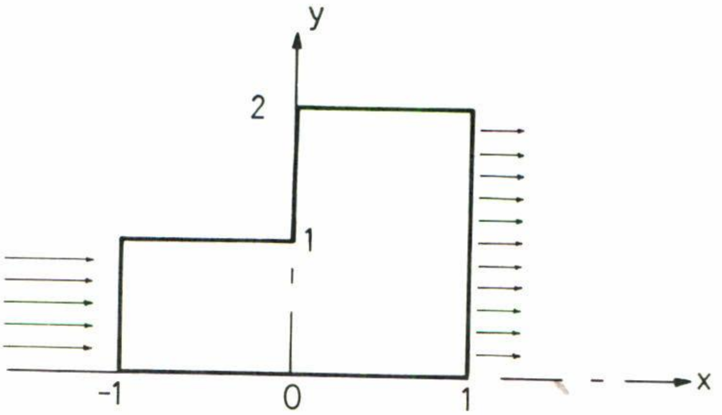

FIGURE 3. A two-dimensional body with uniform incoming and outgoing current densities. The dimensions are given in arbitrary units and the resistivity is assumed to be equal to one.

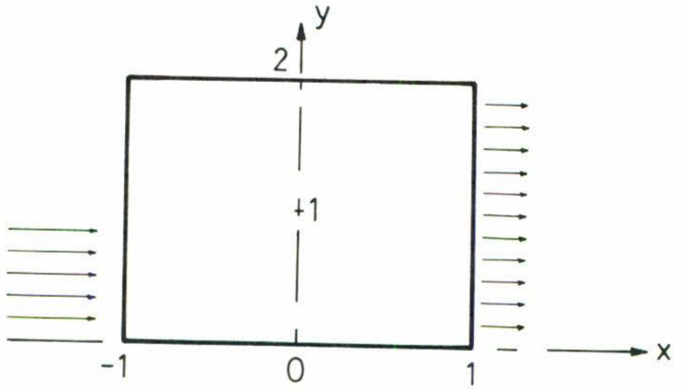

FIGURE 4. To the body shown in Fig. 3 a piece of the same material is added in "parallel".

Finally we present the calculation of the electrical resistance of the two-dimensional homogeneous body shown in Fig. 3. We assume that the incoming and outgoing current densities are uniform. Using the numerical procedure described in Ref. [4] we calculate its electrical resistance taking the value of the conductivity σ as unit and the dimension of the body shown in Fig. 3 in arbitrary units. The procedure requires a discretization of the frontier of the body on the y-axis, the conditions imposed on these points give raise to a set of algebraic equations whose number is equal to the number of points in the discretization. Taking 9 points the value of the resistance is $R = 1.732$. Figure 4 shows the body already considered with the added piece in "parallel". A series expansion can be written for the electrical resistance in this case

$$R_1 = 1 + \sum_{n=1}^{\infty} (K_n)^{-3} \sin^2(K_n) \coth(2K_n), \tag{26}$$

with $K_n = \frac{n\pi}{2}$. Taking only up to $n = 11$ in this expansion we get the approximate value: $R_1 = 1.2719 < R$ as it should be.

3. Conclusions

To conclude, we have given two operational ways to define the electrical resistance of a homogeneous isotropic solid body. In the first one the current density is fixed on part of the surface of the body and in the second one it is the electrostatic potential that is assumed to be fixed, also, on part of the surface of the body. It is shown, in both cases, that the electrical resistance decreases if a piece of material, with the same conductivity σ, is added in "parallel" to the body. Finally an application has been done to a two-dimensional body with a non-conventional shape.

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