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Solution of the Klein-Gordon equation in the Carter A solution

G. SILVA-ORTIGOZA
Facultad de Ciencias Físico Matemáticas
Universidad Autónoma de Puebla
Apartado postal 1152, 72001 Puebla, Pue., México
and
Departamento de Física, Centro de Investigación y de Estudios Avanzados del IPN
Apartado postal 14–740, 07000 México, D.F., México

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ABSTRACT. It is shown that the separation constant not related to isometries, which appears in the solution of the Klein-Gordon equation in the Carter A background, can be defined in an invariant way as the eigenvalue of a second-order differential operator made out of a two-index Killing spinor, with the eigenfunctions being the separable solutions.

RESUMEN. Se muestra que la constante de separación no relacionada con isometrías, la cual aparece en la solución de la ecuación de Klein-Gordon en la métrica A de Carter, se puede definir en forma invariante como el eigenvalor de un operador diferencial de segundo orden construido de un espinor de Killing de dos índices, con las soluciones separables como las eigenfunciones.

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1. INTRODUCTION

The Klein-Gordon equation and its separability properties on black hole metrics have been the subject of intense study and detailed investigations [1, 2]. Carter [3] pointed out that the Klein-Gordon equation is separable in the Kerr-Newman metric, among others (see also Ref. [4]). Furthermore, he has shown that the separability properties are very closely related to symmetry operators [5]; which are linear differential operators that map the space of solutions into itself. The commuting operators with the differential operator appearing in the field equations are the more familiar examples of symmetry operators. They are called constants of motion.

In the work by Carter [5] it is shown that, when the background space-time is a solution of the Einstein or the Einstein-Maxwell field equations, one can construct a first or second-order commuting operator (for the Klein-Gordon operator) if the space-time admits a Killing vector or a two-index Killing tensor, respectively.

On the other hand, Dudley and Finley [6] have shown that, when the background is the Plebański-Demiański metric, the decoupled perturbation equations for the radiative components corresponding to spin $s = 0$, $1/2$, 1 and 2 admit R-separable solutions. But they did not explain the R-separability properties by means of symmetry operators.

The generalization of Carter's results, on a general curved background for the conformally invariant Klein-Gordon operator, was given by Kamran and McLenaghan [7]. They showed that the most general second-order symmetry operator for the conformally invariant Klein-Gordon operator can be constructed from a two-index conformal Killing tensor and that this operator admits the R-separable solutions as eigenfunctions, where the eigenvalue is the separation constant (not related to isometries) arising from the separation of variables. Symmetry operators for the Dirac and Weyl neutrino equations have also been considered, important examples of which [8, 9] arise from the separability of these equations in the Kerr-Newman solution. For the Maxwell equations, Torres del Castillo [10] has shown that the separable solutions of the source-free Maxwell equations, on a type-D vacuum space-time with cosmological constant, can be characterized by means of a differential operator constructed from a two-index Killing spinor. This operator provides, at the same time, a covariant definition of the Starobinsky constant, which arises from the differential relations that connect the separated functions (the so-called Teukolsky-Starobinsky identities). Furthermore, he found analogous results to the separation constant not related to isometries [11] (see also Ref. [12]). For the Rarita-Schwinger equation in a type-D vacuum background see Ref. [13].

The aim of the present work is to show that when the background space-time is the Carter A metric, which is a solution of the Einstein-Maxwell field equations, the charged Klein-Gordon equation can be solved by the separation-of-variables method and that the separation constant not related to the Killing vectors, can be defined in a covariant way as the eigenvalue of a certain second-order differential operator made out of the two-index Killing spinor admitted by the Carter A solution, with the eigenfunctions being the separable solutions of the Klein-Gordon equation.

The spinor formalism and the Newman-Penrose notation are used throughout [14, 15].

2. KILLING SPINORS

The concept of a two-index Killing spinor has its origin in the work of Walker and Penrose [16], who demonstrated the existence in any type-D vacuum space-time of a second-order symmetric two-spinor satisfying the twistor equation

$$\nabla_{\dot{A}(B}L_{CD)} = 0, \tag{1}$$

where the parenthesis denote symmetrization on the indices enclosed. If L_{AB} is a two-index Killing spinor, it is equivalent to saying that there exists a (complex) vector field $K_{A\dot{B}}$ such that

$$\nabla_{A\dot{R}}L_{BC} = \tfrac{1}{3}\Big(\epsilon_{AB}\,K_{C\dot{R}} + \epsilon_{AC}\,K_{B\dot{R}}\Big). \tag{2}$$

Nowadays, we know that if the Weyl spinor, Ψ_{ABCD}, does not vanish, then the integrability conditions of Eq. (1) imply that Ψ_{ABCD} has to be of type D or N, depending on whether L_{AB} is algebraically general or special, respectively. In a flat space-time the integrability conditions on Eq. (1) are trivially satisfied. (For a proof of these results, see Ref. [17].)

If the space-time is a type-D solution of the Einstein vacuum field equations with cosmological constant or of the Einstein-Maxwell field equations with an algebraically general electromagnetic field whose principal null directions are geodesic and shear-free, then the solution of Eq. (1) is given by

$$L_{AB} = -2\phi^{-1} o_{(A}\iota_{B)}, \tag{3}$$

up to a complex constant factor [17], where o^A and ι^A are principal spinors of Ψ_{ABCD}, and ϕ is such that [18]

$$\begin{aligned} o^B \nabla_{A\dot{C}} o_B &= o_A\, o^B\, \partial_{B\dot{C}} \ln\phi, \\ \iota^B \nabla_{A\dot{C}} \iota_B &= \iota_A\, \iota^B\, \partial_{B\dot{C}} \ln\phi. \end{aligned} \tag{4}$$

3. THE KLEIN-GORDON EQUATION AND THE CARTER A METRIC

In the two-component spinor formalism, the Klein-Gordon equation in a general space-time with a background electromagnetic field is given by

$$\left(\nabla_{A\dot{B}} + ie\Phi_{A\dot{B}}\right)\left(\nabla^{A\dot{B}} + ie\Phi^{A\dot{B}}\right)\Psi = M^2\Psi, \tag{5}$$

where Ψ is the complex scalar field with mass parameter M and electric charge e, while $\Phi_{A\dot{B}}$ is the four-potential of the electromagnetic field. Making use of the notation of Newman-Penrose, Eq. (5) is equivalent to

$$\begin{aligned} \Big[&(D + \epsilon + \overline{\epsilon} - \overline{\rho} - \rho + ie\Phi_{0\dot{0}})(\Delta + ie\Phi_{1\dot{1}}) \\ &\quad -(\delta + \beta - \overline{\alpha} + \overline{\pi} - \tau + ie\Phi_{0\dot{1}})(\overline{\delta} + ie\Phi_{1\dot{0}}) \\ &+(\Delta - \gamma - \overline{\gamma} + \overline{\mu} + \mu + ie\Phi_{1\dot{1}})(D + ie\Phi_{0\dot{0}}) \\ &\quad -(\overline{\delta} - \alpha + \overline{\beta} - \overline{\tau} + \pi + ie\Phi_{1\dot{0}})(\delta + ie\Phi_{0\dot{1}})\Big]\Psi = M^2\Psi, \end{aligned} \tag{6}$$

where the components of $\Phi_{A\dot{B}}$ are taken with respect to a spin-frame $\{o^A, \iota^A\}$ with $o^A\iota_A = 1$, such that the only nonvanishing components of Ψ_{ABCD} and φ_{AB} are Ψ_2 and φ_1, respectively.

The Carter A background space-time [19] is a solution of the Einstein-Maxwell field equations with an aligned electromagnetic field, that possesses two commuting Killing vectors. With respect to a (local) coordinate system $\{p, q, u, v\}$ such that ∂_u and ∂_v are the two commuting Killing vectors; this metric is given by

$$ds^2 = \left\{ \frac{\mathcal{Q}}{p^2+q^2}(du - p^2\,dv)^2 - \frac{p^2+q^2}{\mathcal{Q}}\,dq^2 - \frac{\mathcal{P}}{p^2+q^2}(du + q^2\,dv)^2 - \frac{p^2+q^2}{\mathcal{P}}\,dp^2 \right\}, \tag{7}$$

where

$$\begin{aligned} \mathcal{P} &= b - Q_g^2 + 2np - \epsilon_0 p^2 - (\lambda_0/3)p^4, \\ \mathcal{Q} &= b + Q_e^2 - 2mq + \epsilon_0 q^2 - (\lambda_0/3)q^4. \end{aligned} \tag{8}$$

The parameters m, n, Q_e, Q_g, and λ_0 correspond to mass, NUT parameter, electric and magnetic charge, and cosmological constant, respectively; while ϵ_0 and b are two additional parameters. The Kerr-Newman metric is obtained if one takes $b = a^2$, $g = 0 = n$ and $\epsilon_0 = 1$. In terms of the Boyer-Linquist coordinates $q = r$, $p = -a\cos\theta$, $u = -t + a\varphi$ and $v = \varphi/a$.

Equivalently, the Carter A solution can be specified by the null tetrad

$$\begin{aligned} D &= \partial_q + \frac{1}{\mathcal{Q}}(\partial_v - q^2\partial_u), \\ \Delta &= \tfrac{1}{2}\phi\overline{\phi}\mathcal{Q}\left(-\partial_q + \frac{1}{\mathcal{Q}}(\partial_v - q^2\partial_u)\right), \\ \delta &= \left(\frac{\mathcal{P}}{2}\right)^{1/2}\overline{\phi}\left(\partial_p + \frac{i}{\mathcal{P}}(\partial_v + p^2\partial_u)\right), \\ \overline{\delta} &= \left(\frac{\mathcal{P}}{2}\right)^{1/2}\phi\left(\partial_p - \frac{i}{\mathcal{P}}(\partial_v + p^2\partial_u)\right), \end{aligned} \tag{9}$$

where

$$\phi = \frac{1}{q + ip}. \tag{10}$$

With respect to this tetrad the only nonvanishing spin coefficients are

$$\begin{aligned} &\beta = \delta \ln \mathcal{P}^{1/4}, \quad \alpha = -\overline{\delta}\ln\frac{\mathcal{P}^{1/4}\mathcal{Q}^{1/2}}{q+ip}, \quad \gamma = -\Delta\ln\frac{\mathcal{P}^{1/4}\mathcal{Q}^{1/2}}{q+ip}, \\ &\rho = D\ln\phi, \quad \tau = \delta\ln\phi, \quad \pi = -\overline{\delta}\ln\phi, \quad \mu = -\Delta\ln\phi, \end{aligned} \tag{11}$$

and the curvature and electromagnetic field components different from zero are

$$\Psi_2 = \left\{-(m+in) + (Q_e^2 + Q_g^2)\overline{\phi}\right\}\phi^3, \quad \Lambda = \frac{\lambda_0}{6}, \quad \varphi_1 = \tfrac{1}{2}(Q_e + iQ_g)\phi^2. \tag{12}$$

The four-potential associated with (12), is given by [20]

$$\Phi_{0\dot{0}} = -\frac{Q_e q}{\mathcal{Q}}, \quad \Phi_{0\dot{1}} = \frac{iQ_g p\overline{\phi}}{\sqrt{2\mathcal{P}}}, \quad \Phi_{1\dot{0}} = \overline{\Phi_{0\dot{1}}}, \quad \Phi_{1\dot{1}} = -\frac{Q_e q\phi\overline{\phi}}{2}, \tag{13}$$

where the bar denotes complex conjugation. This four-potential satisfies the Lorenz gauge condition (*i.e.*, $\nabla^{A\dot{B}}\Phi_{A\dot{B}} = 0$, and therefore, $\varphi_{AB} = \nabla_{\dot{A}A}\Phi^{\dot{A}}_{B}$).

4. CHARACTERIZATION OF THE SEPARATION CONSTANTS

As we can see, the spin-coefficients and four-potential of the electromagnetic field are independent of the variables u and v. Therefore, this means that Eq. (5) admits solutions with a dependence in the variables u and v of the form

$$e^{i(ku+lv)}, \tag{14}$$

where k and l are separation constants. Acting on functions with a dependence of the form (14), the tetrad vectors can be replaced according to

$$\begin{aligned} D &\to \mathcal{D}_0, & \Delta &\to -\tfrac{1}{2}\phi\bar{\phi}\,\mathcal{Q}\mathcal{D}_0{}^{\dagger}, \\ \delta &\to \tfrac{1}{\sqrt{2}}\bar{\phi}\,\mathcal{L}_0{}^{\dagger}, & \bar{\delta} &\to \tfrac{1}{\sqrt{2}}\phi\,\mathcal{L}_0, \end{aligned} \tag{15}$$

where

$$\begin{aligned} \mathcal{D}_n &\equiv \partial_q + \frac{i}{\mathcal{Q}}(l - kq^2) + n\frac{\dot{\mathcal{Q}}}{\mathcal{Q}}, \\ \mathcal{D}_n^{\dagger} &\equiv \partial_q - \frac{i}{\mathcal{Q}}(l - kq^2) + n\frac{\dot{\mathcal{Q}}}{\mathcal{Q}}, \\ \mathcal{L}_n &\equiv \sqrt{\mathcal{P}}\left(\partial_p + \frac{1}{\mathcal{P}}(l + kp^2) + \frac{n}{2}\frac{\dot{\mathcal{P}}}{\mathcal{P}}\right), \\ \mathcal{L}_n^{\dagger} &\equiv \sqrt{\mathcal{P}}\left(\partial_p - \frac{1}{\mathcal{P}}(l + kp^2) + \frac{n}{2}\frac{\dot{\mathcal{P}}}{\mathcal{P}}\right). \end{aligned} \tag{16}$$

Using Eqs. (8)–(16), one finds that the solution of Eq. (5) is given by

$$\Psi = e^{i(ku+lv)}R_0(q)\,S_0(p), \tag{17}$$

where the one-variable functions $R_0(q)$ and $S_0(p)$ satisfy

$$\begin{aligned} &\left[\mathcal{Q}(\mathcal{D}_1\mathcal{D}_0^{\dagger} + \mathcal{D}_1^{\dagger}\mathcal{D}_0) + 2ieQ_e q(\mathcal{D}_0 - \mathcal{D}_0^{\dagger}) + \frac{2e^2Q_e^2q^2}{\mathcal{Q}} + 2q^2M^2 + A\right]R_0(q) = 0, \\ &\left[(\mathcal{L}_1^{\dagger}\mathcal{L}_0 + \mathcal{L}_1\mathcal{L}_0^{\dagger}) + \frac{2eQ_g p}{\sqrt{\mathcal{P}}}(\mathcal{L}_0^{\dagger} - \mathcal{L}_0) - \frac{2e^2Q_g^2p^2}{\mathcal{P}} + 2p^2M^2 - A\right]S_0(p) = 0. \end{aligned} \tag{18}$$

In these equations A is another separation constant.

We see that the separable solutions of the Klein-Gordon equation involve three separation constants, denoted as k, l and A. It is clear that the two first constants are related to the two-dimensional Abelian isometry group admitted by the Carter A solution and, apart from a factor $(-i)$, they can be defined in a covariant way as the eigenvalues of the Lie derivatives with respect to the Killing vectors ∂_u and ∂_v, respectively. The separation constant A, which is analogous to Carter's "fourth-constant" found in the case of tha Hamilton-Jacobi equation in the Kerr background, is not related to the space-time symmetries; it turns out that A is the eigenvalue of a certain second-order differential operator made out from the two-index Killing spinor admitted by the Carter A solution. In fact: if Ψ is a separable solution of Eq. (5) then

$$\Big[(\phi\overline{\phi})^{-1}\,\nabla_{A\dot{B}}\phi\overline{\phi}\,L^{A}{}_{C}\,L^{\dot{B}}{}_{\dot{D}}\,\nabla^{C\dot{D}} + 2ie\Phi_{E\dot{F}}\,L^{E}{}_{S}\,L^{\dot{F}}{}_{\dot{H}}\,\nabla^{S\dot{H}} - e^2\Phi_{T\dot{B}}\,L^{T}{}_{C}\,L^{\dot{B}}{}_{\dot{D}}\,\Phi^{C\dot{D}} - \tfrac{1}{2}M^2(\phi^{-2}+\overline{\phi^{-2}})\Big]\Psi = A\Psi. \tag{19}$$

In the Newman-Penrose notation Eq. (19) is given by

$$(\phi\overline{\phi})^{-1}\Big[(D+\epsilon+\overline{\epsilon}-\overline{\rho}-\rho+2ie\Phi_{0\dot{0}})\Delta + (\delta+\beta-\overline{\alpha}+\overline{\pi}-\tau+2ie\Phi_{0\dot{1}})\overline{\delta} + (\Delta-\gamma-\overline{\gamma}+\overline{\mu}+\mu+2ie\Phi_{1\dot{1}})D + (\overline{\delta}-\alpha+\overline{\beta}-\overline{\tau}+\pi+2ie\Phi_{1\dot{0}})\delta - 2e^2(\Phi_{0\dot{0}}\,\Phi_{1\dot{1}}+\Phi_{0\dot{1}}\,\Phi_{1\dot{0}}) - \frac{M^2}{2}(\phi^{-1}\,\overline{\phi}+\overline{\phi^{-1}}\,\phi)\Big]\Psi = A\Psi. \tag{20}$$

Using the expresions for the null tetrad, spin-coefficients, four-potential of the electromagnetic field and Eqs. (18), one can readily verify that Eq. (19) holds.

5. Concluding remarks

According to the previously expounded, we see that the validity of Eq. (19) implies that the operator acting on Ψ, on the left-hand side of Eq. (19), maps a solution of the Klein-Gordon equation into another solution (*i.e.*, is a *symmetry operator*). This fact, can be expressed by saying that the differential operator in Eq. (19) commutes with the Klein-Gordon operator modulo the Klein-Gordon operator itself. Furthermore, we see that the existence of a two-index Killing spinor is very closely related to the separability properties.

Since all the type-D electrovac metrics admit a two-index Killing spinor, it seems that a procedure similar to the one given above can be applied to all of them, or at least to those without acceleration.

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