

# Scattering of weak gravitational waves by a sphere

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**ABSTRACT.** By considering the gravitational field in the linearized Einstein theory, the scattering of a weak gravitational plane wave by a sphere is studied. The spherical components of the field and the differential scattering cross section are given in terms of the spin-weighted spherical harmonics and the effect on the polarization of the waves is analyzed.

**RESUMEN.** Considerando el campo gravitacional en la teoría de Einstein linealizada, se estudia la dispersión de una onda plana gravitacional débil por una esfera. Las componentes esféricas del campo y la sección de dispersión diferencial se dan en términos de los armónicos esféricos con peso de espín y se analiza el efecto sobre la polarización de las ondas.

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## 1. INTRODUCTION

As is well known, the Einstein field equations predict the existence of gravitational waves. In fact, the equations for weak gravitational fields, obtained by linearizing the Einstein field equations about the Minkowski metric, imply that the space-time curvature satisfies equations analogous to the Maxwell equations and that the weak gravitational waves share several properties with the electromagnetic waves.

In this paper we study the scattering of a plane weak gravitational wave by a sphere, neglecting the gravitational field produced by the sphere, *i.e.*, we assume that the mass of the scatterer is so small that the space-time is approximately flat. We make use of the gauge-independent description of the gravitational field given by the curvature tensor and of the fact that the solutions of the spin-2 Helmholtz equation can be expressed in terms of two scalar potentials that satisfy the Helmholtz equation [1,2]. A further simplification is obtained by making use of the spin-weighted components of the curvature (which are, essentially, the spinor components of the conformal curvature, denoted by  $\Psi_0, \dots, \Psi_4$  in the Newman-Penrose notation) and of the spin-weighted spherical harmonics [3]. In Sect. 2, the curvature perturbations are expressed in terms of two scalar potentials and using the expansion of a spin-2 plane wave in spherical waves, the scattering of a weak

gravitational wave by a sphere is analyzed. The scattered field is obtained by assuming that the gravitational waves do not penetrate the scatterer, which means that the scatterer is rigid. We find that, in the long wavelength limit, the differential cross section is strongly peaked in the backward direction ( $\theta = \pi$ ), but is finite at  $\theta = \pi$  (i.e., there is no "glory effect" (see, e.g., Refs. [4, 5] and the references cited therein)).

## 2. SCATTERING OF WEAK GRAVITATIONAL WAVES

The Einstein vacuum field equations linearized about the Minkowski metric lead to the equations

$$\begin{aligned} \partial_i E_{ij} &= 0, & \partial_i B_{ij} &= 0, \\ \frac{1}{c} \partial_t E_{ij} &= \varepsilon_{ikl} \partial_k B_{lj}, & \frac{1}{c} \partial_t B_{ij} &= -\varepsilon_{ikl} \partial_k E_{lj}, \end{aligned} \quad (1)$$

where

$$E_{ij} \equiv K_{0i0j}, \quad B_{ij} \equiv -\frac{1}{2} K_{0ikl} \varepsilon_{jkl}, \quad (2)$$

and  $K_{\alpha\beta\gamma\delta}$  is the curvature tensor to first order in the metric perturbation (Latin indices run from 1 to 3 and Greek indices run from 0 to 3) (see, e.g., Refs. [1, 6]). Equations (1), which are analogous to the source-free Maxwell equations, imply that if the traceless symmetric tensor fields (2) have a harmonic time dependence with frequency  $\omega$ , then they satisfy the Helmholtz equation; therefore, there exist solutions to the scalar Helmholtz equation,  $\psi_1$  and  $\psi_2$ , such that [1, 2]

$$E_{ij} = \text{Re} \left[ (kU_{ij}(\psi_1) + V_{ij}(\psi_2)) e^{-i\omega t} \right], \quad (3)$$

where  $k = \omega/c$ ,

$$U_{ij} \equiv iL_i X_j + iL_j X_i, \quad V_{ij} \equiv \varepsilon_{imn} \partial_m U_{nj}, \quad (4)$$

with

$$\mathbf{L} \equiv \frac{1}{i} \mathbf{r} \times \nabla, \quad \mathbf{X} \equiv i\nabla \times \mathbf{L} - \nabla. \quad (5)$$

Then, from Eqs. (1) it follows that

$$B_{ij} = \text{Re} \left[ \frac{1}{i} (kU_{ij}(\psi_2) + V_{ij}(\psi_1)) e^{-i\omega t} \right], \quad (6)$$

whence the complex traceless symmetric tensor field

$$F_{ij} \equiv E_{ij} + iB_{ij} \quad (7)$$

is given by

$$F_{ij} = kU_{ij} \left( \frac{\phi_1 + \phi_2}{2} + \frac{\bar{\phi}_1 - \bar{\phi}_2}{2} \right) + V_{ij} \left( \frac{\phi_1 + \phi_2}{2} - \frac{\bar{\phi}_1 - \bar{\phi}_2}{2} \right), \tag{8}$$

where

$$\phi_1 \equiv \psi_1 e^{-i\omega t}, \quad \phi_2 \equiv \psi_2 e^{-i\omega t}, \tag{9}$$

satisfy the scalar wave equation.

The components of any traceless symmetric tensor field  $t_{ij}$  with respect to the basis  $\{\hat{e}_\theta, \hat{e}_\varphi, \hat{e}_r\}$  induced by the spherical coordinates can be combined into the five quantities

$$\begin{aligned} t_{\pm 2} &\equiv t_{\theta\theta} - t_{\varphi\varphi} \pm 2it_{\theta\varphi}, \\ t_{\pm 1} &\equiv \mp t_{r\theta} - it_{r\varphi}, \\ t_0 &\equiv t_{rr}, \end{aligned} \tag{10}$$

in such a way that  $t_s$  has spin-weight  $s$  [3, 7], *i.e.*, under the rotation

$$\hat{e}_\theta + i\hat{e}_\varphi \rightarrow e^{i\alpha}(\hat{e}_\theta + i\hat{e}_\varphi) \tag{11}$$

the components  $t_s$  transform as

$$t_s \rightarrow e^{is\alpha} t_s. \tag{12}$$

From Eqs. (8) and (10) it follows that the spin-weighted components of the field  $F_{ij}$  are given by [2]

$$\begin{aligned} F_{+2} &= k^2 \partial \bar{\partial} \left\{ \left( -\frac{2i}{kr^2} \partial_r r^2 + \frac{1}{k^2 r^2} \partial_r^2 r^2 - 1 \right) \frac{\phi_1 + \phi_2}{2} \right. \\ &\quad \left. - \left( \frac{2i}{kr^2} \partial_r r^2 + \frac{1}{k^2 r^2} \partial_r^2 r^2 - 1 \right) \frac{\bar{\phi}_1 - \bar{\phi}_2}{2} \right\}, \\ F_{+1} &= -\frac{k}{r} \bar{\partial} \partial \bar{\partial} \left\{ \left( -i + \frac{1}{kr} \partial_r r \right) \frac{\phi_1 + \phi_2}{2} - \left( i + \frac{1}{kr} \partial_r r \right) \frac{\bar{\phi}_1 - \bar{\phi}_2}{2} \right\}, \\ F_0 &= \frac{1}{r^2} \bar{\partial} \bar{\partial} \partial \bar{\partial} \left( \frac{\phi_1 + \phi_2}{2} - \frac{\bar{\phi}_1 - \bar{\phi}_2}{2} \right), \\ F_{-1} &= \frac{k}{r} \partial \bar{\partial} \bar{\partial} \left\{ \left( i + \frac{1}{kr} \partial_r r \right) \frac{\phi_1 + \phi_2}{2} - \left( -i + \frac{1}{kr} \partial_r r \right) \frac{\bar{\phi}_1 - \bar{\phi}_2}{2} \right\}, \\ F_{-2} &= k^2 \bar{\partial} \bar{\partial} \left\{ \left( \frac{2i}{kr^2} \partial_r r^2 + \frac{1}{k^2 r^2} \partial_r^2 r^2 - 1 \right) \frac{\phi_1 + \phi_2}{2} \right. \\ &\quad \left. - \left( -\frac{2i}{kr^2} \partial_r r^2 + \frac{1}{k^2 r^2} \partial_r^2 r^2 - 1 \right) \frac{\bar{\phi}_1 - \bar{\phi}_2}{2} \right\}, \end{aligned} \tag{13}$$



where

$$\begin{aligned}\partial\eta &\equiv -\sin^s\theta(\partial_\theta + i\csc\theta\partial_\varphi)(\eta\sin^{-s}\theta), \\ \bar{\partial}\eta &\equiv -\sin^{-s}\theta(\partial_\theta - i\csc\theta\partial_\varphi)(\eta\sin^s\theta),\end{aligned}\quad (14)$$

if  $\eta$  has spin-weight  $s$ ; then  $\partial\eta$  and  $\bar{\partial}\eta$  have spin-weight  $s+1$  and  $s-1$ , respectively. Owing to the definitions (7) and (10), the components  $F_s$  amount to

$$\begin{aligned}F_{\pm 2} &= E_{\theta\theta} - E_{\varphi\varphi} \mp 2B_{\theta\varphi} + i(B_{\theta\theta} - B_{\varphi\varphi} \pm 2E_{\theta\varphi}), \\ F_{\pm 1} &= \mp E_{r\theta} + B_{r\varphi} + i(\mp B_{r\theta} - E_{r\varphi}), \\ F_0 &= E_{rr} + iB_{rr}.\end{aligned}\quad (15)$$

The energy flux for a wave with frequency  $\omega$  is given by the vector field [6]

$$S_i = \frac{c^7}{8\pi G\omega^2}\epsilon_{ijk}E_{jm}B_{km}\quad (16)$$

hence, from Eqs. (15) it follows that the radial component of this vector field is

$$S_r = \frac{c^5}{64\pi Gk^2}\left(|F_{-2}|^2 - |F_{+2}|^2 + 2|F_{-1}|^2 - 2|F_{+1}|^2\right).\quad (17)$$

### 2.1. Asymptotic behavior of the solutions

The scalar Helmholtz equation,  $(\nabla^2 + k^2)\psi_n = 0$ , admits separable solutions in spherical coordinates of the form

$$\psi_n = A_n i^j h_j^{(1)}(kr) Y_{jm}(\theta, \varphi) + B_n i^j h_j^{(2)}(kr) Y_{jm}(\theta, \varphi), \quad (n = 1, 2),\quad (18)$$

where  $h_j^{(1)}$  and  $h_j^{(2)}$  are spherical Hankel functions,  $A_n, B_n$  are constants and the factors  $i^j$  are introduced for convenience. Making use of the asymptotic form of the spherical Hankel functions:

$$h_j^{(1)}(kr) \sim (-i)^{j+1} \frac{e^{ikr}}{kr} \left(1 + \frac{i(j+1)!}{2(j-1)!} \frac{1}{kr} - \frac{1}{2^2 2!} \frac{(j+2)!}{(j-2)!} \frac{1}{(kr)^2} + \dots\right),\quad (19)$$

one finds that

$$\left(i + \frac{1}{kr}\partial_{rr}\right) h_j^{(1)}(kr) \sim 2i(-i)^{j+1} \frac{e^{ikr}}{kr} \left(1 + O\left(\frac{1}{kr}\right)\right),\quad (20a)$$

$$\left(-i + \frac{1}{kr}\partial_{rr}\right) h_j^{(1)}(kr) \sim -\frac{i}{2}(-i)^{j+1} j(j+1) \frac{e^{ikr}}{(kr)^3} \left(1 + O\left(\frac{1}{kr}\right)\right),\quad (20b)$$

$$\left(\frac{2i}{kr^2}\partial_{rr^2} + \frac{1}{k^2 r^2}\partial_r^2 r^2 - 1\right) h_j^{(1)}(kr) \sim -4(-i)^{j+1} \frac{e^{ikr}}{kr} \left(1 + O\left(\frac{1}{kr}\right)\right),\quad (20c)$$

$$\left(-\frac{2i}{kr^2}\partial_{rr^2} + \frac{1}{k^2 r^2}\partial_r^2 r^2 - 1\right) h_j^{(1)}(kr) \sim -\frac{(j+2)!}{4(j-2)!} (-i)^{j+1} \frac{e^{ikr}}{(kr)^5} \left(1 + O\left(\frac{1}{kr}\right)\right),\quad (20d)$$

therefore, if Eq. (18) contains terms with  $h_j^{(1)}$  only, *i.e.*,

$$\phi_n = A_n i^j h_j^{(1)}(kr) Y_{jm}(\theta, \varphi) e^{-i\omega t} \tag{21}$$

[see Eq. (9)], substituting Eqs. (21) into Eqs. (13) and making use of Eqs. (20) one finds that

$$F_s = O\left(\frac{1}{r^{3+s}}\right). \tag{22}$$

Therefore, for outgoing waves,  $F_{-2}$  is the dominant component. (Relations similar to Eq. (22) apply to the massless fields of any spin in an asymptotically simple space-time and this result is known as the “peeling theorem” (see, *e.g.*, Ref. [8].) From Eqs. (17) and (22) it follows that the outgoing energy flux per unit time and unit solid angle is

$$\frac{d^2 E_{\text{out}}}{dt d\Omega} = \lim_{r \rightarrow \infty} \frac{c^5}{64\pi G k^2} r^2 |F_{-2}|^2. \tag{23}$$

Similarly, one finds that for ingoing waves,  $F_{+2}$  is the dominant component,

$$F_s = O\left(\frac{1}{r^{3-s}}\right), \tag{24}$$

and the ingoing energy flux per unit time and unit solid angle is

$$\frac{d^2 E_{\text{in}}}{dt d\Omega} = \lim_{r \rightarrow \infty} \frac{c^5}{64\pi G k^2} r^2 |F_{+2}|^2. \tag{25}$$

Thus, in the radiation zone  $F_{-2}$  represents the outgoing field and  $F_{+2}$  represents the ingoing field.

### 2.2. Polarization

The polarization of the radiation can be also determined very easily from the spin-weighted components  $F_{\pm 2}$ . In fact, if  $F_s$  has a time dependence of the form

$$F_s(t) = F_s(0) e^{i\omega t}, \tag{26}$$

where  $F_s(0)$  is the value of  $F_s$  at  $t = 0$ , comparison with Eq. (12) shows that the time evolution of  $F_s$  amounts to rotate the vectors  $\hat{e}_\theta$  and  $\hat{e}_\varphi$ , about  $\hat{e}_r$ , with an angular velocity  $-\omega/s$  or, equivalently, to rotate  $F_{ij}$  about  $\hat{e}_r$  with an angular velocity  $\omega/s$ . Therefore, if  $F_s$ , with  $s \neq 0$ , is proportional to  $e^{i\omega t}$  or to  $e^{-i\omega t}$ , the field has circular polarization; while the presence of both factors,  $e^{i\omega t}$  and  $e^{-i\omega t}$ , means that the radiation has elliptic polarization.

If in the radiation zone  $F_{-2}$  is proportional to  $e^{i\omega t}$ , the outgoing radiation has right circular polarization (negative helicity) if  $\omega > 0$  or left circular polarization (positive helicity) if  $\omega < 0$ . Since  $F_{+2}$  has spin-weight opposite to that of  $F_{-2}$  and corresponds to waves propagating in the direction  $-\hat{e}_r$ , the foregoing conclusions are equally valid for  $F_{+2}$ ; that is, if  $F_{+2}$  is proportional to  $e^{i\omega t}$ , the ingoing radiation has right or left circular polarization according to whether  $\omega$  be positive or negative, respectively.

2.3. Expansion of a plane wave

For a circularly polarized plane wave propagating in the  $\hat{e}_z$  direction, the “electric part” of the curvature is proportional to the real part of the complex tensor field with cartesian components

$$(E_{ij}^c) = \begin{pmatrix} 1 & \pm i & 0 \\ \pm i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} e^{i(kz-\omega t)}. \tag{27}$$

By expressing this tensor field in the form

$$E_{ij}^c = kU_{ij}(\phi_1) + V_{ij}(\phi_2) \tag{28}$$

(cf. Eq. (3)), one finds that  $x_i x_j E_{ij}^c = \bar{\partial} \bar{\partial} \partial \partial \phi_2$  and from Eq. (27),  $x_i x_j E_{ij}^c = (r \sin \theta e^{\pm i\varphi})^2 \times e^{i(kz-\omega t)}$ . Using now the well known expansion

$$e^{ikz} = \sum_{j=0}^{\infty} i^j (2j+1) j_j(kr) P_j(\cos \theta) = \sum_{j=0}^{\infty} i^j \sqrt{4\pi(2j+1)} j_j(kr) Y_{j0}(\theta, \varphi), \tag{29}$$

and the relations

$$\begin{aligned} \sin \theta e^{\pm i\varphi} Y_{jm} &= \pm \left[ \frac{(j \mp m)(j \mp m - 1)}{(2j-1)(2j+1)} \right]^{1/2} Y_{j-1, m\pm 1} \\ &\mp \left[ \frac{(j \pm m + 1)(j \pm m + 2)}{(2j+1)(2j+3)} \right]^{1/2} Y_{j+1, m\pm 1}, \end{aligned} \tag{30}$$

$$j_{j-1}(x) + j_{j+1}(x) = \frac{2j+1}{x} j_j(x) \tag{31}$$

(see, e.g., Ref. [9]) and

$$\begin{aligned} \partial_s Y_{jm} &= [(j-s)(j+s+1)]^{1/2} {}_{s+1}Y_{jm}, \\ \bar{\partial}_s Y_{jm} &= -[(j+s)(j-s+1)]^{1/2} {}_{s-1}Y_{jm}, \end{aligned} \tag{32}$$

where the  ${}_s Y_{jm}$  are spin-weighted spherical harmonics, with  ${}_0 Y_{jm} = Y_{jm}$ , one finds that

$$\phi_2 = -\frac{1}{k^2} \sum_{j=2}^{\infty} \left[ \frac{4\pi(2j+1)(j-2)!}{(j+2)!} \right]^{1/2} i^j j_j(kr) Y_{j,\pm 2} e^{-i\omega t}. \tag{33}$$

On the other hand, calculating  $\epsilon_{ikl} \partial_k E_{ij}^c$  from Eqs. (27) and (28), it turns out that

$$E_{ij}^c = \pm [kU_{ij}(\phi_2) + V_{ij}(\phi_1)]. \tag{34}$$



Comparing the expressions (28) and (34) one concludes that  $\phi_1 = \pm\phi_2$ , *i.e.*,

$$\phi_1 = \mp \frac{1}{k^2} \sum_{j=2}^{\infty} \left[ \frac{4\pi(2j+1)(j-2)!}{(j+2)!} \right]^{1/2} i^j j_j(kr) Y_{j,\pm 2} e^{-i\omega t}. \tag{35}$$

By substituting Eqs. (33) and (35) into Eqs. (8) or (13) one obtains the multipole expansion of a plane wave.

2.4. Scattering of a plane wave by a sphere

Now we can consider the scattering of a plane wave by a sphere. As in the case of the field of the incoming wave, the scattered field and the total field can be expressed in the form (8) in terms of appropriate solutions of the scalar wave equation. For instance, the scattered field has the form

$$F_{ij}^{sc} = kU_{ij} \left( \frac{\phi_3 + \phi_4}{2} + \frac{\overline{\phi_3 - \phi_4}}{2} \right) + V_{ij} \left( \frac{\phi_3 + \phi_4}{2} - \frac{\overline{\phi_3 - \phi_4}}{2} \right), \tag{36}$$

where  $\phi_3$  and  $\phi_4$  are solutions of the scalar wave equation of the form

$$\begin{aligned} \phi_3 &= \mp \frac{1}{2k^2} \sum_{j=2}^{\infty} \left[ \frac{4\pi(2j+1)(j-2)!}{(j+2)!} \right]^{1/2} \alpha_{\pm}(j) i^j h_j^{(1)}(kr) Y_{j,\pm 2} e^{-i\omega t}, \\ \phi_4 &= -\frac{1}{2k^2} \sum_{j=2}^{\infty} \left[ \frac{4\pi(2j+1)(j-2)!}{(j+2)!} \right]^{1/2} \beta_{\pm}(j) i^j h_j^{(1)}(kr) Y_{j,\pm 2} e^{-i\omega t}, \end{aligned} \tag{37}$$

and the coefficients  $\alpha_{\pm}(j)$ ,  $\beta_{\pm}(j)$  are determined from the boundary conditions at the surface of the sphere. From Eqs. (13), (20c), (32) and (37) it follows that in the radiation zone the scattered field is given by

$$\begin{aligned} F_{-2}^{sc} \sim & -\frac{i}{kr} \sum_{j=2}^{\infty} \sqrt{4\pi(2j+1)} \left[ (\beta_{\pm}(j) \pm \alpha_{\pm}(j)) e^{i(kr-\omega t)} {}_{-2}Y_{j,\pm 2} \right. \\ & \left. - (\overline{\beta_{\pm}(j)} \mp \overline{\alpha_{\pm}(j)}) e^{-i(kr-\omega t)} {}_2Y_{j,\pm 2} \right], \end{aligned} \tag{38}$$

and from Eq. (23) it follows that the time-averaged energy flux of the scattered field is

$$\begin{aligned} \left\langle \frac{d^2 E^{sc}}{dt d\Omega} \right\rangle = & \frac{c^5}{16Gk^4} \left\{ \left| \sum_{j=2}^{\infty} \sqrt{2j+1} (\alpha_{\pm}(j) \pm \beta_{\pm}(j)) {}_{-2}Y_{j,\pm 2} \right|^2 \right. \\ & \left. + \left| \sum_{j=2}^{\infty} \sqrt{2j+1} (\alpha_{\pm}(j) \mp \beta_{\pm}(j)) {}_2Y_{j,\pm 2} \right|^2 \right\}. \end{aligned} \tag{39}$$

Since the incident energy flux is  $c^5/4\pi Gk^2$  [see Eq. (16)], the scattering differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\pi}{4k^2} \left\{ \left| \sum_{j=2}^{\infty} \sqrt{2j+1} (\alpha_{\pm}(j) \pm \beta_{\pm}(j)) {}_{-2}Y_{j,\pm 2} \right|^2 + \left| \sum_{j=2}^{\infty} \sqrt{2j+1} (\alpha_{\pm}(j) \mp \beta_{\pm}(j)) {}_2Y_{j,\pm 2} \right|^2 \right\} \tag{40}$$

and by virtue of the orthonormality of the spin-weighted spherical harmonics the total scattering cross section is

$$\sigma = \frac{\pi}{2k^2} \sum_{j=2}^{\infty} (2j+1) [|\alpha_{\pm}(j)|^2 + |\beta_{\pm}(j)|^2]. \tag{41}$$

From Eq. (38) it follows that the scattered field has, in general, elliptic polarization and that it has circular polarization if and only if  $\alpha_{\pm}(j) = \beta_{\pm}(j)$  or  $\alpha_{\pm}(j) = -\beta_{\pm}(j)$ , for all  $j$ ; in fact, if  $\alpha_{\pm}(j) = \beta_{\pm}(j)$ , the scattered field has the same helicity as the incident wave, while if  $\alpha_{\pm}(j) = -\beta_{\pm}(j)$ , the scattered field has the *opposite helicity* to that of the incident wave (*cf.* Ref. [4]). (The same result is obtained in the case of the scattering of electromagnetic waves [10, 11]; note that in Ref. [11], p. 771, the case  $\alpha_{\pm}(j) = -\beta_{\pm}(j)$  is missing.)

Similarly, one finds that the total time-averaged outgoing power is

$$\left\langle \frac{dE_{\text{out}}}{dt} \right\rangle = \frac{c^5}{8Gk^2} \sum_{j=2}^{\infty} (2j+1) [ |1 + \alpha_{\pm}(j)|^2 + |1 + \beta_{\pm}(j)|^2 ], \tag{42}$$

hence, for each multipole, the total outgoing power is equal to the incoming power if and only if  $1 + \alpha_{\pm}(j)$  and  $1 + \beta_{\pm}(j)$  have modulus 1. (The total incoming power is infinite, since we are dealing with a plane wave; nevertheless, the power in each multipole is finite.)

Assuming that the origin of the system of coordinates is placed at the center of the sphere, the coefficients  $\alpha_{\pm}(j)$  and  $\beta_{\pm}(j)$  in Eqs. (37) are determined by the boundary conditions on the curvature perturbations at  $r = a$ , where  $a$  is the radius of the sphere. We shall assume that the sphere is rigid, in such a way that the curvature perturbations vanish inside the scatterer. Making use of the Stokes theorem and the equation

$$\varepsilon_{ikl} \partial_k E_{lj} = -\frac{1}{c} \partial_t B_{ij} - \frac{4\pi G}{c^4} (\varepsilon_{jkl} \partial_k T_{li} + \frac{1}{3} \varepsilon_{ikj} \partial_k T),$$

where  $T_{\alpha\beta}$  is the energy-momentum tensor,  $T = T_{\alpha}^{\alpha}$  and  $E_{ij}$ ,  $B_{ij}$  are the components of the traceless part of  $K_{\alpha\beta\gamma\delta}$ , which follows from the Bianchi identities and the Einstein



field equations, one finds that  $E_{\theta\theta} = E_{\varphi\varphi}$  and  $E_{\theta\varphi} = 0$  at  $r = a$ . Owing to Eqs. (15), these boundary conditions amount to

$$(F_{+2} + \overline{F_{-2}})\Big|_{r=a} = 0, \tag{43}$$

which, in view of Eqs. (13), are equivalent to

$$\frac{1}{kr^2} \partial_r r^2 (\phi_1 + \phi_3)\Big|_{r=a} = 0, \quad \left( \frac{1}{k^2 r^2} \partial_r^2 r^2 - 1 \right) (\phi_2 + \phi_4)\Big|_{r=a} = 0. \tag{44}$$

Substituting Eqs. (33), (35) and (37) into Eqs. (44) one obtains

$$1 + \alpha_{\pm}(j) = - \frac{\frac{d}{dr} r^2 h_j^{(2)}(kr)}{\frac{d}{dr} r^2 h_j^{(1)}(kr)}\Big|_{r=a}, \quad 1 + \beta_{\pm}(j) = - \frac{\left( \frac{1}{k^2 r^2} \frac{d^2}{dr^2} r^2 - 1 \right) h_j^{(2)}(kr)}{\left( \frac{1}{k^2 r^2} \frac{d^2}{dr^2} r^2 - 1 \right) h_j^{(1)}(kr)}\Big|_{r=a} \tag{45}$$

which have modulus equal to 1, and therefore there is no absorption of energy by the sphere.

In the long wavelength limit,  $ka \ll 1$ , Eqs. (45) yield

$$\alpha_{\pm}(j) \simeq \frac{2(j+2)i(ka)^{2j+1}}{(j-1)(2j+1)[(2j-1)!!]^2}, \quad \beta_{\pm}(j) \simeq - \frac{2(j+1)(j+2)i(ka)^{2j+1}}{j(j-1)(2j+1)[(2j-1)!!]^2} \tag{46}$$

therefore, the dominant terms in Eqs. (38)–(41) correspond to  $j = 2$ . Then, from Eqs. (40), (41) and (46) one obtains

$$\alpha_{\pm}(2) \simeq \frac{8i}{45}(ka)^5, \quad \beta_{\pm}(2) \simeq -\frac{4i}{15}(ka)^5,$$

$$\frac{d\sigma}{d\Omega} \simeq \frac{a^2}{81}(ka)^8 \left( \cos^8(\theta/2) + 25 \sin^8(\theta/2) \right) \tag{47}$$

and

$$\sigma \simeq \frac{104}{405} \pi a^2 (ka)^8. \tag{48}$$

Equation (47) shows that, in this limit, the scattered field is mainly concentrated in the backward direction ( $\theta = \pi$ ); the differential cross section for electromagnetic waves scattered by a perfectly conducting sphere gives a similar pattern, though less asymmetric and less narrow [10, 11]. (In the case of the electromagnetic waves, 80% of the energy of the scattered field gets out through the hemisphere  $\pi/2 \leq \theta \leq \pi$ ; while Eq. (47) implies that more than 93% of the energy of the scattered field is reflected in the hemisphere  $\pi/2 \leq \theta \leq \pi$ .) The cross section for the gravitational waves is much less than that for electromagnetic waves [10, 11] owing to the presence of an extra factor  $(ka)^4$ . These

differences follow from the fact that the lowest order multipole present in the scattered field corresponds to  $j = 1$  in the case of spin 1 and to  $j = 2$  in the case of spin 2.

On the other hand, considering the scattering of a spin-0 field by a sphere, with the boundary condition that the field be equal to zero at the surface of the sphere, one finds that in the long wavelength limit

$$\frac{d\sigma}{d\Omega} \simeq \frac{a^2}{4} \quad (49)$$

and

$$\sigma \simeq \pi a^2. \quad (50)$$

Thus, by contrast with the result (47) and that corresponding to electromagnetic waves, in the case of a spin-0 field, the scattering is approximately isotropic and the cross section is much greater than those for the spin-1 and -2 cases.

A rigid material, in the sense defined above, is the analogue of a perfect conductor in the electromagnetic case and it would behave as a mirror for the gravitational waves. However, Eq. (48) shows that, in the long wavelength limit, the total scattering cross section of a rigid sphere is a very small fraction of its geometric area; therefore, the effect of rigid, or nearly rigid, celestial bodies on the gravitational waves is negligibly small.

### 3. CONCLUDING REMARKS

The differential cross sections obtained here differ considerably from those given in Ref. [4], which correspond to the scattering by the gravitational field of a mass  $m$  in the weak-field approximation. In particular, the differential scattering cross sections for scalar, electromagnetic, and gravitational waves given in Ref. [4] do coincide for small angles  $\theta$ .

From Eqs. (13) it can be readily seen that the gravitational perturbations can be expressed in terms of a single complex scalar potential. Making use of Eqs. (9), one finds that Eqs. (13) can be rewritten as

$$\begin{aligned} F_{+2} &= \frac{1}{r^2} \left( \frac{1}{c} \partial_t + \partial_r \right)^2 r^2 \partial \partial \chi, \\ F_{+1} &= -\frac{1}{r^2} \left( \frac{1}{c} \partial_t + \partial_r \right) r \bar{\partial} \partial \partial \chi, \\ F_0 &= \frac{1}{r^2} \bar{\partial} \bar{\partial} \partial \partial \chi, \\ F_{-1} &= -\frac{1}{r^2} \left( \frac{1}{c} \partial_t - \partial_r \right) r \partial \bar{\partial} \bar{\partial} \chi, \\ F_{-2} &= \frac{1}{r^2} \left( \frac{1}{c} \partial_t - \partial_r \right)^2 r^2 \bar{\partial} \bar{\partial} \chi, \end{aligned}$$

where  $\chi \equiv (\phi_1 + \phi_2 - \bar{\phi}_1 + \bar{\phi}_2)/2$  is a solution of the wave equation.

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