

Optimization of coupled finite-time heat engines

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ABSTRACT. In this paper we present an analysis of two Carnot-like endoreversible thermal engines coupled through a thermal bypass. By means of the definition of a coupling parameter (α) we study some optimization criteria for this array of engines. We show that α works as a driving parameter of the global power output and efficiency of the array. There exist α -values that maximize those quantities and we demonstrate that the coupled engines have performance advantages over the uncoupled case.

RESUMEN. En este trabajo presentamos un análisis de dos máquinas endorreversibles tipo Carnot acopladas mediante un puente térmico. Mediante la definición de un parámetro de acoplamiento α , estudiamos algunos criterios de optimización para este arreglo de máquinas. Mostramos que α funciona como un parámetro que maneja la potencia y eficiencia globales del arreglo. Existen valores de α que maximizan esas cantidades y demostramos que las máquinas acopladas tienen ventajas de operación sobre el caso desacoplado.

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1. INTRODUCTION

Classical equilibrium thermodynamics (CET) imposes limits over diverse process variables as efficiency, work, heat and others. Generally, the CET-limits (which are only achievable in the reversible regime) are far away from real values for the corresponding process variables. A typical example is given by the Carnot's theorem, which establishes the Carnot efficiency ($\eta_C = 1 - \frac{T_2}{T_1}$) as the upper limit for the efficiency of any engine working between two heat reservoirs with temperatures T_1 and T_2 ($T_1 > T_2$). The great departure between η_C and real efficiencies is not the only trouble concerning this reversible limit. Moreover, the Carnot reversible engine has not power output. In 1975, Curzon and Ahlborn (CA) [1] proposed a Carnot-like engine which produces power output and entropy in a finite time. The CA model reached a remarkable agreement between calculated and observed efficiencies for several power sources (see Table I of Ref. [1]) and marked the beginning of a branch

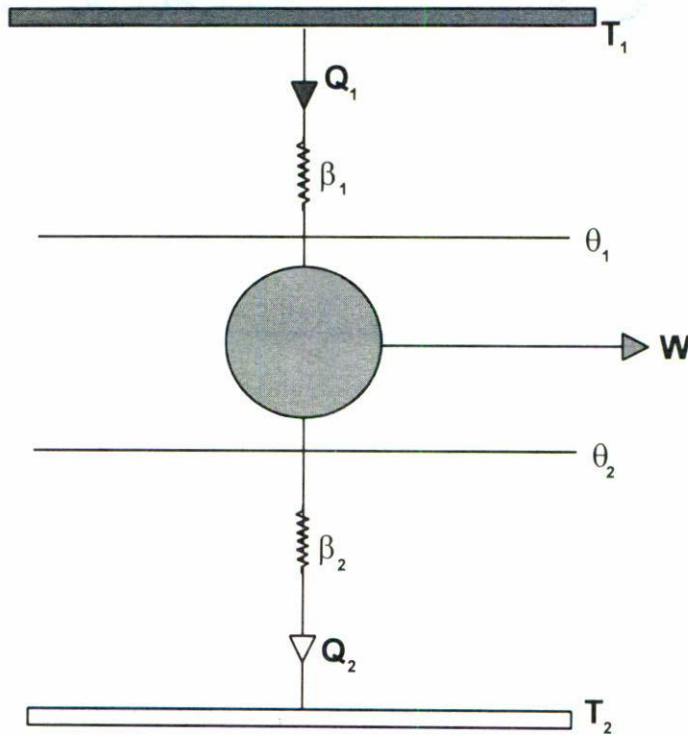


FIGURE 1. Diagram of an endoreversible Curzon-Ahlborn engine.

of thermodynamics known as finite-time thermodynamics (FTT) [2-4]. In the CA-model there is not thermal equilibrium between heat reservoirs and the working fluid along the isothermal branches of the cycle. The heat transfer in these processes is given by the so-called Newton's law of cooling, which is

$$Q_j = \beta_j(\pm T_j \mp \theta_j), \quad j = 1, 2, \quad (1)$$

with Q_j a heat flux per unit time, β_j a thermal conductance, T_j the absolute temperatures of the heat reservoirs, and θ_j the absolute temperatures of the working fluid as it is depicted in Fig. 1.

In the CA-model, the so-called endoreversibility hypothesis plays a very important role and this consists in assuming that the working fluid undergoes only reversible transformations and all the entropy sources are located at the couplings between the heat reservoirs and the working substance [5]. By means of Eq. (1) and the endoreversibility hypothesis, Curzon and Ahlborn [1] found that an engine as that of Fig. 1, working in a maximum power regime has an efficiency given by

$$\eta_{CA} = 1 - \left(\frac{T_2}{T_1}\right)^{1/2}, \quad (2)$$

which only depends on the temperatures of the heat reservoirs (as in the Carnot case), and provides an excellent approximation to efficiencies of certain power sources [1]. Since the

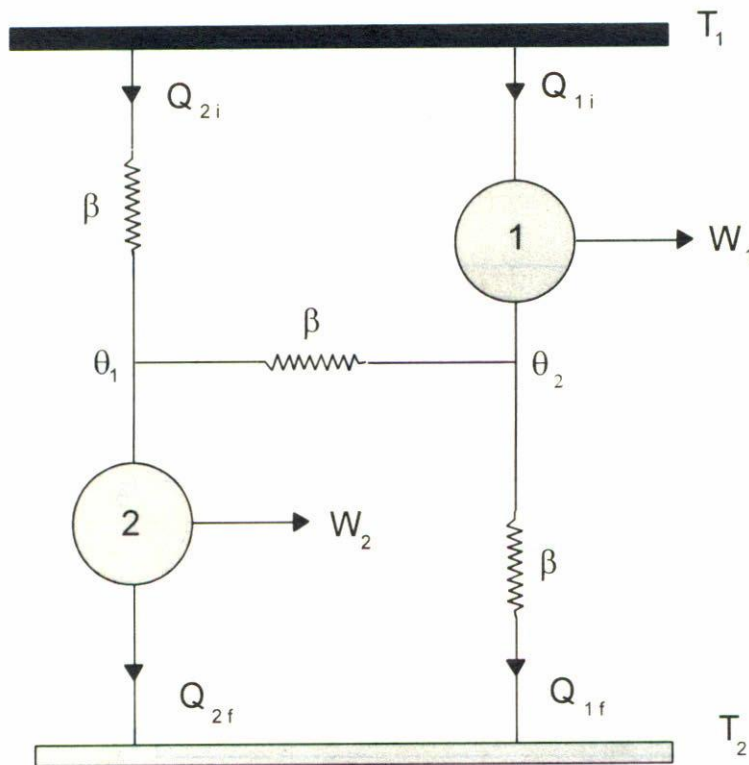


FIGURE 2. De Mey-De Vos array of two endoreversible CA-engines.

CA paper, many other authors have obtained Eq. (2) by means of different approaches [6–8]. Equation (2) has not the same universality that Carnot’s efficiency. In fact, the efficiency in a maximum power regime strongly depends on the heat transfer law used [9] and Eq. (2) is only valid for the Newton’s cooling law case. Recently, De Mey and De Vos [10] have proposed an array of CA engines that in the maximum power regime has an efficiency given by

$$\eta = 1 - \left(\frac{T_2}{T_1}\right)^{1/3}. \tag{3}$$

In the following section we briefly present the De May-De Vos engine and in Sect. 3 we discuss this engine under a new approach consisting in the introduction of a coupling parameter, which gives new insights over parallel arrays of CA-engines.

2. THE DE MEY-DE VOS ENGINE

In Ref. [10], De Mey and De Vos (MV) show how the famous Curzon-Ahlborn formula [Eq. (2)] for the efficiency at maximum power regime is maintained for series arrays of finite-time Carnot-like engines. Nevertheless, these authors found that there exist other

arrays where the maximum power efficiency is given by Eq. (3). We call De Mey-De Vos engine the system depicted in Fig. 2, which consists in two Carnot engines and three equal conductances put in a kind of parallel array. De Mey and De Vos reach their result by means of the following procedure: First, they express the powers W_1 and W_2 (work per unit time) as (see Fig. 2)

$$W_1 = \beta(2\theta_2 - \theta_1 - T_2) \left(\frac{T_1}{\theta_2} - 1 \right) \quad (4)$$

and

$$W_2 = \beta(T_1 + \theta_2 - 2\theta_1) \left(1 - \frac{T_2}{\theta_1} \right), \quad (5)$$

where the Newton's cooling law and the endoreversibility hypothesis have been used. The second step is the application of an extremal condition

$$\frac{\partial(W_1 + W_2)}{\partial\theta_1} = 0 \quad (6)$$

and

$$\frac{\partial(W_1 + W_2)}{\partial\theta_2} = 0. \quad (7)$$

Thus, they get

$$\theta_1 = (T_1 T_2^2)^{1/3} \quad (8)$$

and

$$\theta_2 = (T_1^2 T_2)^{1/3}, \quad (9)$$

which permit to obtain the optimum power output

$$W_{\text{opt}} = \beta(3T_1 + 3T_2 - 3\theta_1 - 3\theta_2). \quad (10)$$

From Fig. 2, MV calculate the optimum supplied heat flux, which is

$$Q_{\text{opt}} = \beta(3T_1 - 2\theta_1 - \theta_2). \quad (11)$$

Finally, they find the optimum efficiency as,

$$\eta_{\text{MV}} = \eta_{\text{opt}} = \frac{W_{\text{opt}}}{Q_{\text{opt}}} = 1 - \left(\frac{T_2}{T_1} \right)^{1/3} \frac{3 \left(\frac{T_2}{T_1} \right)^{1/3} + 2}{2 \left(\frac{T_2}{T_1} \right)^{1/3} + 3}. \quad (12)$$

Thus, their assertion that some endoreversible systems of linear thermal conductors have other efficiencies than the CA-formula (Eq. (2)) is proved. η_{opt} is always smaller than unity. Equation (12) is the maximum efficiency of the overall engine at maximum power regime. Under the same circumstances, the efficiencies of the separate engines are [10],

$$\eta_{1,\text{opt}} = 1 - \frac{\theta_2}{T_1} = 1 - \left(\frac{T_2}{T_1}\right)^{1/3} \tag{13}$$

and

$$\eta_{2,\text{opt}} = 1 - \frac{T_2}{\theta_1} = 1 - \left(\frac{T_2}{T_1}\right)^{1/3} \tag{14}$$

(see Fig. 2). In this way, De Mey and De Vos show how the array of Fig. 2 has an optimum efficiency where the term $\left(\frac{T_2}{T_1}\right)^{1/3}$ plays in fact the role of the term $\left(\frac{T_2}{T_1}\right)^{1/2}$ in CA engines.

3. THE COUPLING PARAMETER

In Ref. [10], De Mey and De Vos only remark the appearance of the cubic root term in the efficiency at maximum power regime for the array of CA- engines in Fig. 2. In this paper we show that a coupling parameter (α) between both CA-engines can be defined, which may be used for driving some engine’s optimum regimes of operation, as the maximum efficiency and the maximum power output regimes. We also propose that the so called ecological regime [11] is a relevant tool for the optimization of the array of Fig. 2. The ecological function E was defined in Ref. [11] as

$$E = W - T_2\sigma, \tag{15}$$

where W is the work per unit time (power output), σ the entropy production rate and T_2 the temperature of the cold reservoir. The E function has for endoreversible heat engines the property that the power of the maximum E regime is around eighty percent that of the maximum power output regime, while the entropy production is reduced down to thirty percent [11].

By means of Eqs. (4) and (5) we have the total power output of the MV-regime

$$P(\theta_1, \theta_2) = W_1 + W_2 = \beta(2\theta_2 - \theta_1 - T_2) \left(\frac{T_1}{\theta_2} - 1\right) + \beta(T_1 + \theta_2 - 2\theta_1) \left(1 - \frac{T_2}{\theta_1}\right). \tag{16}$$

The global entropy production (working fluid plus surroundings) is given by (see Fig. 2),

$$\sigma = \frac{Q_f}{T_2} - \frac{Q_i}{T_1} = \frac{Q_{1f} + Q_{2f}}{T_2} - \frac{Q_{1i} + Q_{2i}}{T_1}, \tag{17}$$

where $Q_{f,i}$ are the heat fluxes between the working fluid and the heat reservoirs. In the obtaining of Eq. (17) we assume the validity of the endoreversibility hypothesis. Thus, by

using Newton's cooling law in Eq. (17) and Eqs. (16) and (15) we obtain an expression for the ecological function,

$$E(\theta_1, \theta_2) = \beta \left[(3T_1 + 9T_2) - \left(\frac{T_2}{T_1} + 1 \right) \theta_1 - 2\theta_2 - (T_1 + T_2) \frac{\theta_1}{\theta_2} - 2T_2 \frac{\theta_2}{\theta_1} - \frac{T_2(T_1 + T_2)}{\theta_2} - \frac{2T_1T_2}{\theta_1} \right], \quad (18)$$

thus, by means of an extremal condition, we obtain θ_{1E} and θ_{2E} which maximize Eq. (18):

$$\theta_{1E} = \left[\frac{2}{\frac{1}{T_1T_2^2} + \frac{1}{T_1^2T_2}} \right]^{1/3} \quad (19)$$

and

$$\theta_{2E} = \left[\frac{T_1^2T_2 + T_1T_2^2}{2} \right]^{1/3}. \quad (20)$$

Thus, the efficiency in the maximum E regime is

$$\eta_E = \frac{P(\theta_{1E}, \theta_{2E})}{Q_{1i} + Q_{2i}} = 1 - \left(\frac{T_2}{T_1} \right)^{1/3} \frac{-3 \left(\frac{T_2}{T_1} \right)^{2/3} + \left[2 + \left[\frac{1}{2} \frac{T_2}{T_1} \left[1 + \frac{T_2}{T_1} \right] \right]^{1/3} \left[\frac{1}{2} \left[1 + \frac{T_2}{T_1} \right] \right]^{1/3}}{3 - 2 \left[\frac{2}{\frac{T_1}{T_2} \left[1 + \frac{T_1}{T_2} \right]} \right]^{1/3} - \left[\frac{4}{\frac{T_1}{T_2} + \frac{T_2}{T_1} + 2} \right]^{1/3}} \quad (21)$$

η_E is also smaller than one, and depends on the factor $\left(\frac{T_2}{T_1} \right)^{1/3}$, as in the MV-case. Equation (21) is between η_C and η_{MV} [Eq. (12)], as can be seen in Fig. 3. The general form of the functions $E(\theta_1, \theta_2)$ and $P(\theta_1, \theta_2)$ are given by Eqs. (18) and (16), respectively. The function $\eta(\theta_1, \theta_2)$ is immediately obtained by means of

$$\eta(\theta_1, \theta_2) = \frac{P(\theta_1, \theta_2)}{Q_{1i} + Q_{2i}} = \frac{3T_1 + 3T_2 - \theta_1 - \theta_2 - T_1 \frac{\theta_1}{\theta_2} - T_2 \frac{\theta_2}{\theta_1} - \frac{T_1T_2}{\theta_1} - \frac{T_1T_2}{\theta_2}}{3T_1 - \theta_1 - T_1 \frac{\theta_1}{\theta_2} - \frac{T_1T_2}{\theta_2}}, \quad (22)$$

where $Q_{1i} + Q_{2i}$ are calculated as we said, by using the Newton's law of cooling, that is

$$Q_{1i} = \beta(T_1 - \theta_1), \quad (23)$$

$$Q_{2i} = \beta \frac{T_1}{\theta_2} (2\theta_2 - \theta_1 - T_2). \quad (24)$$

As it can be seen through Eqs. (16), (18) and (22), for T_1 and T_2 fixed, P , E and η only depend on θ_1 and θ_2 , that is, the temperatures that form a bypass between the two

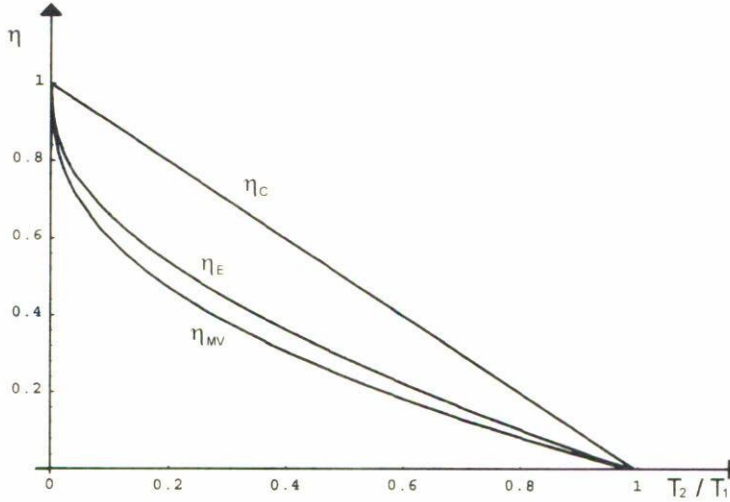


FIGURE 3. Comparison between η_C , η_{MV} and η_E .

CA-engines (see Fig. 2). Evidently, that bypass constitutes a coupling element between the two engines. If $\theta_1 = \theta_2$, then the two engines are uncoupled, since the heat flux is interrupted due to the Newton’s cooling law. Thus, we define a coupling parameter α , as

$$\alpha \equiv \frac{\theta_1}{\theta_2}. \tag{25}$$

When $\alpha = 1$, the two CA-engines of the MV array remain uncoupled. If we substitute the parameter α in Eqs. (16), (18) and (22), then we obtain

$$P(\alpha, \theta_1) = \beta \left[\left(3(T_1 + T_2) - \theta_1 - \frac{T_1 T_2}{\theta_1} \right) - \alpha(T_2 + \theta_1) - \alpha^{-1} \left(T_1 + \frac{T_1 T_2}{\theta_1} \right) \right], \tag{26}$$

$$E(\alpha, \theta_1) = \beta \left[\left(3T_1 + 9T_2 - \left(\frac{T_2}{T_1} + 1 \right) \theta_1 - \frac{2T_1 T_2}{\theta_1} \right) - 2\alpha(\theta_1 + T_2) - \alpha^{-1}(T_1 + T_2) \left(1 + \frac{T_2}{\theta_1} \right) \right] \tag{27}$$

and

$$\eta(\alpha, \theta_1) = \frac{\left[3(T_1 + T_2) - \theta_1 - \frac{T_1 T_2}{\theta_1} \right] - \alpha(T_2 + \theta_1) - \alpha^{-1} \left(T_1 + \frac{T_1 T_2}{\theta_1} \right)}{3T_1 - \theta_1 - \alpha^{-1} \left(T_1 + \frac{T_1 T_2}{\theta_1} \right)}. \tag{28}$$

A typical behavior of these functions is depicted in Figs. 4, 5 and 6, respectively, for arbitrary values of T_1 , T_2 , β and θ_2 . As it is observed in those graphs, in the three cases there exists a particular value of the coupling parameter α where P , E and η reach their maximum values, and these three values are different from one (the uncoupled case). Thus, when two CA-engines are coupled by a thermal resistance, as in Fig. 2, the global power

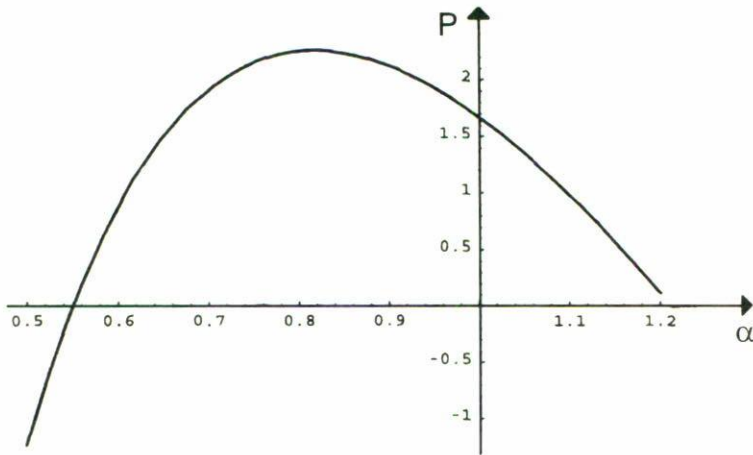


FIGURE 4. Power output versus coupling parameter curve for arbitrary values of T_1, T_2, β and θ_2 .

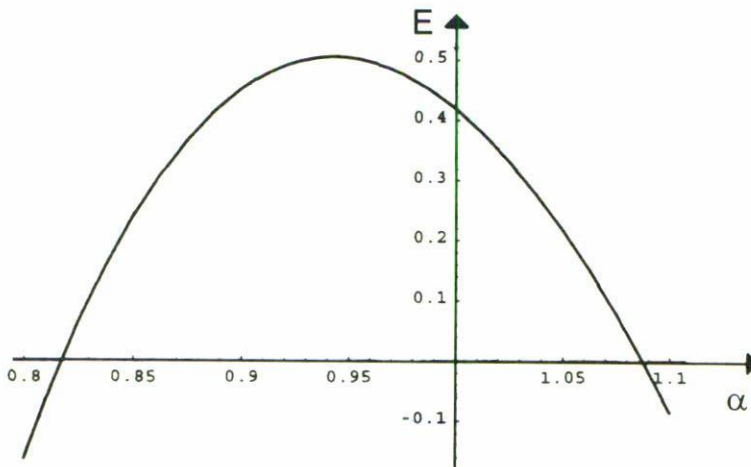


FIGURE 5. Ecological function versus coupling parameter curve for arbitrary values of T_1, T_2, β and θ_2 .

output and efficiency are evidently improved. That is, the coupling of two CA-engines is an advantageous mechanism and α results to be a driver parameter of the efficiency and the power output of the MV-array. This same behavior was observed for a wide data collection for T_1, T_2, β and θ_2 . A very interesting case is that of Figs. 7 and 8, (for $T_1 = 700$ K, $T_2 = 300$ K, $\theta_2 = 343.6$ K and $\beta = 1$ W/K), where is observed that the values of α_η which maximizes the efficiency is smaller than one and the α_P that maximizes the power output is greater than one. That is, an inversion in the heat flux through the bypass is necessary for the maximizations of η and P , respectively.

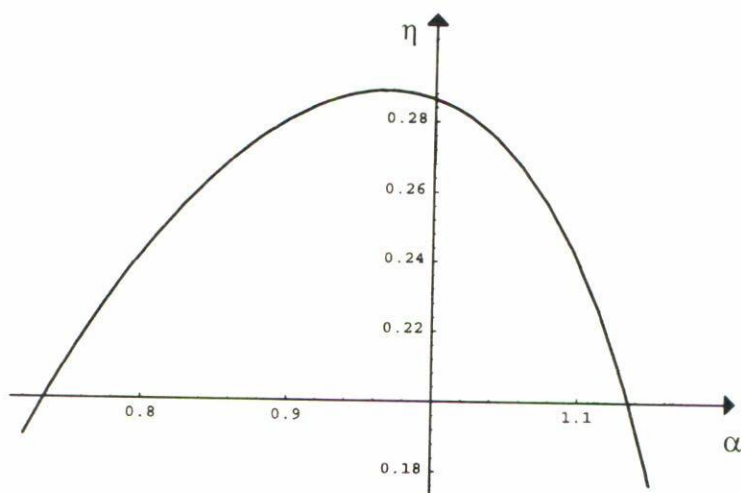


FIGURE 6. Efficiency versus coupling parameter curve for arbitrary values of T_1, T_2, β and θ_2 .

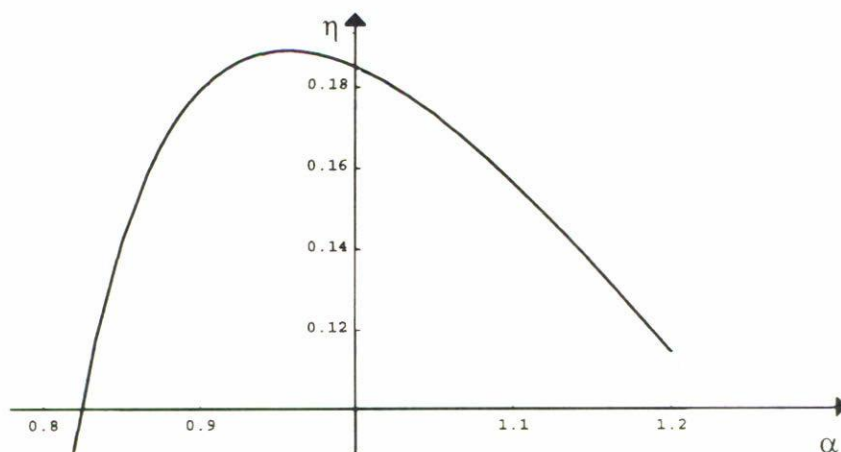


FIGURE 7. Efficiency versus α curve for $T_1 = 700$ K, $T_2 = 300$ K, $\theta_2 = 343.6$ K and $\beta = 1$ W/K.

4. CONCLUSIONS

As it has been discussed by several authors, the CA-formula for the efficiency of an endoreversible Carnot-like engine is not universal and the efficiency for maximum power output conditions strongly depends on the heat transfer law used for modeling the heat fluxes between heat reservoirs and working fluid. In fact, for CA-engines with a Newton's cooling law other expressions than the CA-formula can be obtained. In the array of CA-engines proposed by De Mey and De Vos a dependence on $(T_2/T_1)^{1/3}$ is found. In our work we show that a coupling parameter between the two CA-engines forming the MV-array can be defined. This parameter has relevance as a driving quantity of optimization criteria. We show that for $\alpha \neq 1$ ($\alpha = 1$ is the uncoupled case) the efficiency and the power output

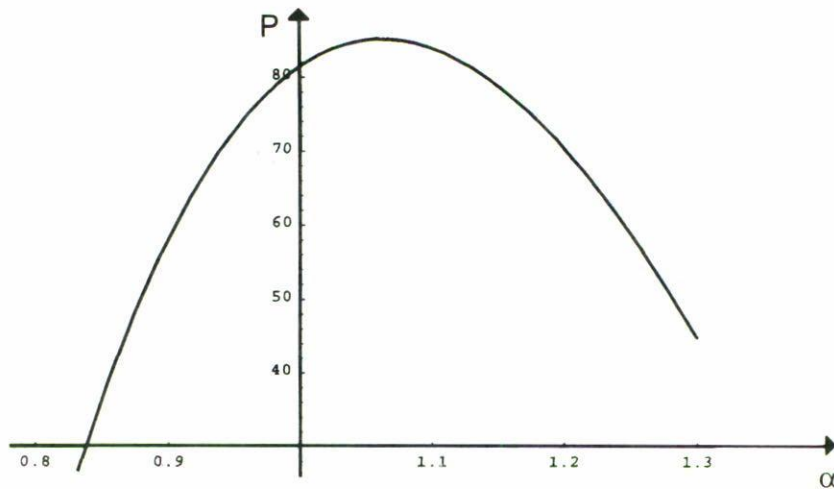


FIGURE 8. Power output versus α curve for $T_1 = 700$ K, $T_2 = 300$ K, $\theta_2 = 343.6$ K, and $\beta = 1$ W/K.

of the MV-array are remarkably improved. That is, in terms of P and η maximizations, the coupling situation is better than the uncoupled case.

We also find that in some cases the P and η maximizations demand an inversion of the heat flux through the bypass of the MV-array. In Ref. [3], De Vos treats chemical reactions as modeled by chemical engines and discuss the photosynthesis as two coupled engines: One photovoltaic engine and a chemical engine. We believe that our results may have implications for these models. However, a study of those problems will be published elsewhere.

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