Revista Mexicana de Física **42**, *No. 4 (1996) 663–674*

On the evaluation of the capacitance of bispherical capacitors

A. GÓNGORA-T. AND E. LEY-KOO
Instituto de Física, Universidad Nacional Autónoma de México
Apartado postal 20-364, 01000 México, D.F., México

Recibido el 6 de diciembre de 1994; aceptado el 20 de febrero de 1996

ABSTRACT. The electrostatic field and the capacitance of spherical capacitors with concentric electrodes is studied in the introductory physics courses. The study of the same problem when the electrodes are not concentric requires a higher level of electrical and mathematical knowledge. Maxwell in his Treatise solved the problem by the method of images and by expansions in spherical harmonics, but both solutions present slow convergence difficulties. In this paper the solution of the problem is constructed by using bispherical coordinates and showing that the corresponding expansions in bispherical harmonics have no convergence difficulties.

RESUMEN. El campo electrostático y la capacitancia de condensadores esféricos con electrodos concéntricos se estudia en los cursos introductorios de física. El estudio del mismo problema cuando los electrodos no son concéntricos requiere niveles más avanzados de conocimientos eléctricos y matemáticos. Maxwell en su tratado resolvió el problema por el método de imágenes y por medio de desarrollos en armónicos esféricos, pero ambas soluciones presentan dificultades de convergencia lenta. En este artículo la solución del problema se construye usando coordenadas biesféricas y mostrando que los desarrollos correspondientes en armónicos biesféricos no tienen dificultades de convergencia.

PACS: 41.10.Dq

1. INTRODUCTION

The electrostatic field and the capacitance of a charged conducting sphere is a standard topic in the beginning of the study of electricity [1–3]. The corresponding properties for a spherical capacitor with concentric electrodes follow immediately as an application of the previous results and the superposition principle [1–3]. The inquisitive readers may wonder what happens to the field and the capacitance when the spherical electrodes are off center. Their search for an answer in the above references and even in more advanced textbooks will very likely prove frustrating. In this paper we provide a guide to the few references on the problem and present our own quantitative solution.

Maxwell studied the problem of two nonintersecting spheres by the method of images and by expansions in spherical harmonics centered in each sphere, obtaining infinite series representations for the capacity coefficients [4]. However, both solutions present slow convergence difficulties as their use in current research on the electrostatic response of bidimensional arrangements of microspheres has shown [5]. Moon and Spencer [6] studied the problem of two equal spheres using bispherical coordinates, while Arfken [7] assigns the problem of a sphere and a plane as an exercise in the application of the same coordinates.

In this work the general problem of bispherical capacitors is formulated and solved. In Sect. 2 we introduce the bispherical coordinates as the natural coordinates to describe such capacitors and their electrostatic fields; the emphasis is in the geometry and it may be assimilated by students in the intermediate level. Section 3 gives the formal analytic treatment of the problem constructing the electrostatic potential function using the Green function technique [8], and obtaining from here the electric intensity field, the charge distribution and total charges on the electrodes, and finally the capacitance of the capacitors. Special attention is paid to the analysis of the series representing the capacitance for the different geometrical configurations, in order to exhibit that there are no convergence difficulties. The Green function and some relevant integrals are explicitly constructed in the Appendix. Advanced level knowledge of electrostatics and mathematics are required to appreciate the difficulties and fine points in this treatment. Section 4 contains a discussion of some details of didactic interest.

2. BISPHERICAL COORDINATES TO DESCRIBE BISPHERICAL CAPACITORS

The bispherical coordinates are related to the cartesian coordinates through the equations [8]

$$x = \frac{a \sin\xi \cos\varphi}{\cosh\eta - \cos\xi} \quad y = \frac{a \sin\xi \sin\varphi}{\cosh\eta - \cos\xi} \quad z = \frac{a \sinh\eta}{\cosh\eta - \cos\xi}. \tag{1}$$

The surfaces with fixed values of ξ correspond to surfaces of revolution around the z-axis with meridian cross sections that are circular arcs with a radius $a \csc\xi$ and centers on the xy-plane a distance $a \cot\xi$ from the z-axis; all these surfaces meet at $(x = 0, y = 0, \pm a)$, and include the external part of the z-axis $(x = 0, y = 0, |z| > a)$ corresponding to $\xi = 0$, the sphere of radius a centered at the origin corresponding to $\xi = \frac{\pi}{2}$, and the central part of the z-axis $(x = 0, y = 0, |z| < a)$ corresponding to $\xi = \pi$. The surfaces with fixed values of η correspond to spheres centered at $(x = 0, y = 0, z = a \coth\eta)$ and radius $a|\operatorname{csch}\eta|$; the limiting positions $\eta = \pm\infty$ correspond to the points $(x = 0, y = 0, z = \pm a)$, respectively, and $\eta = 0$ corresponds to the xy-plane. The surfaces with fixed values of φ correspond to the familiar meridian half-planes meeting at the z-axis. These coordinates are illustrated in Fig. 1.

Now we can point out how the bispherical coordinates can serve to describe the capacitors under consideration. Any pair of spherical electrodes can be defined through their respective spherical coordinates η_1 and η_2. Figure 2 illustrates three different possible situations: a) one sphere inside another $\eta_1 > \eta_2 > 0$, b) sphere and plane $\eta_1 > 0$ and $\eta_2 = 0$, which is explicitly studied in [7]; and c) spheres outside each other $\eta_1 > 0$ and $\eta_2 < 0$, where the situation of two equal spheres corresponds to $\eta_2 = -\eta_1$ [6]. The intermediate level student should not have any difficulties in determining the values of a, η_1 and η_2 for two spheres defined by their radii and the distance between their centers as in Ref. [4].

The orthogonality of the bispherical coordinates can be established by evaluating the infinitesimal displacement from Eq. (1) and identifying the respective scale factors and

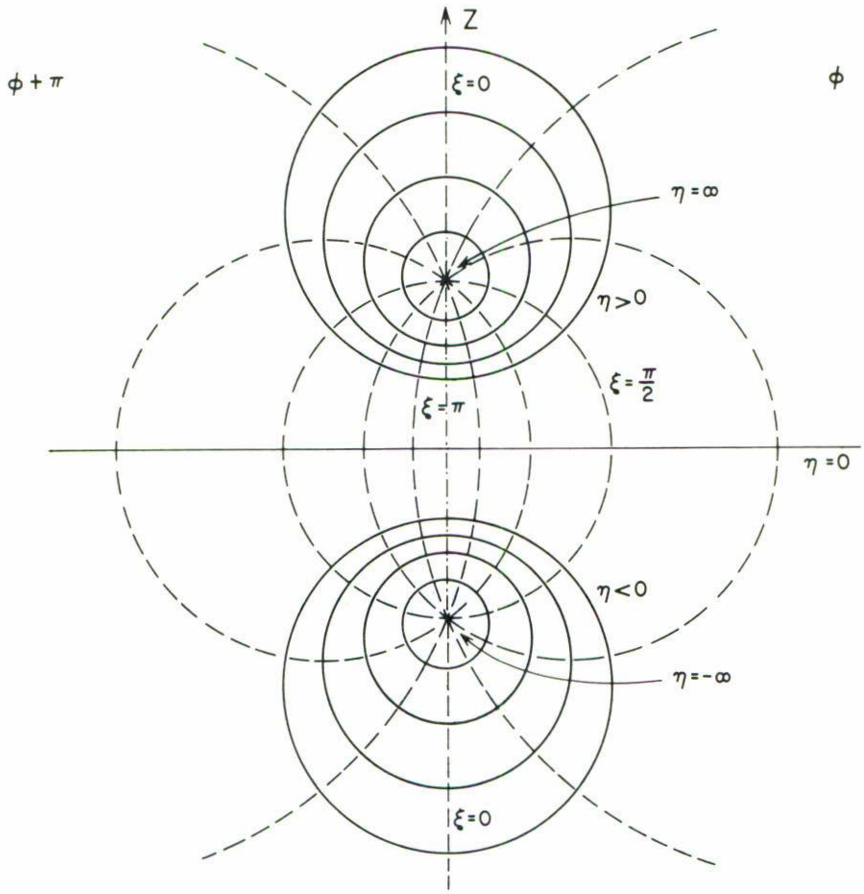

FIGURE 1. Meridian cross section of bispherical coordinates (ξ, η, φ): ξ = constant are circular arcs meeting at $(x = 0, y = 0, z = \pm a)$, η = constant are circles with centers on the z–axis, and φ = constant are half-planes meeting at the z–axis.

unit vectors:

$$d\vec{r} = \hat{\imath}\, dx + \hat{\jmath} dy + \hat{k}\, dz = \hat{\xi} h_\xi\, d\xi + \hat{\eta} h_\eta\, d\eta + \hat{\varphi} h_\varphi\, d\varphi, \tag{2}$$

$$h_\xi = h_\eta = \frac{a}{\cosh\eta - \cos\xi}, \qquad h_\varphi = \frac{a\sin\xi}{\cosh\eta - \cos\xi}, \tag{3}$$

$$\begin{aligned}\hat{\xi} &= \frac{(\cosh\eta\cos\xi - 1)(\hat{\imath}\cos\varphi + \hat{\jmath}\sin\varphi) - \hat{k}\sinh\eta\sin\xi}{\cosh\eta - \cos\xi},\\ \hat{\eta} &= \frac{-\sinh\eta\sin\xi(\hat{\imath}\cos\varphi + \hat{\jmath}\sin\varphi) - \hat{k}(\cosh\eta\cos\xi - 1)}{\cosh\eta - \cos\xi},\\ \hat{\varphi} &= -\hat{\imath}\,\mathrm{sen}\,\varphi + \hat{\jmath}\cos\varphi.\end{aligned} \tag{4}$$

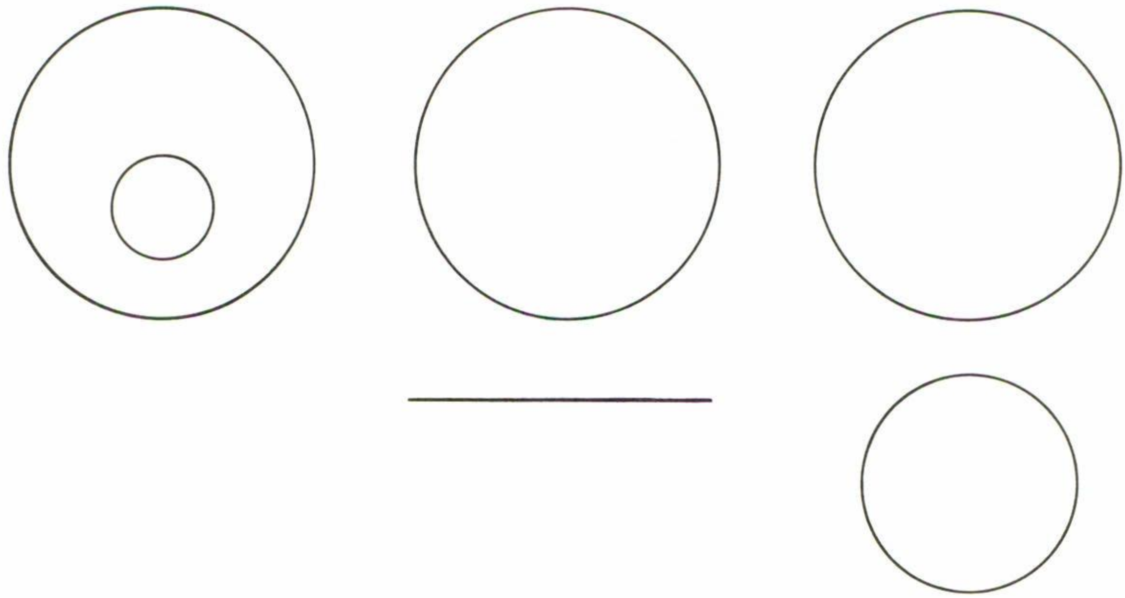

FIGURE 2. Meridian cross sections of bispherical capacitors: a) one sphere inside another, b) sphere and plane and c) spheres outside each other.

3. ANALYTICAL DESCRIPTION OF ELECTROSTATIC FIELD, SOURCES AND CAPACITANCE

The electrostatic potential function must satisfy the Laplace equation

$$\left\{\frac{(\cosh\eta-\cos\xi)^3}{a^2\sin\xi}\left[\frac{\partial}{\partial\xi}\frac{\sin\xi}{\cosh\eta-\cos\xi}\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\frac{\sin\xi}{\cosh\eta-\cos\xi}\frac{\partial}{\partial\eta}\right]\right.$$
$$\left.+\frac{(\cosh\eta-\cos\xi)^2}{a^2\sin^2\xi}\frac{\partial^2}{\partial\varphi^2}\right\}\phi(\xi,\eta,\varphi)=0, \tag{5}$$

and also the boundary conditions at the respective electrodes

$$\phi(\xi,\eta=\eta_1,\varphi)=V_1, \tag{6}$$
$$\phi(\xi,\eta=\eta_2,\varphi)=V_2=0. \tag{7}$$

Equation (5) is R-separable [9], having general solutions of the form

$$\phi(\xi,\eta,\varphi)=(\cosh\eta-\cos\xi)^{1/2}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell}\left[A_{\ell m}P_\ell^m(\cos\xi)+B_{\ell m}Q_\ell^m(\cos\xi)\right]$$
$$\times\left[C_\ell\sinh(\ell+\tfrac{1}{2})\eta+D_\ell\cosh(\ell+\tfrac{1}{2})\eta\right]\left[E_m\sin m\varphi+F_m\cos m\varphi\right]. \tag{8}$$

The presence of the square-root of the binomial factor is the sign of the R separability, which makes the fulfillment of the boundary condition of Eq. (6) a nontrivial matter.

For this reason we resort to the Green function technique to construct the electrostatic potential function [8]:

$$\nabla^2 G_{\mathrm{D}}(\vec{r},\vec{r}\,') = -4\pi\delta(\vec{r}-\vec{r}\,') \tag{9}$$

with Dirichlet boundary conditions

$$G_{\mathrm{D}}(\vec{r},\vec{r}\,')\Big|_S = 0, \tag{10}$$

$$\phi(\vec{r}) = -\frac{1}{4\pi}\oint_S da'\,\phi(\vec{r}\,')\frac{\partial G_{\mathrm{D}}(\vec{r},\vec{r}\,')}{\partial n'}, \tag{11}$$

where n' is the displacement perpendicular to the boundary surface S. The Green function for the bispherical capacitors is constructed in the Appendix. Its normal derivative at the sphere η_1, Eq. (35), is substituted in Eq. (11) to obtain

$$\begin{aligned}\phi(\xi,\eta,\varphi) &= \frac{V_1}{a}(\cosh\eta-\cos\xi)^{1/2}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}Y_{\ell m}(\xi,\varphi)\,\frac{\sinh[(\ell+\frac{1}{2})(\eta-\eta_2)]}{\sinh[(\ell+\frac{1}{2})(\eta_1-\eta_2)]}\\ &\quad\int_0^{\pi}h_{\xi'}\,d\xi'\int_0^{2\pi}h_{\varphi'}\,d\varphi'\frac{1}{h_{\eta'}}Y^*_{\ell m}(\xi',\varphi')\,(\cosh\eta_1-\cos\xi')^{1/2}\\ &= V_1(\cosh\eta-\cos\xi)^{1/2}\sum_{\ell=0}^{\infty}\frac{\sinh[(\ell+\frac{1}{2})(\eta-\eta_2)]}{\sinh[(\ell+\frac{1}{2})(\eta_1-\eta_2)]}\\ &\quad\times N_\ell P_\ell(\cos\xi)\int_0^{\pi}\frac{d\xi'\,\sin\xi' N_\ell P_\ell(\cos\xi')}{(\cosh\eta_1-\cos\xi')^{1/2}},\end{aligned} \tag{12}$$

where the explicit forms of the scale factors, Eqs. (3), have been used. Here the integration over the azimuthal angle selects the terms with $m=0$ only, reflecting the symmetry of the system under rotations around the z–axis, and N_ℓ is the normalization factor of the Legendre polynomials. It is recognized that Eq. (12) is a special case of Eq. (8), satisfying obviously the boundary condition of Eq. (7). The integral over ξ' is represented as $C_\ell(\cosh\eta_1)$ and evaluated in the Appendix. The fulfillment of the boundary condition of Eq. (6) is not so immediately obvious, but it can be verified as follows:

$$\begin{aligned}\phi(\xi,\eta=\eta_1,\varphi) &= V_1(\cosh\eta_1-\cos\xi)^{1/2}\int_0^{\pi}\frac{d\xi'\sinh\xi'}{(\cosh\eta_1-\cos\xi')^{1/2}}\sum_{\ell=0}^{\infty}N_\ell P_\ell(\cos\xi)N_\ell P_\ell(\cos\xi')\\ &= V_1(\cosh\eta_1-\cos\xi)^{1/2}\int_0^{\pi}\frac{d\xi'\sinh\xi'}{(\cosh\eta_1-\cos\xi')^{1/2}}\delta(\cos\xi'-\cos\xi) = V_1.\end{aligned} \tag{13}$$

The summation over ℓ can be identified as the representation of the Dirac-delta function, then the integration can be done immediately, and the cancellation of the square root of the binomial leads to the potential of Eq. (6). The same result can be established by using Eqs. (36) and (37) directly.

The electric intensity field is obtained by taking the negative gradient of Eq. (12):

$$\begin{aligned}\vec{E}(\xi,\eta,\varphi) &= -\left[\hat{\xi}\frac{\partial}{h_\xi\partial_\xi}+\hat{\eta}\frac{\partial}{h_\eta\partial_\eta}+\hat{\varphi}\frac{\partial}{h_\varphi\partial_\varphi}\right]\phi(\xi,\eta,\varphi)\\ &= -\frac{V_1}{a}(\cosh\eta-\cos\xi)^{3/2}\sum_{\ell=0}^{\infty}\frac{C_\ell(\cosh\eta_1)}{\sinh[(\ell+\frac{1}{2})(\eta_1-\eta_2)]}\\ &\quad\left\{\hat{\xi}\left[N_\ell\frac{dP_\ell(\cos\xi)}{d\xi}+\frac{\sin\xi}{2(\cosh\eta-\cos\xi)}\cdot N_\ell P_\ell(\cos\xi)\right]\sinh\left[(\ell+\tfrac{1}{2})(\eta-\eta_2)\right]\right.\\ &\quad+\hat{\eta}\left[(\ell+\tfrac{1}{2})\cosh\left[(\ell+\tfrac{1}{2})(\eta-\eta_2)\right]+\frac{\sinh\eta\sinh[(\ell+\frac{1}{2})(\eta-\eta_2)]}{2(\cosh\eta-\cos\xi)}\right]\\ &\quad\left.\times\ N_\ell P_\ell(\cos\xi)\right\}. \end{aligned} \tag{14}$$

It is obvious that the direction of the electric field lines at the sphere η_2 is that of vector $\hat{\eta}$ perpendicular to the electrode. It is not so obvious, but the same holds at the sphere η_1 as the following examination of the $\hat{\xi}$ component of the electric intensity shows:

$$\begin{aligned}\hat{\xi}\cdot\vec{E}(\xi,\eta=\eta_1,\varphi) &= -\frac{V_1}{a}(\cosh\eta_1-\cos\xi)^{3/2}\sum_{\ell=0}^{\infty}C_\ell(\cosh\eta_1)\\ &\quad\left[N_\ell\frac{dP_\ell(\cosh\xi)}{d\xi}+\frac{\sin\xi}{2(\cosh\eta_1-\cos\xi)}N_\ell P_\ell(\cos\xi)\right].\end{aligned} \tag{15}$$

According to Eq. (37) the sum in the second term in this expression is

$$\sin\xi\sum_{\ell=0}^{\infty}\frac{C_\ell(\cos\eta_1)N_\ell P_\ell(\cos\xi)}{2(\cosh\eta_1-\cos\xi)}=\frac{\sin\xi}{2(\cosh\eta_1-\cos\xi)^{3/2}}, \tag{16}$$

while the sum in the first term is

$$\sum_{\ell=0}^{\infty}C_\ell(\cosh\eta_1)N_\ell\frac{dP_\ell(\cos\xi)}{d\xi}=\frac{d}{d\xi}\frac{1}{(\cosh\eta_1-\cos\xi)^{1/2}}=-\frac{\sin\xi}{2(\cosh\eta_1-\cos\xi)^{3/2}}, \tag{17}$$

and the result is the mutual cancellation of both sums. Therefore the electric intensity at the sphere η_1 has only its $\hat{\eta}$ component perpendicular to the electrode.

The electric charge distribution on the electrodes is evaluated by using Gauss' law applied to Eq. (14) at the respective spheres:

$$\begin{aligned}\sigma(\xi,\eta=\eta_1,\varphi) &= -\frac{\hat{\eta}\cdot\vec{E}(\xi,\eta=\eta_1,\varphi)}{4\pi}\\ &= \frac{V_1}{4\pi a}(\cosh\eta_1-\cos\xi)^{3/2}\sum_{\ell=0}^{\infty}\Bigg[(\ell+\tfrac{1}{2})\cosh[(\ell+\tfrac{1}{2})(\eta_1-\eta_2)]\\ &\quad+\frac{\sinh\eta_1\sinh[(\ell+\frac{1}{2})(\eta_1-\eta_2)]}{2(\cosh\eta_1-\cos\xi)}\Bigg]\frac{C_\ell(\cosh\eta_1)P_\ell(\cos\xi)N_\ell}{\sinh[(\ell+\frac{1}{2})(\eta_1-\eta_2)]}, \qquad (18)\\ \sigma(\xi,\eta=\eta_2,\varphi) &= \frac{\hat{\eta}\cdot\vec{E}(\xi,\eta=\eta_2,\varphi)}{4\pi}\\ &= -\frac{V_1}{4\pi a}(\cosh\eta_2-\cos\xi)^{3/2}\sum_{\ell=0}^{\infty}\frac{(\ell+\frac{1}{2})C_\ell(\cosh\eta_1)}{\sinh[(\ell+\frac{1}{2})(\eta_1-\eta_2)]}N_\ell P_\ell(\cos\xi). \qquad (19)\end{aligned}$$

The total charges on the electrodes are obtained by integrating Eqs. (18)–(19) over the respective spheres:

$$\begin{aligned}Q_2 &= \int_0^\pi\int_0^{2\pi}\sigma(\xi,\eta,\varphi)\,h_\xi\,d\xi\,h_\varphi\,d\varphi\Big|_{\eta=\eta_2}\\ &= -\frac{V_1 2\pi a^2}{4\pi a}\sum_{\ell=0}^{\infty}\frac{(\ell+\frac{1}{2})C_\ell(\cosh\eta_1)}{\sinh[(\ell+\frac{1}{2})(\eta_1-\eta_2)]}\int_0^\pi\frac{d\xi\sin\xi N_\ell P_\ell(\cos\xi)}{(\cosh\eta_2-\cos\xi)^{1/2}},\\ &= -\frac{V_1 a}{2}\sum_{\ell=0}^{\infty}\frac{(\ell+\frac{1}{2})C_\ell(\cosh\eta_1)C_\ell(\cosh\eta_2)}{\sinh[(\ell+\frac{1}{2})(\eta_1-\eta_2)]} \qquad (20)\\ Q_1 &= \int_0^\pi\int_0^{2\pi}\sigma(\xi,\eta,\varphi)\,h_\xi\,h_\varphi\,d\xi\,d\varphi\Big|_{\eta=\eta_1}\\ &= \frac{V_1 a}{2}\sum_{\ell=0}^{\infty}\frac{C_\ell(\cosh\eta_1)}{\sinh[(\ell+\frac{1}{2})(\eta_1-\eta_2)]}\\ &\quad\Bigg\{(\ell+\tfrac{1}{2})\cosh[(\ell+\tfrac{1}{2})(\eta_1-\eta_2)]\int_0^\pi\frac{d\xi\sin\xi N_\ell P_\ell(\cos\xi)}{(\cosh\eta_1-\cos\xi)^{1/2}}\\ &\quad+\sinh[(\ell+\tfrac{1}{2})(\eta_1-\eta_2)]\int_0^\pi\frac{d\xi\sinh\xi N_\ell P_\ell(\cos\xi)\sinh\eta_1}{2(\cosh\eta_1-\cos\xi)^{1/2}}\Bigg\}\\ &= \frac{V_1 a}{2}\sum_{\ell=0}^{\infty}\frac{C_\ell(\cosh\eta_1)}{\sinh[(\ell+\frac{1}{2})(\eta_1-\eta_2)]}\Bigg\{(\ell+\tfrac{1}{2})\cosh[(\ell+\tfrac{1}{2})(\eta_1-\eta_2)]C_\ell(\cosh\eta_1)\\ &\quad-\sinh[(\ell+\tfrac{1}{2})(\eta_1-\eta_2)]\frac{dC_\ell(\cosh\eta_1)}{d\eta_1}\Bigg\}. \qquad (21)\end{aligned}$$

In the last line the integrals over ξ are identified with $C_\ell(\cosh\eta_1)$ and its derivative by using Eq. (36). Physically the magnitudes of the charges in Eqs. (20) and (21) are expected to be equal. The advanced level students may prove that the expression inside the curly brackets is in fact $C_\ell(\cosh\eta_2)$.

The capacitance of the bispherical capacitor follows from either Eqs. (20), (21),

$$C = \frac{a}{2}\sum_{\ell=0}^{\infty}(\ell+\tfrac{1}{2})C_\ell(\cosh\eta_1)C_\ell(\cosh\eta_2)\,\mathrm{csch}[(\ell+\tfrac{1}{2})(\eta_1-\eta_2)]. \tag{22}$$

It is instructive to analyze the convergence of the series in Eq. (22) for some particular cases.

a) If one of the spheres has a very small radius, say $\eta_1 \to \infty$, Eq. (38) indicates that the summation in Eq. (22) contains only the $\ell = 0$ term, and the capacitance is reduced to

$$C = \frac{a}{2}e^{-\frac{\eta_1}{2}}C_0(\cosh\eta_2)\,\mathrm{csch}\,\frac{\eta_1-\eta_2}{2}. \tag{23}$$

b) If the other sphere also has a small radius and encloses the first one $\eta_2 \to \infty$, with $\eta_2 \leq \eta_1$

$$C = \frac{2ae^{-\eta_1/2}\,e^{-\eta_2/2}}{e^{\frac{\eta_1-\eta_2}{2}} - e^{-\frac{\eta_1-\eta_2}{2}}} = \frac{2a}{e^{\eta_1}-e^{\eta_2}} \approx \frac{1}{\frac{1}{a\,\mathrm{csch}\,\eta_1} - \frac{1}{a\,\mathrm{csch}\,\eta_2}} = \frac{1}{\frac{1}{r_1}-\frac{1}{r_2}}, \tag{24}$$

reproducing the well-known result of concentric spheres [1–3].

c) If one of the spheres has a very large radius say $\eta_2 \to 0_+$, Eq. (39) indicates that the summation in Eq. (22) will have to include many terms,

$$C = \frac{a}{2}\sum_{\ell=0}^{\infty}\sqrt{2\ell+1}\,C_\ell(\cosh\eta_1)\,\mathrm{csch}\left[(\ell+\tfrac{1}{2})(\eta_1-\eta_2)\right], \tag{25}$$

but the series will converge fairly rapidly because the factor $C_\ell(\cosh\eta_1)$ has a value smaller than that of Eq. (39), and the hyperbolic cosecant factor will decrease exponentially as ℓ increases. Obviously, the case of Fig. 2b with $\eta_2 = 0$ is a particular case of Eq. (25).

d) The parallel plate capacitor corresponds to the limit $\eta_1 = -\eta_2 \to 0$,

$$C = a\sum_{\ell=0}^{\infty}\mathrm{csch}[(2\ell+1)\eta_1], \tag{26}$$

with an infinite capacity due to the infinite area.

e) The case of spheres external to each other does not present any convergence problem because the hyperbolic cosecant factor in Eq. (22) will exponentially decrease with ℓ more rapidly than in Eq. (25).

4. DISCUSSION

The electrostatic field for bispherical capacitors has been described in bispherical coordinates, through its electrostatic potential function, Eqs. (12) and (36), and its electric intensity field, Eq. (14). The charge distributions and total charges on the electrodes, Eqs. (18)–(19), and (20)–(21), respectively, and the capacitance, Eq. (22), were also obtained. All these quantities are given as superpositions of an infinite number of bispherical harmonic contributions. The reason behind this infinite number is the R-separability of the Laplace equation, Eqs. (5) and (8). The convergence of the series for the capacitance, Eq. (22), was also explicitly analyzed and its numerical implementation for specific spheres should not meet with any difficulties. This can be contrasted with the difficulties of Maxwell's solutions mentioned in the Introduction [4, 5]. Some readers may recognize the similarity and differences of this problem and those of the toroidal and spherical-cap electrode capacitors [10, 11]; it is very instructive to read these papers in succession in order to appreciate their relationships and peculiarities. From the didactic point of view the suggested order for senior undergraduate or graduate students is to start with the present paper and follow with the other two, on account of the closeness in form of the bispherical harmonics to the familiar spherical harmonics, and the increasing remoteness of the toroidal and spherical cap harmonics.

APPENDIX

The construction of the Green function requires the solution of Eq. (9) in bispherical coordinates, for which the Laplacian has the explicit form of Eq. (5), and the Dirac-delta function is written as

$$\delta(\vec{r}-\vec{r}\,')=\frac{\delta(\xi-\xi')\,\delta(\eta-\eta')\,\delta(\varphi-\varphi')}{h_\xi\,h_\eta\,h_\varphi}=\frac{\delta(\eta-\eta')}{h_\xi\,h_\eta\,h_\varphi}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}Y^{*}_{\ell m}(\xi',\varphi')Y_{\ell m}(\xi,\varphi). \quad (27)$$

Here we use the completeness of the bi-spherical harmonics in the angles (ξ,φ) to represent the corresponding Dirac-delta functions.

The Dirichlet boundary condition on the Green function, Eq. (10), takes the explicit form

$$G_{\rm D}(\xi,\eta=\eta_1,\varphi;\xi',\eta',\varphi')=0,\quad G_{\rm D}(\xi,\eta=\eta_2,\varphi;\xi',\eta',\varphi')=0. \quad (28)$$

The combined analysis of Eqs. (9) and (5) with the explicit forms of Eqs. (8) and (27) suggests the harmonic expansion of the Green function,

$$\begin{aligned}G_{\rm D}(\xi,\eta,\varphi;\xi',\eta',\varphi') &= (\cosh\eta'-\cos\xi')^{1/2}(\cosh\eta-\cos\xi)^{1/2}\\ &\times\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}Y^{*}_{\ell m}(\xi',\varphi')Y_{\ell m}(\xi,\varphi)g_\ell(\eta,\eta'), \quad (29)\end{aligned}$$

where we have also incorporated the symmetry under the exchange of the field-point and the source-point, and

$$g_\ell(\eta,\eta') = A_\ell \sinh[(\ell+\tfrac{1}{2})(\eta_1-\eta_>)]\sinh[(\ell+\tfrac{1}{2})(\eta_<-\eta_2)], \tag{30}$$

guaranteeing that the boundary conditions of Eq. (28) are satisfied, and that the Green function is continuous at $\eta=\eta'$. There remains to determine the coefficients A_ℓ. This is accomplished by substitution of Eqs. (27) and (29)–(30) in the explicit form of Eq. (9), making use of the R-separability of the Laplace equation, and of the linear independence of the bispherical harmonics, to obtain

$$\left[\frac{d^2}{d\eta^2}-(\ell+\tfrac{1}{2})^2\right]g_\ell(\eta,\eta') = -\frac{4\pi}{a}\delta(\eta-\eta'). \tag{31}$$

The integration of this equation leads in turn to the discontinuity in the η-derivatives of the Green function

$$\frac{d}{d\eta}g_\ell(\eta,\eta')\Big|_{\eta=\eta'_+} - \frac{d}{d\eta}g_\ell(\eta,\eta')\Big|_{\eta=\eta'_-} = -\frac{4\pi}{a}. \tag{32}$$

Use of the explicit form of Eq. (30) gives the value of the coefficients,

$$A_\ell = \frac{4\pi}{a(\ell+\frac{1}{2})\sinh[(\ell+\frac{1}{2})(\eta_1-\eta_2)]}. \tag{33}$$

The explicit form of the Green function, Eqs. (29)–(30), becomes

$$\begin{aligned} G_{\mathrm{D}}(\xi,\eta,\varphi;\xi',\eta',\varphi') &= \frac{4\pi}{a}(\cosh\eta'-\cos\xi')^{1/2}(\cosh\eta-\cos\xi)^{1/2} \\ &\times \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\xi',\varphi')Y_{\ell m}(\xi,\varphi) \\ &\times \frac{\sinh[(\ell+\frac{1}{2})(\eta_1-\eta_>)]\sinh[(\ell+\frac{1}{2})(\eta_<-\eta_2)]}{(\ell+\frac{1}{2})\sinh[(\ell+\frac{1}{2})(\eta_1-\eta_2)]}. \end{aligned} \tag{34}$$

The construction of the electrostatic potential function requires the normal derivative at the spherical electrode η_1:

$$\begin{aligned} \frac{1}{h_{\eta'}}\frac{\partial G_{\mathrm{D}}}{\partial\eta'}\Big|_{\eta'=\eta_1} &= \frac{1}{h_{\eta'}}\frac{4\pi}{a}(\cosh\eta-\cos\xi)^{1/2} \\ &\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\xi',\varphi')Y_{\ell m}(\xi,\varphi)\frac{\sinh[(\ell+\frac{1}{2})(\eta-\eta_2)]}{\sinh[(\ell+\frac{1}{2})(\eta_1-\eta_2)]} \end{aligned}$$

$$\left\{-(\cosh\eta' - \cos\xi')^{1/2}\cosh[(\ell+\frac{1}{2})(\eta_1-\eta')]\right.$$
$$\left.+\ \frac{1}{2(\ell+\frac{1}{2})}(\cosh\eta' - \cos\xi')^{-1/2}\sinh\eta'\sinh[(\ell+\tfrac{1}{2})(\eta_1-\eta')]\right\}\Bigg|_{\eta'=\eta_1}$$
$$= -\frac{1}{h_{\eta'}}\frac{4\pi}{a}(\cosh\eta - \cos\xi)^{1/2}$$
$$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\xi',\varphi')Y_{\ell m}(\xi,\varphi)\frac{\sinh[(\ell+\frac{1}{2})(\eta-\eta_2)]}{\sinh[(\ell+\frac{1}{2})(\eta_1-\eta_2)]}$$
$$\times(\cosh\eta_1 - \cos\xi')^{1/2}. \tag{35}$$

The integral appearing in Eq. (12),

$$\int_0^\pi \frac{d\xi'\,\sin\xi' N_\ell P_\ell(\cosh\xi')}{(\cosh\eta_1 - \cos\xi')^{1/2}} = C_\ell(\cosh\eta_1)$$
$$= \frac{1}{2^\ell N_\ell}(\operatorname{sech}\eta_1)^{\frac{1}{2}+\ell}\,{}_2F_1\left(\tfrac{\ell}{2}+\tfrac{1}{4},\tfrac{\ell}{2}+\tfrac{3}{4};\ell+\tfrac{3}{2};\operatorname{sech}^2\eta_1\right), \tag{36}$$

can be interpreted as the coefficients in the Legendre polynomial expansion of the inverse of the square root of the binomial,

$$\frac{1}{(\cosh\eta_1 - \cos\xi')^{1/2}} = \sum_{\ell=0}^{\infty} C_\ell(\cosh\eta_1)N_\ell P_\ell(\cos\xi')$$
$$= \sum_{\ell=0}^{\infty} N_\ell P_\ell(\cos\xi)\frac{1}{2^\ell N_\ell}(\operatorname{sech}\eta_1)^{\frac{1}{2}+\ell}\,{}_2F_1\left(\tfrac{\ell}{2}+\tfrac{1}{4},\tfrac{\ell}{2}+\tfrac{3}{4};\ell+\tfrac{3}{2};\operatorname{sech}^2\eta_1\right). \tag{37}$$

The explicit form of this expansion can be obtained by using successively the binomial expansion, expressing the powers of $\cos\xi'$ as a linear combination of Legendre polynomials, exchanging the order of the summations and identifying one of them as the hypergeometric function ${}_2F_1$. Obviously, the explicit form of the integral in Eq. (36) is written by reading it off from Eq. (37).

The numerical values of the $C_\ell(\cosh\eta)$ functions vary monotonically between

$$C_\ell(\cosh\eta)\Big|_{\eta\to\pm\infty} = \frac{1}{2^\ell N_\ell}(\operatorname{sech}\eta)^{\ell+\frac{1}{2}} \to \delta_{\ell 0}\sqrt{2}\,(\operatorname{sech}\eta)^{1/2} = 2\delta_{\ell 0}e^{-|\eta|/2} \to 0 \tag{38}$$

and

$$C_\ell(\cosh\eta)\Big|_{\eta=0} = \frac{1}{2^\ell N_\ell}\,{}_2F_1\left(\tfrac{\ell}{2}+\tfrac{1}{4},\tfrac{\ell}{2}+\tfrac{3}{4};\ell+\tfrac{3}{2};1\right) = \frac{2}{\sqrt{2\ell+1}} \tag{39}$$

when the radius goes from zero to infinity.

ACKNOWLEDGEMENTS

The writing of this paper had been contemplated for the future by the authors. However, the presentation of the work on the spherical-cap-electrode capacitors [11] at the Canadian-American-Mexican Physics Meeting, motivated the writing at the present time. We thank Dr. Jorge José of Northeastern University for letting us know about the current interest in the problem and for Refs. [4, 6].

REFERENCES

1. D. Halliday and R. Resnick, *Physics for Students of Science and Engineering*, Wiley, New York (1960) Chap. 30.
2. M. Alonso and E.J. Finn, *Fundamental Physics*, Vol. II, Fields and Waves, Addison-Wesley, Reading, Mass. (1967) chap. 16.
3. E.M. Purcell, *Electricity and Magnetism*, McGraw-Hill, New York (1963).
4. J.C. Maxwell, *A Treatise on Electricity and Magnetism*, Third Edition, Oxford (1892), p. 266-273 and 224-231.
5. Jorge V. José, private communication (1994).
6. P. Moon and D.E. Spencer, *Field Theory for Engineers*, Van Nostrand, Princeton (1960), p. 376-384.
7. G. Arfken, *Mathematical Methods for Physicists*, Second Edition, Academic Press, New York (1970) Chap. 2.
8. J.D. Jackson, *Classical Electrodynamics*, Second Edition, Wiley, New York (1975) Chap. 1.
9. W. Miller Jr., *Symmetry and Separation of Variables*, Addison-Wesley, Reading, Mass (1977), Chap. 3.
10. A. Góngora T. y E. Ley-Koo, *Rev. Mex. Fís.* **40** (1994) 814.
11. A. Góngora T. y E. Ley-Koo, *Supl. Bol. Soc. Mex. Fís.* **8** (1994) 152.