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Contributions to the optical aberration coefficients of high order I: using Aldis Theorem and the Method of Andersen

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ABSTRACT. T.B. Andersen proposed an alternative method for the calculation of high order aberration coefficients for an axially symmetrical optical system; however the method does not give information of the contributions of each surface to those coefficients. The purpose of this work is to present an alternative method for the calculation of these contributions based on the method of Andersen, where the theorem of Aldis is used.

RESUMEN. T.B. Andersen propuso un método alternativo para el cálculo de coeficientes de aberración de alto orden para un sistema óptico axialmente simétrico; sin embargo el método no proporciona información acerca de la contribución de cada superficie a esos coeficientes. El propósito de este trabajo, es presentar un alternativa para el cálculo de las contribuciones basados en el método de Andersen, para lograr este propósito, se usa el teorema de Aldis.

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1. INTRODUCTION

The main purpose of this paper is to find a set of equations for the contributions from each surface of an optical system to its total transverse aberration. The achieve this goal the method developed by Andersen [1] is used, employing the so called theorem of Aldis [2] presented by Cox [2], in his book on optical design. Another aim of the formulation here established is to obtain more comprehensive equations than those obtained by Cox [2] and Buchdahl [3], for finding the aforementioned contributions of each optical surface. It is important to state that the method of Andersen only permits knowledge of the total amount of transverse aberration in the optical system, whereas in this paper each surface contributions is obtained.

In a comparative study between the works of Buchdahl, Cox, and Andersen, it was found that the total amount of transverse aberration in a triplet lens was the same for the three cases. Considering that Cox's results were obtained by using the Aldis theorem to find the contributions of each optical surface to the total transverse aberration of a system, a decision was made to use the method of Andersen together with the same theorem of Aldis, in order to find the necessary equations for the surface contributions.

2. ANDERSEN'S METHOD

Andersen developed a method based on ray tracing, that allows the computation of polynomial functions from knowledge of the ray coordinate intersections (x, y), and the direction tangents (ξ, η) of any skew ray with respect to the image plane. While this method may be applied to both symmetric and non-symmetric systems, our present study is valid only for the former. In the equations derived in this paper, the aberration coefficients are not obtained explicitly, but instead a series of quantities are derived as functions of several variables that can be represented in a computer program as arrays, having the coefficients of a power series of those variables up to a certain order. The arithmetic operations between the quantities just mentioned can be performed by means of the subroutines contained in Andersen's paper [1].

2.1. Notation and conventions

As has already been mentioned, the optical system is considered axially symmetric, with a Cartesian coordinate system whose z axis lies along the optical axis. The system in general has Γ surfaces, with the zero surface at the entrance pupil. The vertex for each surface has the coordinates $(0, 0, z_i)$. The focal plane is assumed perpendicular to the optical axis, and has coordinates $(0, 0, z_{\Gamma+1})$. The distance between the surface vertices is given by

$$d_i = z_{i+1} - z_i, \quad i = 0, 1, \ldots, \Gamma. \tag{1}$$

The sagittate for the i–th surface $(z - z_i)$ is represented by the function

$$z - z_i = f_i(x, y), \tag{2}$$

where the functions f_i are

$$f_i(x, y) = f_i(\rho) = \sum_{j=1}^{\infty} a_{ij} \rho^j, \tag{3}$$

with $\rho = x^2 + y^2$.

The index of refraction before and after each of the refracting surfaces is written as v_{i-1} and v_i, respectively. The relative index of refraction for the i–th surface is

$$\mu_i = \begin{cases} \frac{v_{i-1}}{v_i}, & \text{if the surface is refracting,} \\ -1, & \text{if the surface is reflecting.} \end{cases} \tag{4}$$

The intersection points and the direction tangents of a skew ray with the osculating plane at the vertex of each refracting or reflecting surface, have the coordinates (x_i, y_i) and (ξ, η_i), respectively. For the case of a symmetric optical system it is not necessary to know all four quantities since it is well known that $\rho_i = x_i^2 + y_i^2$, $\psi_i = \xi_i^2 + \eta_i^2$ and $\kappa_i = x_i\xi_i + y_i\eta_i$. Hence with ρ_i, ψ_i, κ_i known, the ray is fully characterized along its entire path.

2.2. General ray tracing equations

There are in the literature several methods for calculating the ray trace of an optical system [4–6]. The method that is presented in this paper is similar to the one developed by Spencer [4] in what he called equations for a general method of ray tracing.

For an ray incident at the i-th surface, the unit vector along the ray direction is

$$\tilde{\sigma}_i = (L_i, M_i, N_i); \tag{5}$$

L_i, M_i, N_i are the direction cosines of the ray; and the "crossing" point of the ray with the osculating plane $z = z_i$ is $P_i(x_i, y_i, z_i)$.

Hence, the coordinates of the intersection point, $\tilde{P}_i(\tilde{x}, \tilde{y}, \tilde{z})$ of the ray with the optical surface are related with other parameters by means of the equation

$$\frac{\tilde{x}_i - x_i}{L_i} = \frac{\tilde{y}_i - y_i}{M_i} = \frac{\tilde{z}_i - z_i}{N_i} = \frac{f_i(\tilde{x}_i, \tilde{y}_i)}{N_i}. \tag{6}$$

However, instead of the direction cosines, let us use the direction tangents defined by

$$\xi_i = \frac{L_i}{N_i}, \qquad \eta_i = \frac{M_i}{N_i}; \tag{7}$$

substituting Eq. (7) into Eq. (6) the coordinates $\tilde{x}_i, \tilde{y}_i$ are equal to

$$\begin{aligned} \tilde{x}_i &= x_i + \xi_i f_i(\tilde{x}_i, \tilde{y}_i), \\ \tilde{y}_i &= y_i + \eta_i f_i(\tilde{x}_i, \tilde{y}_i). \end{aligned} \tag{8}$$

The solution for Eq. (8) in general is obtained by the iterative Newton-Raphson method. Instead, here we will follow the Andersen method that defines the four functions $S_i = (\rho_0, \psi_0, \kappa_0)$, $T_i = (\rho_0, \psi_0, \kappa_0)$, $V_i = (\rho_0, \psi_0, \kappa_0)$, and $W_i = (\rho_0, \psi_0, \kappa_0)$, such that the coordinates (x_i, y_i) and the direction tangents (ξ_i, η_i) can be determined by means of the equations

$$\begin{bmatrix} x_i \\ y_i \end{bmatrix} = S_i(\rho_0, \psi_0, \kappa_0) \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + T_i(\rho_0, \psi_0, \kappa_0) \begin{bmatrix} \xi_0 \\ \eta_0 \end{bmatrix}, \tag{9}$$

$$\begin{bmatrix} \xi_i \\ \eta_i \end{bmatrix} = V_i(\rho_0, \psi_0, \kappa_0) \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + W_i(\rho_0, \psi_0, \kappa_0) \begin{bmatrix} \xi_0 \\ \eta_0 \end{bmatrix}. \tag{10}$$

Here the parameters ρ_0, ψ_0 and κ_0 are defined for the entrance pupil of the optical system, that is, for the first considered surface of the system.

On the other hand, the functions S_i, T_i, V_i, W_i can be calculated according to

$$\begin{bmatrix} S_{i+1} & T_{i+1} \\ V_{i+2} & W_{i+1} \end{bmatrix} = \begin{bmatrix} 1 - \chi_i(d_i - f_i) & f_i + (d_i - f_i)\left(\mu_i \dfrac{N_i}{N_{i+1}} - \chi_i f_i\right) \\ -\chi_i & \mu_i \dfrac{N_i}{N_{i+1}} - \chi_i f_i \end{bmatrix} \cdot \begin{bmatrix} S_i & T_i \\ V_i & W_i \end{bmatrix}, \quad (11)$$

where[1]

$$\chi_i = 2\frac{\gamma_i}{N_{i+1}}(\cos\theta_i' - \mu_i \cos\theta_i), \quad (12)$$

and N_{i+1} is the direction cosine with respect to the z axis for the refracted ray at the i–th surface; θ_i and θ_i' are the incident and refracted angles at the same i-th surface; and

$$\gamma_i = \left[1 + \left(\frac{\partial f_i}{\partial x}\right)^2 + \left(\frac{\partial f_i}{\partial y}\right)^2\right]^{-1/2}. \quad (13)$$

Other equations used by Andersen for calculating the defined functions S, T, V and W through the parameters ρ, ψ, κ are

$$G(\rho_0, \psi_0, \kappa_0) = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \sum_{k=0}^{j} G_{n,n-j,j-k,k}\, \rho_0^{n-j}\, \psi_0^{j-k}\, \kappa_0^{k}. \quad (14)$$

For example, $\rho_i = \rho_i(\rho_0, \psi_0, \kappa_0)$ is equal to

$$\rho_i(\rho_0, \psi_0, \kappa_0) = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \sum_{k=0}^{j} \rho_{in,n-j,j-k,k}\, \rho_0^{n-j}\, \psi_0^{j-k}\, \kappa_0^{k}, \quad (15)$$

where $\rho_{in,n-j,j-k,k}$ are the associated coefficients to the variable ρ_i, that can be stored in the computer as an array. The reader is referred to the appendix A of Andersen [1] work for the computer subroutines to calculate sums, products, etc. of the quantities G.

In what follows, mathematical equations representing the optical surfaces will be written since they will be used later on. Considering axially symmetric optical surfaces, the conic surfaces can be represented by their sagittate by means of the equation

$$\tilde{z}_i = f_i(\tilde{\rho}_i) = \frac{c_i \tilde{\rho}_i}{1 + \sqrt{1 - (1 + k_i) c_i^2 \tilde{\rho}_i}}, \quad (16)$$

with c_i as the paraxial curvature, and the conic constant k_i takes the values shown in Table I [8].

[1] In this paper we use χ_i for the parameter β_i in Andersen's work.

TABLE I. Eccentricities of the conics.

Eccentricity	Surface
$k_i < -1$	Hyperboloid
$k_i = -1$	Paraboloid
$-1 < k_i < 0$	Ellipsoid-major axis
$k_i = 0$	Sphere
$k_i > 0$	Ellipsoid-minor axis

The aspheric surface equation is equal to

$$c_i^2 \rho_i + 2c_i \sum_{i=2}^{N} \bar{\alpha}_{ij}\, \rho_i^j + [1 - c_i f_i(\rho_i)]^2 = 1, \tag{17}$$

where $\bar{\alpha}_{ij}$ are the deformation coefficients of j–th order.

Another way to describe the sagittate $\tilde{z}_i$ of Eq. (16) is as follows:

$$\tilde{z}_i = f_i(\tilde{\rho}) = \frac{1}{c_i}\left[1 - \left(1 - c_i^2 \tilde{\rho}_i - 2c_i \sum_{j=2}^{N} \bar{a}_{ij}\, \tilde{\rho}_i^j\right)^{1/2}\right]. \tag{18}$$

Considering that the derivatives of $f_i(\rho)$ will be used, from Eqs. (16) and (17) it can be shown that for a conic surface

$$f_i'(\tilde{\rho}_i) = \frac{c_i}{2\sqrt{1 - (1 + k_i)c_i^2 \tilde{\rho}_i}}; \tag{19}$$

and for aspheric surfaces

$$f_i'(\tilde{\rho}_i) = \frac{c_i + \sum_{j=2}^{N} 2j\bar{a}_{ij}\, \tilde{\rho}_i^{j-1}}{2[1 - c_i f_i(\tilde{\rho}_i)]}. \tag{20}$$

3. THE ALDIS THEOREM

The Aldis theorem allows us to have a mathematical equation that gives the transverse aberration for any ray, from the knowledge of the contribution of each surface to such a transverse aberration. This theorem was fully explained by Cox [2], who used it to obtain the aberration coefficients for an axially symmetric system, but including, where required, the decentering of the optical components. Welford [9] also mentiones in his book the Aldis theorem, but he does not use it for any practical purpose. One reason why the Aldis theorem is not used more frequently could be the fact that it gives information for just one ray, and not for a bundle of rays. Therefore, if it is desired to know the crossing points of a beam of rays with the image plane, it is necessary to obtain polynomial equations for such crossing points for each optical surface and the image plane; and this requires calculation of the polynomial aberrations of the optical system. In this section the Aldis theorem will be briefly described. A more complete explanation can be found in the book by Cox [2].

For an axially symmetric optical system, considering the notation of Sect. 2, the following paraxial quantities can be defined:

$$l_i = v_{i-1}L_i, \quad m_i = v_{i-1}M_i, \quad n_i = v_{i-1}N_i, \tag{21}$$

$$b_i = b_{i-1} - c_i\beta_i(v_i - v_{i-1}), \tag{22}$$

$$\beta_i = \beta_{i-1} + \frac{b_{i-1}t_{i-1}}{v_{i-1}}, \tag{23}$$

$$B_i = b_i + v_i\beta_i c_i = b_{i-1} + v_{i-1}\beta_i c_i, \tag{24}$$

$$y^* b_0 = y_1' b_1 = \cdots = y_\Gamma' b_\Gamma = H, \tag{25}$$

where b_i and β_i are the paraxial angles and heights of the rays; y^* is the object position at the object plane, whose actual value is chosen such that $x^* = 0$; and B_i is a paraxial invariant for the i-th surface. Therefore, it is possible to obtain

$$\begin{aligned} b_\Gamma N_{\Gamma+1}(\tilde{y}_{\Gamma+1} - y') = {} & \sum_{i=1}^{\Gamma} B_i\left[\tilde{y}(N_{i+1} - N_i) - \tilde{z}_i(M_{i+1} - M_i)\right] \\ & - \sum_{i=1}^{\Gamma} \beta_i\left[c_i\tilde{y}(n_{i+1} - n_i) + (1 - c_i\tilde{z}_i)(m_{i+1} - m_i\right] \\ & - H\sum_{i=1}^{\Gamma}(N_{i+1} - N_i); \end{aligned} \tag{26}$$

and

$$\begin{aligned} b_\Gamma N_{\Gamma+1}\tilde{x}_{\Gamma+1} = {} & \sum_{i=1}^{\Gamma} B_i\left[\tilde{x}_i(L_{i+1} - L_i) - \tilde{z}_i(M_{i+1} - M_i)\right] \\ & - \sum_{i=1}^{\Gamma} \beta_i\left[c_i\tilde{y}_i(l_{i+1} - l_i) + (1 - c_i\tilde{z}_i)(m_{i+1} - m_i)\right]. \end{aligned} \tag{27}$$

The above Eqs. (26) and (27) are known as the Aldis' equations [2]. The method to obtain such a set of equations is known as the Aldis theorem.

For the case of conic and aspheric surfaces, the Aldis' equations are equal to

$$\tilde{x}_{\Gamma+1} - x' = \sum_{i=1}^{\Gamma}\left\{\frac{B_i}{b_\Gamma}[E_{ix} + F_{ix}] - \frac{\beta_i}{b_\Gamma}\Omega_{ix}\right\}, \tag{28}$$

$$\tilde{y}_{\Gamma+1} - y' = \sum_{i=1}^{\Gamma}\left\{\frac{B_i}{b_\Gamma}[E_{iy} + F_{iy}] - \frac{\beta_i}{b_\Gamma}\Omega_{iy} - \frac{H}{b_\Gamma}P_i\right\}, \tag{29}$$

where

$$P = \frac{N_{i+1} - N_i}{N_{\Gamma+1}}, \tag{30}$$

$$E_{ix} = \tilde{x}_i P_i, \tag{31}$$

$$E_{iy} = \tilde{y}_i P_i, \tag{32}$$

$$F_{ix} = -\frac{L_{i+1} - L_i}{N_{\Gamma+1}} f_i(\tilde{\rho}_i), \tag{33}$$

$$F_{iy} = -\frac{M_{i+1} - M_i}{N_{\Gamma+1}} f_i(\tilde{\rho}_i), \tag{34}$$

$$\Omega_{ix} = \begin{cases} k_i c_i f_i(\tilde{\rho}_i)(l_{i+1} - l_i)\dfrac{1}{N_{\Gamma+1}}, & \text{if surface } i \text{ is conic;} \\ -\tilde{x}_i \left(\displaystyle\sum_{j=2}^{N} 2j\tilde{a}_{ij}\,\tilde{p}_i^{j-1}\right)\dfrac{n_{i+1} - n_i}{N_{\Gamma+1}}, & \text{if surface } i \text{ is aspheric;} \end{cases} \tag{35}$$

$$\Omega_{iy} = \begin{cases} k_i c_i f_i(\tilde{\rho}_i)(m_{i+1} - m_i)\dfrac{1}{N_{\Gamma+1}}, & \text{if surface } i \text{ is conic;} \\ -\tilde{y}_i \left(\displaystyle\sum_{j=2}^{N} 2j\tilde{a}_{ij}\,\tilde{p}_i^{j-1}\right)\dfrac{n_{i+1} - n_i}{N_{\Gamma+1}}, & \text{if surface } i \text{ is aspheric;} \end{cases} \tag{36}$$

with

$$\tilde{z}_i = f_i(\tilde{\rho}_i). \tag{37}$$

If a ray tracing is performed, Eqs. (28) and (29) allow us to determine the contribution to the transverse aberration for each surface for a particular ray through the optical system. As an alternative method to ray tracing, Andersen's method can be used for the calculations of the ray trace by means of his polynomial functions to any order of approximation. In the next section, Eqs. (28) to (37) will be used, in the method developed by Andersen.

4. Formulae using Andersen's method with the theorem of Aldis

Andersen has used his mathematical equations for calculating the aberration coefficients, to determine focal surfaces [10], caustics [11], and the derivatives of the aberration functions with respect to the axial distances [12], or with respect to the refractive index [13], or some other surface parameters [14]. It should be said that Andersen's method has not been used profusely. Only Tam [15] has reported this method for the study of the diffraction of an optical system with symmetry about the optical axis. Hence, after analyzing some properties of Andersen's work, it was decided to use the Aldis theorem together with the equations derived by Andersen. In order to achieve this task, in the first place, a finite approximation order is considered for Eq. (14). In this case the quantities obtained from Eqs. (28) and (29) must have the same approximation order. Therefore, starting with Eqs. (28) and (29) and after some algebraic manipulation, each term can be written as a function of the Andersen quantities S, T, V and W.

4.1. Aberration equation for object at infinity

From the definitions

$$\Delta x^{(M)} = \tilde{x}_{\Gamma+1} - x', \tag{38}$$

$$\Delta x^{(M)} = \tilde{y}_{\Gamma+1} - y', \tag{39}$$

for the transverse ray aberrations at the Gaussian image plane, to the order M; the quantities from Eq. (30) to Eq. (37) appear implicitly or explicitly within the method developed by Andersen as briefly explained in Sect. 2. Therefore, using Eqs. (28) and (29), then Eqs. (38) and (39) can be written as

$$\begin{bmatrix} \Delta x^{(M)} \\ \Delta y^{(M)} \end{bmatrix} = \sum_{i=1}^{\Gamma} \left\{ \frac{B_i}{b_\Gamma} \begin{bmatrix} E_{ix} + F_{ix} \\ E_{iy} + F_{iy} \end{bmatrix} - \frac{1}{b_\Gamma} P_i \begin{bmatrix} 0 \\ H \end{bmatrix} - \frac{b_i}{b_\Gamma} \begin{bmatrix} \Omega_{ix} \\ \Omega_{iy} \end{bmatrix} \right\}; \tag{40}$$

and each term of Eq. (40) will be analyzed in what follows.

From Eqs. (31) and (32) it is possible to write

$$\begin{bmatrix} E_{ix} \\ E_{iy} \end{bmatrix} = P_i \begin{bmatrix} \tilde{x}_i \\ \tilde{y}_i \end{bmatrix}, \tag{41}$$

and from Eq. (8), then

$$\begin{bmatrix} E_{ix} \\ E_{iy} \end{bmatrix} = P_i \left\{ \begin{bmatrix} x_i \\ y_i \end{bmatrix} + f_i \begin{bmatrix} \xi_i \\ \eta_i \end{bmatrix} \right\}. \tag{42}$$

Combining Eqs. (9) and (10) and (42) we have, after further algebraic manipulation,

$$\begin{bmatrix} E_{ix} \\ E_{iy} \end{bmatrix} = (S_i + f_i V_i) P_i \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + (T_i + f_i W_i) P_i \begin{bmatrix} \xi_0 \\ \eta_0 \end{bmatrix}. \tag{43}$$

Conversely, from Eqs. (7) and (10) the following equation can be obtained:

$$\begin{bmatrix} (L_{i+1} - L_i) \big/ N_{\Gamma+1} \\ (M_{i+1} - M_i) \big/ N_{\Gamma+1} \end{bmatrix} = \frac{N_{i+1} V_{i+1} - N_i V_i}{N_{\Gamma+1}} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \frac{(N_{i+1} W_{i+1} - N_i W_i}{N_{\Gamma+1}} \begin{bmatrix} \xi_0 \\ \eta_0 \end{bmatrix}. \tag{44}$$

By substitution of Eq. (44) into Eqs. (33) and (34), we have

$$\begin{bmatrix} F_{ix} \\ F_{iy} \end{bmatrix} = \frac{f_i (N_{i+1} V_{i+1} - N_i V_i)}{N_{\Gamma+1}} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \frac{f_i (N_{i+1} W_{i+1} - n_i W_i)}{N_{\Gamma+1}} \begin{bmatrix} \xi_0 \\ \eta_0 \end{bmatrix}. \tag{45}$$

Therefore, combining Eqs. (43) and (45)

$$\begin{bmatrix} E_{ix} + F_{ix} \\ E_{iy} + F_{iy} \end{bmatrix} = \left((S_i + f_i V_i) P_i - \frac{f_i (N_{i+1} V_{i+1} - N_i V_i)}{N_{\Gamma+1}} \right) \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \left((T_i + f_i W_i) P_i - \frac{f_i (N_{i+1} W_{i+1} - N_i W_i)}{N_{\Gamma+1}} \right) \begin{bmatrix} \xi_0 \\ \eta_0 \end{bmatrix}. \tag{46}$$

As can be noticed, Eq. (46) finally contains the quantities defined by Andersen, and will be used later on.

Now, from Eqs. (7), (10) and (21) it is possible to obtain

$$\begin{bmatrix} l_{i+1} - l_i \\ m_{i+1} - m_i \end{bmatrix} = (v_i N_{i+1} V_{i+1} - v_{i-1} N_i V_i) \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + (v_i N_{i+1} W_{i+1} - v_{i-1} N_i W_i) \begin{bmatrix} \xi_0 \\ \eta_0 \end{bmatrix}. \tag{47}$$

If the i–th surface is a conic, then by means of Eqs. (35), (36) and (47)

$$\begin{bmatrix} \Omega_{ix} \\ \Omega_{iy} \end{bmatrix} = \frac{v_i N_{i+1} V_{i+1} - v_{i-1} N_i V_i}{N_{\Gamma+1}} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} k_i c_i f_i + \frac{v_i (N_{i+1} W_{i+1} - v_{i-1} N_i W_i}{N_{\Gamma+1}} \begin{bmatrix} \xi_0 \\ \eta_0 \end{bmatrix} k_i c_i f_i; \tag{48}$$

for the case of an aspheric surface and considering that

$$(n_{i+1} - n_i) = v_i N_{i+1} - v_{i-1} N_i, \tag{49}$$

and using Eqs. (8) ,(9) and (10)

$$\begin{bmatrix} \tilde{x}_i \\ y_i \end{bmatrix} = (S_i + f_i V_i) \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + (T_i + f_i W_i) \begin{bmatrix} \xi_0 \\ \eta_0 \end{bmatrix}, \tag{50}$$

we have

$$\begin{bmatrix} \Omega_{ix} \\ \Omega_{iy} \end{bmatrix} = (S_i + f_i V_i) \left(\sum_{j=2}^{N} 2j \bar{a}_{ij} \, \tilde{\rho}_i^{j-1} \right) (v_i N_{i+1} - v_{i-1} N_i) \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} (T_i + f_i W_i) \left(\sum_{j=2}^{N} 2j \bar{a}_{ij} \, \tilde{\rho}_i^{j-1} \right) (v_i N_{i+1} - v_{i-1} N_i) \begin{bmatrix} \xi_0 \\ \eta_0 \end{bmatrix}. \tag{51}$$

If the i–th surface is conic or aspheric, then Eqs. (48) and (51) can be written as

$$\begin{bmatrix} \Omega_{ix} \\ \Omega_{iy} \end{bmatrix} = A_i \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + F_i \begin{bmatrix} \xi_0 \\ \eta_0 \end{bmatrix}, \tag{52}$$

where

$$A_i = \begin{cases} \dfrac{k_i c_i f_i (v_i N_{i+1} V_{i+1} - v_{i-1} N_i V_i)}{N_{\Gamma+1}}, & \text{if surface } i \text{ is conic;} \\ -(S_i + f_i V_i)\left(\displaystyle\sum_{j=2}^{N} 2j\tilde{a}_{ij}\,\tilde{p}_i^{j-1}\right), & \text{if surface } i \text{ is aspheric.} \end{cases} \tag{53}$$

$$F_i = \begin{cases} \dfrac{k_i c_i f_i (v_i N_{i+1} W_{i+1} - v_{i-1} N_i W_i)}{N_{\Gamma+1}}, & \text{if surface } i \text{ is conic;} \\ -(T_i + f_i W_i)\left(\displaystyle\sum_{j=2}^{N} 2j\tilde{a}_{ij}\,\tilde{p}_i^{j-1}\right), & \text{if surface } i \text{ is aspheric.} \end{cases} \tag{54}$$

As can be seen, Eq. (52) has been written using the quantities defined by Andersen.

With the results obtained in Eqs. (46) and (52) the contributions to the transverse aberration contained in Eq. (40) are now equal to

$$\begin{bmatrix} \Delta x^{(M)} \\ \Delta y^{(M)} \end{bmatrix} = \sum_{i=1}^{\Gamma} \left\{ \frac{B_i}{b_\Gamma}\left(Q_i \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + R_i \begin{bmatrix} \xi_0 \\ \eta_0 \end{bmatrix}\right) - \frac{\beta_i}{b_\Gamma}\left(A_i \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + F_i \begin{bmatrix} \xi_0 \\ \eta_0 \end{bmatrix}\right) - \frac{\beta_i}{b_\Gamma} P_i \begin{bmatrix} 0 \\ H \end{bmatrix} \right\}, \tag{55}$$

with

$$Q_i = (S_i + f_i V_i) P_i - \frac{f_i (N_{i+1} V_{i+1} - N_i V_i)}{N_{\Gamma+1}}, \tag{56}$$

$$R_i = (T_i + f_i W_i) P_i - \frac{f_i (N_{i+1} W_{i+1} - N_i W_i)}{N_{\Gamma+1}}. \tag{57}$$

For an object at infinity, from Eq. (55) the Smith Helmholtz invariant $H = y' b_\Gamma$.Taking into account that $y' = f'\eta_0$, and the conditions $x' = 0$, and $\xi_0 = 0$, then

$$\begin{bmatrix} 0 \\ H \end{bmatrix} = b_\Gamma f' \begin{bmatrix} 0 \\ \eta_0 \end{bmatrix}, \tag{58}$$

where f' is the effective focal distance. With the previous results Eq. (55) for this case can be written as

$$\begin{bmatrix} \Delta x^{(M)} \\ \Delta y^{(M)} \end{bmatrix} = \sum_{i=1}^{\Gamma} \left\{ \left(\frac{B_i}{b_\Gamma} Q_i - \frac{\beta_i}{b_\Gamma} A_i\right) \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \left(\frac{B_i}{b_\Gamma} R_i - \frac{\beta_i}{b_\Gamma} F_i + \frac{\beta_1}{b_\Gamma} P_i\right) \begin{bmatrix} 0 \\ \eta_0 \end{bmatrix} \right\}. \tag{59}$$

Equation (59) implies that for an object at infinity, η_0 is constant for the entrance pupil plane. This means that incident rays are parallel to the optical axis and therefore they have the same direction tangent.

TABLE II. Cooke triplet design from H.A. Buchdahl with the stop positioned at a distance $d =$ 0.041353 behind rear to the fourth surface.

	j					
	1	2	3	4	5	6
c	4.82439	−.753929	−1.64505	5.11794	.310726	−1.46116
N	1.	1.6162	1.	1.5725	1.	1.6162
d	.040278	.016851	.0096145	.138738	.0313246	—

TABLE III. Contributions of the surface $i = 1$, $M = 3$ from the Eq. (56).

$\frac{B_i}{b_\Gamma}Q_i - \frac{\beta_i}{b_\Gamma}A_i$	$\frac{B_i}{b_\Gamma}R_i - \frac{\beta_i}{b_\Gamma}(F_i - P_i)$	n	$n-j$	$j-k$	k
$.000000E+00$	$.000000E+00$	0	0	0	0
$-.132443E+02$	$-.132443E+02$	1	1	0	0
$-.103684E+01$	$-.975177E-01$	1	0	1	0
$-.249132E+01$	$-.234315E+00$	1	0	0	1
$-.160872E+03$	$-.151304E+02$	2	2	0	0
$-.103029E+02$	$-.969017E+00$	2	1	1	0
$-.456051E+02$	$-.428928E+01$	2	1	0	1
$.242500E+00$	$.228077E-01$	2	0	2	0
$.172104E+01$	$.161868E+00$	2	0	1	1
$.103580E+02$	$.974192E+00$	2	0	0	2
$-.223980E+04$	$-.210659E+03$	3	3	0	0
$-.146164E+03$	$-.137471E+02$	3	2	1	0
$-.104364E+04$	$-.981565E+02$	3	2	0	1
$.384475E+01$	$.361608E+00$	3	1	2	0
$.725705E+01$	$.682544E+00$	3	1	1	1
$.144315E+03$	$.135731E+02$	3	1	0	2
$-.119538E-01$	$-.112429E-02$	3	0	3	0
$-.111649E+01$	$-.105009E+00$	3	0	2	1
$-.976423E+00$	$-.918350E-01$	3	0	1	2
$.342496E+02$	$.322126E+01$	3	0	0	3

5. RESULTS

As a means of checking that the equations derived in this work are correct, a comparative study of the surface aberration contributions between our results and those obtained by Buchdahl [16] was performed. The particular optical system chosen was a Cooke triplet, since Buchdahl's analysis of this system was performed in depth. In Table II are shown the values for the curvatures c; refractive index, N; and surface separations, d, for the studied triplet.

The results shown in Tables III to VIII are for an object at infinity, and where obtained in such a way that the coefficients are functions exclusively of the entrance pupil coordinates and the ray direction tangents. However, for a closer object, the transformation coefficient described by Andersen [10] can be used. The optical surface contributions are

TABLE IV. Contributions of the surface $i = 2$, $M = 3$ from the Eq. (56).

$\frac{B_i}{b_\Gamma}Q_i - \frac{\beta_i}{b_\Gamma}A_i$	$\frac{B_i}{b_\Gamma}R_i - \frac{\beta_i}{b_\Gamma}(F_i - P_i)$	n	$n-j$	$j-k$	k
$.000000E+00$	$.000000E+00$	0	0	0	0
$-.176346E+02$	$.616619E+01$	1	1	0	0
$-.229983E+01$	$.804168E+00$	1	0	1	0
$.123324E+02$	$-.431220E+01$	1	0	0	1
$-.416083E+03$	$.110354E+03$	2	2	0	0
$-.317229E+02$	$.642671E+01$	2	1	1	0
$.247819E+03$	$-.559880E+02$	2	1	0	1
$-.102849E+01$	$.328188E+00$	2	0	2	0
$.182043E+02$	$-.507424E+01$	2	0	1	1
$-.455446E+02$	$.101644E+02$	2	0	0	2
$-.982425E+04$	$.225995E+04$	3	3	0	0
$-.803335E+03$	$.163665E+03$	3	2	1	0
$.528891E+04$	$.101000E+04$	3	2	0	1
$-.218306E+02$	$.500120e+01$	3	1	2	0
$.497986E+03$	$-.130741E+03$	3	1	1	1
$-.106611E+04$	$.253545E+03$	3	1	0	2
$-.233176E+00$	$.595896E-01$	3	0	3	0
$.993218E+01$	$-.307263E+01$	3	0	2	1
$-.107915E+03$	$.347683E+02$	3	0	1	2
$.125529E+03.$	$.461004E+02$	3	0	0	3
				0	

TABLE V. Contributions of the surface $i = 3$, $M = 3$ from the Eq. (56).

$\frac{B_i}{b_\Gamma}Q_i - \frac{\beta_i}{b_\Gamma}A_i$	$\frac{B_i}{b_\Gamma}R_i - \frac{\beta_i}{b_\Gamma}(F_i - P_i)$	n	$n-j$	$j-k$	k
$.000000E+00$	$.000000E+00$	0	0	0	0
$.239033E+02$	$-.725795E+01$	1	1	0	0
$.250325E+01$	$-.760084E+00$	1	0	1	0
$-.145159E+02$	$.440759E+01$	1	0	0	1
$.584009E+03$	$-.134570E+03$	2	2	0	0
$.413787E+02$	$-.813729E+01$	2	1	1	0
$-.335433E+03$	$.694388E+0$	2	2	1	0
$.978258E+00$	$-.282152E+00$	2	0	2	0
$-.199853E+02$	$.478994E+01$	2	0	1	1
$.571142E+02$	$-.108737E+02$	2	0	0	2
$.143935E+05$	$-.289681E+04$	3	3	0	0
$.109331E+04$	$-.204544E+03$	3	2	1	0
$-.787120E+04$	$.141438E+04$	3	2	0	1
$.228317E+02$	$-.482148E+01$	3	1	2	0
$-.608788E+03$	$.140463E+03$	3	1	1	1
$.156989E+04$	$-.333224E+03$	3	1	0	2
$.123421E-01$	$.101229E-01$	3	0	3	0
$-.650828E+01$	$.170811E+01$	3	0	2	1
$.104717E+03$	$-.280439E+02$	3	0	1	2
$-.144915E+03$	$.411811E+02$	3	0	0	3

TABLE VI. Contributions of the surface $i = 4$, $M = 3$ from the Eq. (56).

$\frac{B_i}{b_\Gamma}Q_i - \frac{\beta_i}{b_\Gamma}A_i$	$\frac{B_i}{b_\Gamma}R_i - \frac{\beta_i}{b_\Gamma}(F_i - P_i)$	n	$n-j$	$j-k$	k
$.000000E+00$	$.000000E+00$	0	0	0	0
$.748422E+01$	$.162409E+01$	1	1	0	0
$.128408E+01$	$.278647E+00$	1	0	1	0
$.324818E+01$	$.704863E+00$	1	0	0	1
$.925538E+02$	$.127191E+02$	2	2	0	0
$.112641E+02$	$.228684E+01$	2	1	1	0
$.285733E+02$	$.774751E+01$	2	1	0	1
$-.202153E-01$	$.316647E-01$	2	0	2	0
$.136201E+01$	$.383413E+00$	2	0	1	1
$-.625545E+01$	$-.159615E+01$	2	0	0	2
$.234894E+04$	$.196532E+03$	3	3	0	0
$.170550E+03$	$.143242E+02$	3	2	1	0
$-.248757E+03$	$.803025E+02$	3	2	0	1
$.349216E+00$	$.632201E+00$	3	1	2	0
$.100851E+03$	$.238990E+02$	3	1	1	1
$-.192327E+03$	$-.455022E+02$	3	1	0	2
$.795529E-01$	$.478387E-01$	3	0	3	0
$-.202804E+00$	$-.385413E+00$	3	0	2	1
$.674156E+00$	$-.117125E+00$	3	0	1	2
$.154016E+02$	$.584220E+01$	3	0	0	3

TABLE VII. Contributions of the surface $i = 5, M = 3$ from the Eq. (56).

$\frac{B_i}{b_\Gamma}Q_i - \frac{\beta_i}{b_\Gamma}A_i$	$\frac{B_i}{b_\Gamma}R_i - \frac{\beta_i}{b_\Gamma}(F_i - P_i)$	n	$n-j$	$j-k$	k
$.000000E+00$	$.000000E+00$	0	0	0	0
$-.107635E-03$	$-.746417E-03$	1	1	0	0
$-.644107E-01$	$-.446667E+00$	1	0	1	0
$-.149283E-02$	$-.103523E-01$	1	0	0	1
$-.548399E-02$	$-.432382E-03$	2	2	0	0
$.426406E-02$	$.445376E+00$	2	1	1	0
$-.536994E-01$	$-.445053E+00$	2	1	0	1
$-.829507E-02$	$-.475696E-01$	2	0	2	0
$-.284127E-01$	$-.577579E+00$	2	0	1	1
$.189726E+00$	$.128188E+01$	2	0	0	2
$-.105194E+01$	$-.949128E+00$	3	3	0	0
$.582717E+00$	$.230006E+02$	3	2	1	0
$-.119320E+02$	$-.869543E+02$	3	2	0	1
$-.857759E-01$	$-.177094E+00$	3	1	2	0
$-.131879E+01$	$-.215298E+02$	3	1	1	1
$.134777E+02$	$.910599E+02$	3	1	0	2
$-.197288E-02$	$-.817659E-02$	3	0	3	0
$-.926076E-01$	$-.902162E-01$	3	0	2	1
$-.260575E+00$	$-.180360E+01$	3	0	1	2
$-.208160E+01$	$-.130423E+02$	3	0	0	3

TABLE VIII. Contributions of the surface $i = 6$, $M = 3$ from the Eq. (56).

$\frac{B_i}{b_\Gamma}Q_i - \frac{\beta_i}{b_\Gamma}A_i$	$\frac{B_i}{b_\Gamma}R_i - \frac{\beta_i}{b_\Gamma}(F_i - P_i)$	n	$n-j$	$j-k$	k
$.000000E+00$	$.000000E+00$	0	0	0	0
$-.186761E+01$	$.699379E+00$	1	1	0	0
$-.540447E+00$	$.202386E+00$	1	0	1	0
$.139876E+01$	$-.523805E+00$	1	0	0	1
$-.867857E+01$	$.345533E+01$	2	2	0	0
$-.186154E+01$	$.660529E+00$	2	1	1	0
$.125579E+02$	$-.459316E+01$	2	1	0	1
$.212374E+00$	$-.117461E+00$	2	0	2	0
$.223562E+00$	$.156459E+00$	2	0	1	1
$-.267421E+01$	$.597293E+00$	2	0	0	2
$-.237852E+02$	$.378898E+01$	3	3	0	0
$.908724E+01$	$-.443309E+01$	3	2	1	0
$-.147188E+02$	$.239501E+02$	3	2	0	1
$-.559330E+00$	$.161893E+00$	3	1	2	0
$-.412069E+01$	$.771570E+01$	3	1	1	1
$.368407E+02$	$-.338075E+02$	3	1	0	2
$.645830E-01$	$-.363432E-01$	3	0	3	0
$-.139305E+01$	$.264718E+00$	3	0	2	1
$.195437E+01$	$-.577715E+00$	3	0	1	2
$-.957890E+01$	$.665780E+01$	3	0	0	3

TABLE IX. Sum of the contributions, $M = 3$ from the Eq. (56).

$\frac{B_i}{b_\Gamma}Q_i - \frac{\beta_i}{b_\Gamma}A_i$	$\frac{B_i}{b_\Gamma}R_i - \frac{\beta_i}{b_\Gamma}(F_i - P_i)$	n	$n-j$	$j-k$	k
$.000000E+00$	$.000000E+00$	0	0	0	0
$-.135912E+01$	$-.146952E-01$	1	1	0	0
$-.154197E+00$	$-.190677E-01$	1	0	1	0
$-.293905E-01$	$.317779E-01$	1	0	0	1
$.909239E+02$	$-.231728E+02$	2	2	0	0
$.875966E+01$	$.713145E+00$	2	1	1	0
$-.921417E+02$	$.118708E+02$	2	1	0	1
$.376129E+00$	$-.645229E-01$	2	0	2	0
$.149721E+01$	$-.160137E+00$	2	0	1	1
$.131876E+02$	$.547905E+00$	2	0	0	2
$.465358E+04$	$-.648148E+03$	3	3	0	0
$.324031E+03$	$-.217346E+02$	3	2	1	0
$-.390134E+04$	$.323519E+03$	3	2	0	1
$.454999E+01$	$.115833E+01$	3	1	2	0
$-.813383E+01$	$.204897E+02$	3	1	1	1
$.506089E+03$	$-.543557E+02$	3	1	0	2
$-.906244E-01$	$.719071E-01$	3	0	3	0
$.618948E+00$	$-.168044E+01$	3	0	2	1
$-.180697E+01$	$.413408E+01$	3	0	1	2
$.186042E+02$	$-.224028E+01$	3	0	0	3

TABLE X. Numerical values for the optical surfaces contributions to the aberrations.

	i						
	1	2	3	4	5	6	$\sum$
S^{*i}_{1100}	−13.244	−17.634	23.903	7.4842	−.03108	−1.8676	−1.3591
S^{*i}_{1010}	−1.0368	−2.2998	2.5032	1.2841	−.06441	−.54045	−.15419
T^{*i}_{1100}	1.2457	−6.16622	7.2579	−1.6241	.037464	−.69937	.014696
T^{*i}_{1010}	−.09752	.80416	−.76	.278647	−.44667	.202385	−.01907
S^{*i}_{1001}	−.23432	−4.3122	4.4075	.704867	−.01035	−.5238	.031778
S^{*i}_{2200}	−160.87	−416.08	584.01	92.5538	−.025483	−8.6786	90.9239

shown in Tables III to VIII, and the total amount of transverse aberration is presented in Table IX. The upper left heading in each table represents the specific aberration coefficients, while the upper right heading indicates the order of approximation. The numerical values for each aberration are listed according to the particular coefficient with the index n, j and k.

For example, if we consider the third order spherical aberration coefficient, then $n = 1$, $k = 0$, $j = 0$. From Table IX, for these index values, the total aberration coefficient is equal to -1.35912. A comparison with the study realized by Buchdahl on the Cook triplet (Table X) shows that the numerical value for the same aberration coefficient is remarkably similar.

6. CONCLUSIONS

The main purpose of this paper was to investigate the application of the Aldis theorem combined with the mathematical results of the method proposed by Andersen. As result we have derived a set of equations that allow the calculation of each surface contribution to the transverse aberration, for an axially symmetrical optical system. The results have been verified by comparison with those obtained by Buchdahl for a Cooke triplet optical system.

One should bear in mind that although the example selected contains only spherical and refractive elements, our results are more general, since the equations obtained are valid for conic, aspheric, and also reflective surfaces.

Given that with the combination of the Aldis theorem with the Andersen method a symbolic treatment it is not possible, the authors will try in the future to use only the theory of Andersen with the symbolic representation for studying the surface contributions to the aberration coefficients of an optical system.

APPENDIX. OPTICAL ABERRATION COEFFICIENTS DETERMINED BY A. COX

In the development of what is called paraxial optics, it is well known that from the knowledge of the height and angle of any two rays, any other ray can be defined and calculated. By convention, the two chosen rays are the one starting at an object point on axis and starting the edge of the entrance pupil; and that starting at the edge of the object and passing through the center of the same pupil.

In this Appendix, the coefficients derived by Cox [2] are written using the paraxial quantities defined in Sect. 3, mainly Eqs. (23), (22) and (24). Therefore

$$\begin{aligned} \alpha_i &= \alpha_{i-1} + \frac{a_{i-1}t_{i-1}}{v_{i-1}}, \\ a_i &= a_{i-1} + c_i\alpha_i(v_i - v_{i-1}), \\ A_i &= a_i + v_i\alpha_i c_i = a_{i-1} + v_{i-1}\alpha_i c_i, \\ B'_i &= \frac{b_i}{v_i^2} - \frac{b_{i-1}}{v_{i-1}^2}, \\ A'_i &= \frac{a_i}{v_i^2} - \frac{a_{i-1}}{v_{i-1}^2}, \end{aligned} \tag{60}$$

where α_i is the ray height, a_i is the incident angle for the i–th surface; and A_i is a quantity defined in a similar way to B_i represented in Eq. (24).

Following Andersen's method of defining the aberration coefficients, see Cox [18], the following equations can be derived:

$$S^{*i}_{1100} = \frac{1}{2b_r} B_i^2 B'_i \beta_i,$$

$$S^{*i}_{1010} = -\frac{b_i}{b_\Gamma}\left[\beta_i\left(\frac{a_i^2}{2v_i^2} - \frac{a_{i-1}^2}{2v_{i-1}^2}\right) + \left(\frac{b_i}{v_i} - \frac{b_{i-1}}{v_{i-1}}\right)\frac{1}{2}c_i\alpha_1^2\right],$$

$$T^{*i}_{1100} = -\frac{1}{2}A_i B_i B'_i \beta_i,$$

$$T^{*i}_{1010} = \frac{1}{2}b_\Gamma^2 A_i(A'_i + A_i B'_i \alpha_i),$$

$$T^{*i}_{1001} = \frac{1}{b_\Gamma}\left[B_i\left\{\alpha_i\left(\frac{a_i b_i}{v_i^2} - \frac{a_{i-1}b_{i-1}}{v_{i-1}^2}\right) + c_i\alpha_i\beta_i\left(\frac{a_i}{v_i} - \frac{a_{i-1}}{v_{i-1}}\right)\right\} - \left(\frac{a_i b_i}{v_i^2} - \frac{a_{i-1}b_{i-1}}{v_{i-1}^2}\right)\right],$$

$$\begin{aligned} S^{*i}_{2200} = -\frac{1}{2b_\Gamma}\Bigg\{ &\left[\frac{3}{2}A_i B_i B'_I \beta_i - \frac{1}{2}B_i B'_i + \frac{1}{2}\beta_i c_i B_i\left(\frac{1}{v_i} - \frac{1}{v_{i-1}}\right)\right]\sum_{j=1}^{i-1} B_j^2 B'_j \beta_j \\ &-\frac{3}{2}B_i^2 B'_i \beta_i \sum_{j=1}^{i-1} A_j B_j B^i_j \beta_j + \left(\frac{1}{2}\frac{b_\Gamma^2}{v_\Gamma^2} - \frac{3}{2}\frac{b_0}{v_0^2}\right) B_i^2 B'_i \beta_i \\ &- \frac{3}{4}B_i\left(\frac{b_i}{v_i^2} - \frac{b_{i-1}}{v_{i-1}^2}\right) - \frac{3}{4}(\beta_i c_i)^2 B_i^2 B'_i \beta_i\Bigg\}. \end{aligned}$$

As a comparison example of an optical system, a Cooke triplet analyzed by Buchdahl [16] was chosen, whose numerical values for the optical surface contributions to the aberrations are shown in Table X.

In this case the initial values for the entrance ray are, $b_0 = 0$; $\beta_1 = 1$; $a_0 = 1$; $\alpha_i = -.113227$; and the chief ray reaches the first optical surface with a cosine angle equal to 1 and of height $-.113227$.

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