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Alternative proof of Bertrand's theorem using a phase space approach

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ABSTRACT. Using the phase-space technique to analyze the dynamics of two-body classical systems introduced in a previous paper we present an alternative proof of Bertrand's theorem on the existence of only two types of potential all of whose bounded orbits are closed.

RESUMEN. Utilizando la técnica del espacio fase para analizar la dinámica de sistemas clásicos de dos cuerpos, introducidos en un artículo anterior, presentamos una demostración alternativa sobre la existencia de sólo dos tipos de potenciales para los cuales todas las órbitas acotadas son cerradas.

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1. INTRODUCTION

Bertrand's theorem is a topic scarcely discussed in the classical mechanics courses, due to the fact that its demonstration is lengthy and difficult to find in textbooks. The original demonstration was published one hundred twenty two years ago [1], and it is difficult to obtain this reference, so that the more accessible demonstration is that found in Appendix A of the second edition of Goldstein's textbook [2]. Another demonstration was published in the *Revista Mexicana de Física*, about twenty years ago by Berrondo *et al.* [3].

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More recently Martínez y Romero *et al.* [4], published an elegant demonstration of the theorem. They showed that the existence of closed and bounded plane orbits implies the existence of an extra constant of motion **A**, this quantity can be a vector or a second rank tensor, depending on how many dynamical axis of symmetry the orbit has. The symmetry properties of **A** determine the functional form of the potential producing plane close orbits. Thus, when the orbit has only one dynamical axis of symmetry, they obtained the Newtonian potential, and when the orbit has two dynamical symmetry axis, the result is the harmonic oscillator potential, and proved Bertrand's theorem.

Another compact demonstration was given by V.I. Arnold in one of his books [5]. This last demonstration was disregarded by Western physicists, maybe because the presentation is unusual and directed to the mathematicians. However Arnold's demonstration is a rich one, being necessary to translate it into the physicists' language, which is, in essence, the goal of this paper.

Recently the authors introduced a phase space approach to the orbits in central force fields [6], this is the natural scheme to study the closed orbits in central fields and the convenient scheme to understand the alternative proof of Bertrand's theorem made by Arnold.

The structure of the paper is as follows. In Sect. 2 we summarize the results of Ref. [6] rederived in a slightly different language, more suitable for our purposes here and, in Sect. 3, we present the demonstration of Bertrand's theorem using the phase space approach. Finally in Sect. 4, we comment about Arnold's ideas behind his demonstration.

2. THE PHASE SPACE FOR THE ORBITS IN CENTRAL FIELDS

We begin by showing that the energy and angular momentum conservation for central fields allow to construct a phase space.

The energy conservation for central fields is given by

$$\tfrac{1}{2}m\vec{v}^2 + V(r) = E. \tag{1}$$

Since the angular momentum is conserved, the particle moves in a plane and, therefore, we can describe the motion using polar coordinates, in terms of which the particle's vector position and velocity are given by

$$\begin{aligned} \vec{r} &= r\hat{e}_r, \\ \vec{v} &= \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta, \end{aligned} \tag{2}$$

and the angular momentum is

$$\vec{\ell} = \vec{r} \times \vec{p} = mr^2\dot{\theta}\hat{k} = \ell\hat{k}. \tag{3}$$

Introducing the fact that $\ell = mr^2\dot{\theta}$ into Eq. (2) we have that

$$\vec{v} = r\hat{e}_r + \frac{\ell}{mr}\hat{e}_\theta.$$

The orbit in plane polar coordinates is given by $r = r(\theta)$, using this fact, we have that

$$\dot{r} = \frac{dr}{d\theta}\dot{\theta} = \frac{dr}{d\theta}\frac{\ell}{mr^2} = -\frac{\ell}{m}\frac{d}{d\theta}\left(\frac{1}{r}\right), \tag{4}$$

where we use the conservation of the angular momentum. Thus, this conservation law implies that the velocity may be written in the form

$$\vec{v} = -\frac{\ell}{m}\frac{du}{d\theta}\hat{e}_r + \frac{\ell}{m}u\hat{e}_\theta, \tag{5}$$

where $u = 1/r$. Introducing the Eq. (5) into the energy equation, given by Eq. (1), we have that

$$\frac{1}{2}\left(\frac{du}{d\theta}\right)^2 + \frac{1}{2}u^2 + \alpha V\left(\frac{1}{u}\right) = \alpha E, \tag{6}$$

where $\alpha = m/\ell^2$. This equation is formally identical to the energy equation for a particle of unit mass, moving in one dimension under a potential

$$W(u) = \frac{1}{2}u^2 + \alpha V\left(\frac{1}{u}\right), \tag{7}$$

and with energy $E' = \alpha E$. Then we can rewrite Eq. (6) in the form

$$\frac{1}{2}\left(\frac{du}{d\theta}\right)^2 + W(u) = E'. \tag{8}$$

We introduce a phase space of coordinates u and $u' = \frac{du}{d\theta}$. The portrait of an orbit in this space consist of a continuous curve, symmetric with respect to the u-axis and crossing it at the apsidal positions. Clearly, only the part of this curve lying in the half plane $u > 0$ will be physically meaningful. For a closed path completely contained in the region $u > 0$ the area enclosed by it is given by

$$A = 2\int_{u_{\min}}^{u_{\max}} u'\,du = 2\int_{u_{\min}}^{u_{\max}} \sqrt{2(E' - W(u))}\,du. \tag{9}$$

On the other hand, the angle between two consecutive apocenters (or pericenters) is given by

$$\Delta\Theta = 2\int_{u_{\min}}^{u_{\max}} \frac{du}{u'}. \tag{10}$$

From Eqs. (9) and (10), it follows that

$$\Delta\Theta = \frac{\partial A}{\partial E'}. \tag{11}$$

For some values of the energy, the phase curve defined by Eq. (8) and the condition $u > 0$, is not closed. In this case we have only one apsidal point, due to the fact the $u_{\min} = \frac{1}{r_{\max}} = 0$, and the quantity

$$A = 2\int_0^{u_{\max}} u'\,du \tag{12}$$

corresponds to the area enclosed by the phase curve and the line $u' = 0$, then Eq. (10) takes on the form

$$\Delta\Theta = 2\int_0^{u_{\max}} \frac{du}{\sqrt{2(E' - W(u))}} = 2\theta_0, \tag{13}$$

where θ_0 is the angle between the asymptotic line to the orbit and the line from the center to the nearest point in the orbit.

3. PROOF OF BERTRAND'S THEOREM

In this section we use the phase space of orbits in central fields to prove Bertrand's theorem, following Arnold's ideas.

We obtain the phase portrait of the orbits deviating slightly from a stable circular orbit. The circular orbits are obtained when $u' = 0$. Therefore their energy is determined by

$$W(u_0) = E'_0,$$

where u_0 denotes the extreme point for $W(u)$, *i.e.*, $W'(u_0) = 0$ and these orbits will be stable when $W''(u_0) > 0$.

Using the form of $W(u)$ given by Eq. (7), we have that u_0 is determined by

$$u_0 + \alpha V'(u_0) = 0, \tag{14}$$

the energy for a circular orbit is given by

$$\tfrac{1}{2}u_0^2 + \alpha V(u_0) = E'_0 \tag{15}$$

and the stability condition implies that

$$1 + \alpha V''(u_0) > 0. \tag{16}$$

The phase portrait of a circular orbit is the point $(u_0, 0)$.

When the energy is increased by a small amount δE in such a manner as to keep ℓ and therefore α constant, we have that $u = u_0 + \delta u, E' = E'_0 + \delta E'$, then the phase curve is given by

$$\frac{1}{2}\left(\frac{d\,\delta u}{d\theta}\right)^2 + W(u_0 + \delta u) = E'_0 + \delta E'. \tag{17}$$

Expanding the potential up to second order in u around u_0, and using the conditions for circular orbits, we obtain that

$$\frac{1}{2}\left(\frac{d\,\delta u}{d\theta}\right)^2 + \frac{1}{2}(\delta u)^2[1 + \alpha V''(u_0)] = \delta E'. \tag{18}$$

Finally, using the fact that $\delta u = u - u_0$, we have that the phase portrait is an ellipse with center in $(u_0, 0)$ and semiaxes

$$a = \sqrt{2\delta E'}, \tag{19}$$

$$b = \sqrt{\frac{2\delta E'}{1 + \alpha V''(u_0)}}. \tag{20}$$

Now we calculate the angle between the apocenter and the consecutive pericenter, using that

$$\Phi = \tfrac{1}{2}\Delta\Theta. \tag{21}$$

The area enclosed by the phase curve is

$$A = \frac{2\pi\,\delta E'}{\sqrt{1 + \alpha V''(u_0)}}, \tag{22}$$

and from Eq. (11) we have

$$\Phi = \frac{\pi}{\sqrt{2 + \alpha V''(u_0)}}. \tag{23}$$

In general the angle Φ depends of the energy and angular momentum, *i.e.*, $\Phi = \Phi(E_0, \ell)$. We are concerned with finding the functional form for the potential such that Φ would be independent of these quantities. This is satisfied when

$$1 + \alpha V''(u_0) = c,$$

where c is a constant. Using Eq. (14) this condition takes the form

$$\left[1 - u\frac{d\ln V'(u)}{du}\right]_{u_0} = c. \tag{24}$$

We require that Eq. (24) would be valid over the entire domain of u_0. Under these circumstances this equation can be looked on as a differential equation for $V(u)$, it is a simple matter to show that its solution is given by

$$V(u) = \frac{A}{2 - c}u^{2-c};$$

we assume that $c \neq 2$, in this case the orbit is bounded but not closed, therefore we discard this value. For simplicity we rewrite the potential in the following way:

$$V(u) = ku^s, \tag{25}$$

when c is selected as $c = 2 - s > 0$, the inequality comes from the stability condition, given by Eq. (16).

Substituting this value of c into Eq. (23) we have that

$$\Phi = \frac{\pi}{\sqrt{2-s}}, \tag{26}$$

and $\sqrt{2-s}$ must be a rational number if these perturbed orbits are to be closed. Nevertheless, in order for a potential to allow bounded orbits, all of which are closed, it is necessary that Φ be commensurable with π, not only for the orbits obtained by the above perturbative method but for any bounded orbit.

Since Φ given by Eq. (26), is valid for E' close to E'_0 we need to find the general conditions under which all bounded orbits will remain closed.

When the potential is given by Eq. (25), u_0 is determined by Eq. (14) and

$$u_0^{s-2} = -\frac{1}{\alpha k s}. \tag{27}$$

The parameters k and s which appear in Eq. (25) are not independent, as follows from Eq. (27). Due to the fact that $u_0 > 0$ and $\alpha > 0$, it is clear that $ks < 0$, then we need to analyze two cases: i) $k > 0$ and $s < 0$; ii) $k < 0$ and $s > 0$.

In the first case, $V(u)$ goes to zero when u goes to infinity, and it goes to infinity when u goes to zero. The energy for a circular orbit is given by

$$E'_0 = W(u_0) = \frac{u_0^2}{2} + \alpha k u_0^s$$

and using Eq. (27) we have that

$$E'_0 = u_0^2 \left[\frac{1}{2} - \frac{1}{s}\right] > 0, \tag{28}$$

where the inequality is due to the fact that $s < 0$. The one dimensional potential is given by

$$W(u) = \tfrac{1}{2}u^2 + \alpha k u^s. \tag{29}$$

Then for small values of $u, W(u) \approx \alpha k u^s$ and for large values $W(u) \approx \frac{1}{2}u^2$. This behavior of the potential and the fact that $E_0 > 0$, implies that for energy $E' > E'_0$ the orbits are always bounded. This is shown in Fig. 1.

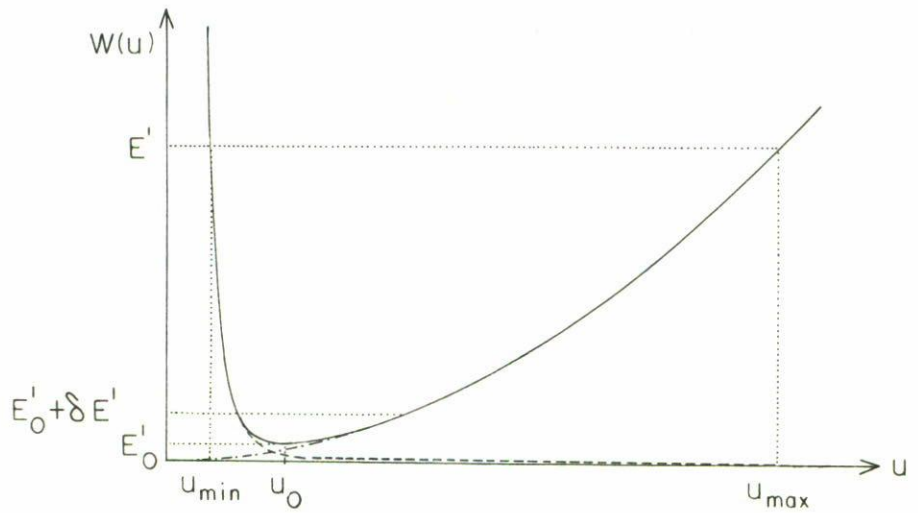

FIGURE 1. The semi-dotted (-·-) curve represents the $\frac{1}{2}u^2$ term; the dotted (···) curve represents the potential $V(u) = ku^s$ with $k > 0$ and $s > 0$, and the solid (—) curve represents the one-dimensional potential $W(u)$.

Now we will analyze the limit of the phase portrait associated with bounded orbits when the energy goes to infinity. The turning points when $E \gg E'_0$, has the following behavior:

$$u_{\min} \to 0, \tag{30}$$

$$\tfrac{1}{2}u^2_{\max} \cong E'. \tag{31}$$

To obtain the limit curve when E' goes to infinity we define the variable $x = \frac{u}{u_{\max}}$, and using the energy equation

$$\frac{1}{2}\left(\frac{du}{d\theta}\right)^2 + \frac{1}{2}u^2 + \alpha k u^s = E'$$

dividing by $u^2_{\max}$ we obtain

$$\frac{1}{2}\left(\frac{dx}{d\theta}\right)^2 + \frac{1}{2}x^2 + u^{s-2}_{\max}\alpha k x^s = \frac{E'}{u^2_{\max}}. \tag{32}$$

Since $s < 0$ it is clear that we can ignore the third term when $E' \to \infty$ and obtain

$$\frac{1}{2}\left(\frac{dx}{d\theta}\right)^2 + \frac{1}{2}x^2 = \frac{1}{2}, \tag{33}$$

or, in terms of u, we have the phase portrait

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = u^2_{\max} = 2E' \tag{34}$$

and we conclude that the limit of the phase portrait when E' goes to infinity is a semicircle of radius $\sqrt{2E'}$. Then the area enclosed when $E' \to \infty$ is given by $A = \pi E'$, and the limit behavior of the angle $\Phi(E')$ can be obtained from Eqs. (11) and (21), thus

$$\lim_{E' \to \infty} \Phi(E') = \frac{1}{2} \frac{\partial(\pi E')}{\partial E'} = \frac{\pi}{2}. \tag{35}$$

Therefore, from Eqs. (26) and (35), the condition for all bounded orbits to be closed is

$$\sqrt{2 - s} = 2 \tag{36}$$

or $s = -2$, yielding the potential

$$V = ku^{-2} = kr^2, \tag{37}$$

which corresponds to isotropic harmonic oscillator.

Now, we analyze the second case in which $k < 0$ and $s > 0$. Due to the stability condition, we only need to analyze values of the parameter s in the range $0 < s < 2$. The one dimensional potential has the form

$$W(u) = \tfrac{1}{2}u^2 - \alpha|k|u^s. \tag{38}$$

For large values of u, the leading term is the first one, *i.e.* $W(u) \sim \frac{1}{2}u^2$, on the other hand for small values of u, the potential goes to zero following the second term, *i.e.* $W(u) \sim -\alpha|k|u^s$. Therefore the minimum value of $W(u)$, which determines the circular orbit, has a negative value, this fact also follows from Eq. (28), which is negative because $s < 2$. This situation is shown in Fig. 2.

All the bounded orbits have energy in the range $0 > E' > E'_0$ and its limit is found when $E' \to 0^-$, like for $E' = 0$ we have only one finite apsidal point, because $u_{\min} \to 0$ when $E' \to 0$, and this implies that for $E' = 0$, we have an open orbit. Thus

$$\lim_{E' \to 0} \Phi(E') = \theta_0,$$

where θ_0 is given by Eq. (13) with $E' = 0$, then

$$\lim_{E' \to 0} \Phi(E') = \int_0^{u_{\max}} \frac{du}{\sqrt{-2W(u)}}; \tag{39}$$

the value of $u_{\max}$ is determined by the condition

$$W(u_{\max}) = 0. \tag{40}$$

Using Eqs. (38) and (40) we obtain that

$$u_{\max}^{s-2} = \frac{1}{2\alpha|k|} \tag{41}$$

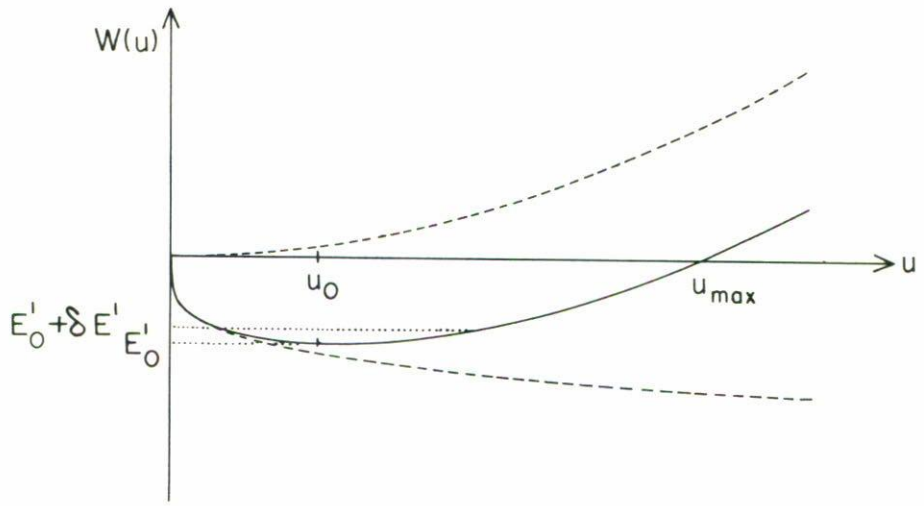

FIGURE 2. The semi-dotted (-·-) curve represents the $\frac{1}{2}u^2$ term; the dotted (···) curve represents the potential $V(u) = ku^s$ with $k < 0$ and $s > 0$, and the solid (—) curve represents the one-dimensional potential $W(u)$.

and the one dimensional potential can be expressed as

$$W(u) = \frac{1}{2}u^2\left[1 - \left(\frac{u}{u_{\max}}\right)^{s-2}\right]. \tag{42}$$

Using Eq. (42) in Eq. (39) we have that

$$\lim_{E'\to 0}\Phi(E') = \int_0^{u_{\max}} \frac{du}{u\sqrt{\left(\frac{u}{u_{\max}}\right)^{s-2} - 1}}. \tag{43}$$

If we make the following change of variable:

$$x^2 = \left(\frac{u}{u_{\max}}\right)^{2-s}, \tag{44}$$

then it is simple to show that

$$\frac{du}{u} = \frac{2}{2-s}\frac{dx}{x}. \tag{45}$$

Introducing Eqs. (44) and (45) into Eq. (43) we obtain

$$\lim_{E'\to 0}\Phi(E') = \frac{2}{2-s}\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2-s}. \tag{46}$$

This result is equal to Eq. (26), only when

$$\sqrt{2-s} = 2 - s; \tag{47}$$

which is satisfied only for $s = 1$. Then the potential for which all bounded orbits are closed is

$$V = -|k|u = -\frac{|k|}{r}, \tag{48}$$

which is the Newtonian potential.

Therefore the only central potentials which give closed orbits are the isotropic harmonic oscillator and the Newtonian potential. This is the statement of Bertrand's theorem.

4. CONCLUDING REMARKS

In Arnold's book [5], the demonstration of Bertrand's theorem can be found when the reader solves a sequence of problems; this work saves that effort. The phase space for orbits is behind the solution of the proposed problems. Our treatment is based on the geometrical of the orbits in the phase space generated by the one dimensional potential $W(u)$. This treatment is summarized in the following steps:

i) We determine the family of potentials, such that the angle for an orbit near to the circular does not depend of u_0, like u_0 is a function of ℓ as follows from Eq. (14), then Φ is independent of the angular momentum, but depends on the potential's parameter s.

ii) The members of the family are separated in two groups, according to their behavior with the energy. The first group contains the potentials which generate orbits that are always bounded for any positive value of the energy E', greater than the energy corresponding to the circular orbits, E'_0. In the second group are included the potentials which generate bounded orbits for negative values of $E' > E'_0$.

iii) We analyze the behavior of $\Phi(E')$ when $E' > E'_0$.

When the first group of potentials is considered, the limit of $\Phi(E')$ when $E' \to \infty$, is given by $\pi/2$. The compatibility of this result with the expression for $\Phi(E')$ near the circular orbits, is satisfied only when $s = -2$, and this corresponds to the harmonic oscillator. On the other hand when the second group is considered, the condition $E' > E'_0$, means that $E' \to 0$, and the evaluation of the limit of $\Phi(E')$ when $E' \to 0$, gives a function of the s-parameter. Impossing the compatibility with $\Phi(E')$ given by Eq. (26) which is valid for values of the energy near E'_0, we obtain that it is only possible when $s = 1$, which corresponds to the Newtonian potential.

Therefore, only for these two cases the orbits are bounded and closed for all values of energy and angular momentum.

Now, it is clear that the ideas behind Arnold's demonstration are based on an asymptotic analysis for $\Phi(E', \ell)$.

First, he found that the potentials with the functional form given by Eq. (25), generate bounded orbits for $E \approx E'_0$ and the angle Φ does not depend on the angular momentum.

Also for these potentials, the value of Φ neither depends on the energy, it only depends on the s-parameter of the potential. For simplicity we denote this values by $\Phi_{\text{cir}}(s)$. Clearly, this result is only valid near the circular orbits.

Second, for the above potentials, the behavior of $\Phi(E', \ell)$ is analyzed when E' is far away from E'_0, and we found that $\Phi(E', \ell)$ does not depend on the angular momentum, it only depends on the s-parameter, denoted by $\Phi_{\text{asym}}(s)$, thus

$$\lim_{E' \gg E'_0} \Phi(E', \ell) = \Phi_{\text{asym}}(s).$$

Third, Arnold introduced the condition that only the potentials which generate bounded orbits for any value of E' and ℓ, it is true that

$$\Phi_{\text{cir}}(s) = \Phi_{\text{asym}}(s)$$

This condition determines some s values, so that selecting those values which also produce a closed orbit. Using these arguments Arnold found the alternative proof of Bertrand's theorem.

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