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On the factorization method and ladder operators

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ABSTRACT. Motivated by the existence of second order ladder operators, we implement an algorithm to find ladder operators for one-dimensional hamiltonians H of order $N \geq 2$, and we show that always it is possible to construct the first order ladder operators, but their more general relation with H is not that of the factorization method.

RESUMEN. Motivados por la existencia de operadores escalera de segundo orden, se propone un algoritmo para hallar operadores escalera de orden $N \geq 2$ para hamiltonianos unidimensionales y se muestra que siempre es posible construir los de primer orden aunque su relación más general con H no es la dada por el método de factorización.

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1. INTRODUCTION

Since Dirac [1], based in the early works of Pauli [2] and Weyl [3], showed that the eigenvalue problems for the one-dimensional harmonic oscillator and the angular momentum could be easily solved by the use of raising and lowering operators, the factorization method (FM) acquired great importance in Quantum Mechanics. Many efforts to deepen on it were realized later on, and the main results were found by Infeld and Hull [4]. Even though there were some later important works about the subject, the general belief was that the factorization method had been entirely explored.

Recently however, Mielnik [5], Nieto [6] and Fernández [7] have shown that the factorization method can provide us still more information. This fact motives us to make a review of the FM's reach.

It is a common prejudice to assume that the ladder operators associated to the eigenfunctions of a differential operator must factorize it. Nevertheless, this idea contains some constrains which restrict the method, and therefore it is not general.

The goal of this paper is to clarify the general relation existent between one second order differential operator H with eigenfunctions ψ_n and eigenvalues λ_n, and its associated ladder operator of index n, showing that in general if we know the ladder operators, it does not mean that we know the factorization of H. This can be the starting point for future research on the real reach of the factorization method.

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2. THE FACTORIZATION METHOD

In its most classical form, the FM turns round about of the following ideas. Consider the usual eigenvalue problem

$$H\psi_n = \lambda_n \psi_n, \tag{1}$$

where H is the second order differential operator

$$H = P(x)\frac{d^2}{dx^2} + Q(x)\frac{d}{dx} + R(x), \tag{2}$$

and the subscript n is discrete ($n = 0, 1, 2, \ldots$). Now, suppose that there exist two first order differential operators A_n^+ and A_n^- [8], called ladder operator, such that they connect two consecutive eigenfunctions of H:

$$A_n^\pm \psi_n = c_n^\pm \psi_{n\pm1}, \tag{3}$$

$c_n^\pm$ being constants. Then, in principle, as some times its argued [4, 9], they could be calculated by demanding that H is factorizable as

$$H = A_{n-1}^+\, A_n^- + k_n^1 = A_{n+1}^- A_n^+ + k_n^2, \tag{4}$$

where k_n^1 and k_n^2 are constants related to $c_n^\pm$ by

$$k_n^1 = \lambda_n - c_{n-1}^+ c_n^-; \qquad k_n^2 = \lambda_n - c_{n+1}^- c_n^+. \tag{5}$$

Assuming that

$$A_n^+ = \alpha_n + \beta_n \frac{d}{dx}, \qquad A_n^- = \gamma_n - \delta_n \frac{d}{dx}, \tag{6}$$

and introducing the last expressions into (4) we obtain

$$\begin{aligned} H &= -\beta_{n-1}\delta_n \frac{d^2}{dx^2} + \left(\beta_{n-1}\gamma_n - \beta_{n-1}\delta_n' - \alpha_{n-1}\delta_n\right)\frac{d}{dx} + \left(\alpha_{n-1}\gamma_n + \beta_{n-1}\gamma_n' + k_n^1\right) \\ &= -\beta_n\delta_{n+1}\frac{d^2}{dx^2} + \left(\beta_n\gamma_{n+1} - \beta_n\delta_{n+1}' - \alpha_n\delta_{n+1}\right)\frac{d}{dx} + \left(\alpha_n\gamma_{n+1} - \delta_{n+1}\alpha_n' + k_n^2\right). \end{aligned} \tag{7}$$

Comparing with Eq. (2), it is easy to note that the functions α_n, β_n, γ_n and δ_n satisfy the non linear equations

$$\begin{aligned} -\beta_{n-1}\delta_n &= P, \\ \beta_{n-1}\gamma_n - \beta_{n-1}\delta_n' - \alpha_{n-1}\delta_n &= Q, \\ \alpha_{n-1}\gamma_n + \beta_{n-1}\gamma_n' + k_n^1 &= R, \\ \alpha_{n-1}\gamma_n - \delta_n\alpha_{n-1}' + k_{n-1}^2 &= R. \end{aligned} \tag{8}$$

If they have solutions, then, the original problem stated by (1) is enormously simplified. This is due to the fact that once one knows a solution for the eigenvalues problem (1), it is always possible to determine all the other solutions, with different subscript, through the ladder operators. So, the problem is reduced to specify the first solution.

To find a first solution of (1), it is sufficient to note that whenever we apply the lower operator A_n^- to one eigenfunction ψ_n, we obtain the immediate inferior eigenfunction ψ_{n-1}. But we can't repeat this procedure forever. It has to exist one eigenfunction, the ground state, ψ_0, such that

$$A_0^- \, \psi_0 = 0. \tag{9}$$

Notice that (9) is a first order differential equation, which always has one solution. In this way, once we know ψ_0, we generate the nth eigenfunction ψ_n performing n successive raising operations on ψ_0, that is

$$\psi_n = \prod_{k=0}^{n-1} \frac{1}{c_k^+} A_k^+ \psi_0. \tag{10}$$

Therefore, the general problem of solving the family of second order differential equations (1) is equivalent to find the solution of the first order equation (9). Here is the power of the FM.

FM may be generalized without any problem to the case when the eigenfunctions depend on several subscripts, like the most general problem to solve in quantum mechanics (the problem with degeneration), and the ladder operator raise or low an index different to n [4,9]. In such a case, the constants k_n^1 and k_n^2 will be functions of the parameter λ_n and the index over which the ladder operators act and the FM technique works in a similar way. However, in this paper we shall restrict our discussion to the FM such as was presented above.

3. Ladder operators of order $N > 1$

Certainly, the relations (4) and (5) are sufficient to ensure that $A_n^\pm$ are ladder operators [4, 9], but it is possible that these relations are not necessary, that is the fact we will try to establish.

If one considers the Eq. (3), it can be noted that this may be summarized by

$$[H, A_n^\pm] = \pm \Delta\lambda_n \, A_n^\pm, \tag{11}$$

with $\Delta\lambda_n = \lambda_{n+1} - \lambda_n$, because from the last expression, the equation (3) can be easily obtained. Nevertheless, Eq. (11) has a more general meaning, because it ensures that $A_n^\pm$ are ladder operators, even though they are not first order operators. Actually this kind of operators are not too strange, because the existence of second order ladder operators has been well established in the context of physical problems by Moshinsky *et al.* [10]. In fact, the Eq. (11) may give us one method to calculate ladder operators whose order is $N \geq 2$, by extending the method used by Morales *et al.* [11] for the case of order 2.

Assuming that $A_n^{\pm}$ exist, and with order N, they may be written as

$$A_n^{\pm} = \sum_{k=0}^{N} a_{n,k}^{\pm}(x) \left(\frac{d}{dx}\right)^k . \tag{12}$$

Now, one can build the expressions

$$HA_n^{\pm} = \sum_{k=0}^{N} \left\{ a_{n,k}^{\pm} P \left(\frac{d}{dx}\right)^{k+2} + \left[2P a_{n,k}^{\pm,(1)} + Q a_{n,k}^{\pm}\right] \left(\frac{d}{dx}\right)^{k+1} + \left(H a_{n,k}^{\pm}\right) \left(\frac{d}{dx}\right)^{k} \right\}, \tag{13}$$

and

$$A_n^{\pm} H = \sum_{k=0}^{N} a_{n,k}^{\pm} \sum_{l=0}^{k} \binom{k}{l} \left[P^{(k-l)} \left(\frac{d}{dx}\right)^{l+2} + Q^{(k-l)} \left(\frac{d}{dx}\right)^{l+1} R^{(k-l)} \left(\frac{d}{dx}\right)^{l} \right], \tag{14}$$

where the derivative of order k over one function f has been written as $f^{(k)}$. The last expression, after changing the order in the sum takes the form

$$A_n^{\pm} H = \sum_{l=0}^{N} \sum_{k=l}^{N} a_{n,k}^{\pm} \binom{k}{l} \left[P^{(k-l)} \left(\frac{d}{dx}\right)^{l+2} + Q^{(k-l)} \left(\frac{d}{dx}\right)^{l+1} R^{(k-l)} \left(\frac{d}{dx}\right)^{l} \right]. \tag{15}$$

Next, when the Eqs. (13) and (15) are introduced into (11), and both left and right sides are compared, we obtain after some algebra the system of equations

$$H a_{n,0}^{\pm} = \pm \Delta\lambda_n \, a_{n,0}^{\pm} + \sum_{k=0}^{N} a_{n,k}^{\pm} \, R^{(k)}, \tag{16}$$

$$2P a_{n,0}^{\pm,(1)} = \pm \Delta\lambda_n \, a_{n,1}^{\pm} + \sum_{l=1}^{N} \left\{ l R^{(l-1)} + Q^{(l)} \right\} a_{n,l}^{\pm} - H a_{n,1}^{\pm}, \tag{17}$$

$$\begin{aligned} 2P a_{n,i-1}^{\pm,(1)} - (i-1) a_{n,i-1}^{\pm} P^{(1)} = \pm \Delta\lambda_n \, a_{n,i}^{\pm} + \sum_{l=i}^{N} \Bigg[\binom{l}{i-2} P^{(l-i+2)} \\ + \binom{l}{i-1} Q^{(l-i+1)} + \binom{l}{i} R^{(l-i)} \Bigg] a_{n,l}^{\pm} - H a_{n,i}^{\pm}; \end{aligned} \tag{18}$$

being $1 \leq i \leq N+1$. Note that Eq. (18) itself is a first order equation for $a_{n,i-1}^{\pm}$, in function of $a_{n,l}^{\pm}$ for $l \geq i$. Setting $i = N+1$ we find

$$2P a_{n,N}^{\pm,(1)} - N a_{n,N}^{\pm} P^{(1)} = 0, \tag{19}$$

which is easy to integrate, to obtain

$$a_{n,N}^{\pm} = P^{N/2} + c_{n,N}^{\pm}, \tag{20}$$

where $c_{n,N}^{\pm}$ is an integration constant. Once we know $a_{n,N}^{\pm}$, it is an easy task find $a_{n,N-1}^{\pm}$ using again the Eq. (18), but now setting $i = N$ to obtain

$$2Pa_{n,N-1}^{\pm,(1)} - (N-1)a_{n,N-1}^{\pm}P^{(1)} = \left[\pm\Delta\lambda_n + \frac{N(N-1)}{2}P^{(2)} + NQ^{(1)} + R - H\right] a_{n,N}^{\pm}, \tag{21}$$

which is a first order linear equation for $a_{n,N-1}^{\pm}$. Once we solve (21), we can iterate this procedure for $a_{n,k}^{\pm}$, $k = N-2, \ldots, 1$. In all of the cases the differential equation for $a_{n,k}^{\pm}$ will be of first order. Finally, $a_{n,0}^{\pm}$ can be calculated from Eq. (17). So, the Eq. (16) will be a constrain over the method, like that established for the FM.

As an example we solve the system (16)–(18) for $N = 2$ for the harmonic oscillator's hamiltonian

$$H = -\frac{d^2}{dx^2} + x^2, \tag{22}$$

whose eigenvalues are $\lambda_n = 2n + 1$. Suppose that exist a second order ladder operator for H, then one may write it down as

$$A^{\pm} = a_0^{\pm} + a_1^{\pm}\frac{d}{dx} + a_2^{\pm}\frac{d^2}{dx^2}. \tag{23}$$

Therefore, from (20) we obtain a constant $a_2^{\pm}$, for which we choose $a_2^{\pm} = 1$. Now the equation for $a_1^{\pm}$ is

$$a_1^{\pm,(1)} = \mp\frac{\Delta}{2} \quad \Rightarrow \quad a_1^{\pm} = \mp\frac{\Delta}{2}x + c_1, \tag{24}$$

where Δ is the distance between the levels connected by $A^{\pm}$. Taking $a_1^{\pm}$ and $a_2^{\pm}$ into (17) one obtain

$$a_0^{\pm,(1)} = \left[\left(\frac{\Delta}{2}\right)^2 - 2\right] x \mp c_1\frac{\Delta}{2}. \tag{25}$$

After the integration

$$a_0^{\pm} = \left(\frac{\Delta^2}{8} - 1\right) x^2 \mp \frac{\Delta}{2}c_1 x + c_0. \tag{26}$$

In this way we have calculated all the functions $a_i^{\pm}$. Finally we must use Eq. (16) in order to fix the constants c_1 and c_0. It can be put in the form

$$\pm\Delta\left(\frac{\Delta^2}{8}-2\right)x^2+c_1\left(2-\frac{\Delta^2}{2}\right)x\pm c_0\Delta+\left(\frac{\Delta}{2}\right)^2=0. \tag{27}$$

Comparing the coefficients with zero, one finds $\Delta=4$, $c_1=0$ and $c_0=\mp 1$. Thus, we have obtained the second order operator

$$A^{\pm}=\frac{d^2}{dx^2}\mp 2x\frac{d}{dx}+x^2\mp 1 \tag{28}$$

that raises $(+)$ or lowers $(-)$ the eigenstates φ_n of H two levels:

$$\varphi_{n\pm 2}\propto A^{\pm}\varphi_n.$$

Once we know $A_n^{\pm}$ with order N, we must emphasize that H is not factorizable by this ladder operators. This fact is clear if we consider the products $A_{n-1}^{+}A_n^{-}$ and $A_{n+1}^{-}A_n^{+}$. Both are differential operator whose order is $N+1$ at least, while H has order 2 only. Hence, the condition (4) will implicate that *the factorization of H is possible only if $N=1$*. Therefore, the FM is useful only when the ladder operators have order 1, because, if $N>1$ then obviously

$$A_{n-1}^{+}A_n^{-}\neq H-k_n^1 \tag{29}$$

and

$$A_{n+1}^{-}A_n^{+}\neq H-k_n^2, \tag{30}$$

But in such a case we may use the method explained before.

Moreover, this conclusion is independent of the method employed to determinate the ladder operators. In fact, we may easily extend it by saying that every L-th order differential operator, may be factored only into the form (4) by operators whose orders N_1 and N_2 are such that $L=N_1+N_2$.

4. General relation of $A_n^{\pm}$ with H

Now then, once we have established the consequence of the existence of ladder operators with order $N>1$, over the FM, we must point out that the FM's strength resides in the reduction of the order of the problem [Eqs. (1) and (9)], in opposition to the case when the order of A_n^{-} is greater than 1. In such a case, we will raise the order of the problem, and in principle, without advantage. However, we always may use the Eq. (3) together with the eigenvalue problem (1), to reduce the differential order of $A_n^\pm$. In order to make clear this point we begin by taking Eq. (1), and differentiating $(l-2)$ times

$$\left(\frac{d}{dx}\right)^{l-2} H\psi_n = \lambda_n \left(\frac{d}{dx}\right)^{l-2} \psi_n. \tag{31}$$

Since H has order two, from (31), the l–th derivative $\psi_n^{(l)}$ can be put as a function of the derivatives whose order is lower than l. So, the last relation may be rewritten as

$$\left(\frac{d}{dx}\right)^{l} \psi_n = F_{ln}\left(\left(\frac{d}{dx}\right), \ldots, \left(\frac{d}{dx}\right)^{l-1}\right)\psi_n, \tag{32}$$

where F_{ln} is a linear differential operator. Hence, the derivative of order l over ψ_n in (3), may be replaced in favor of its lesser orders derivatives.

In this way, all the derivatives with order $l > 1$ may be substituted into (3) to obtain first order ladder operators to ψ_n. For example, the operator considered before in Eq. (28) applied to one eigenfunction φ_n of H in (22), can be transformed using

$$\frac{d^2}{dx^2}\varphi_n = \left(x^2 - 2n - 1\right)\varphi_n \tag{33}$$

into

$$A_n^\pm = \mp 2x\frac{d}{dx} + 2x^2 - (2n+1) \mp 1. \tag{34}$$

Note that in this case $N = 2$ and $F_{2n} = (x^2 - 2n - 1)$.

Thus, we have established that *always it is possible to reduce the order of the problem* (1), even though the original ladder operators have order $N \geq 2$. Therefore it is sufficient to consider the first order ladder operators.

On the other hand, with respect to Eq. (11), it is important to note that this equation contains an intrinsic constrain, associated with the dependence in n of $A_n^\pm$. To be strict, is enough to demand

$$[H, A_n^\pm]\psi_n = \pm\Delta\lambda_n\, A_n^\pm\psi_n \tag{35}$$

to ensure the validity of Eq. (3). In such a case we will have a different operator by each state, as it happens with $A_n^\pm$ in (34), which does not satisfy Eq. (11).

Let us now examine the effect of the previous results on the general relation between $A_n^\pm$ and H.

Because of (29), (30) and (35), the procedure described above may not ensure in general the relation (4). For example, consider the Laguerre's operator defined by

$$HL_n = \left[x\frac{d^2}{dx^2} + (1-x)\frac{d}{dx}\right]L_n = -nL_n, \tag{36}$$

and the recurrence relations [12]

$$\begin{aligned} (n+1)L_{n+1} &= (2n+1-x)L_n - nL_{n-1}, \\ x\frac{d}{dx}L_n &= nL_n - nL_{n-1}. \end{aligned} \tag{37}$$

Using (37) together with (36), we may build the second order ladder operators

$$\begin{aligned} a^+ &= \left[x\frac{d^2}{dx^2} + (1-2x)\frac{d}{dx} + (x-1)\right], \\ a^- &= \left[x\frac{d^2}{dx^2} + \frac{d}{dx}\right], \end{aligned} \tag{38}$$

which can be also obtained by the method showed in the last section [11].

Using (36) into the last one, or directly from (37), we can obtain the first order ladder operators

$$A_n^- = \left[n - x\frac{d}{dx}\right], \qquad A_n^+ = \left[(n+1-x) + x\frac{d}{dx}\right]. \tag{39}$$

It is important to note that

$$A_{n+1}^- A_n^+ = -x^2\frac{d^2}{dx^2} + x(x-1)\frac{d}{dx} - nx + (n+1)^2 \tag{40}$$

and

$$A_{n-1}^+ A_n^- = -x^2\frac{d^2}{dx^2} + x(x-1)\frac{d}{dx} + n(n-x). \tag{41}$$

Hence, even though $A_n^\pm$ are of first order, H can not be factored by these into the form (4). The same happens in the case of the ladder operators in (34) and the hamiltonian for the harmonic oscillator [Eq. (22)]. We are thus leading to the following conclusion: *The more general relation between a second order operator H with its first order ladder operators $A_n^\pm$ is not of the form* (4). Instead, we may write

$$\begin{aligned} D_n &\equiv A_{n+1}^- A_n^+ = H + O_n, \\ E_n &\equiv A_{n-1}^+ A_n^- = H + B_n, \end{aligned} \tag{42}$$

where O_n and B_n are new second order operators, which have the same eigenfunctions of H

$$\begin{aligned} O_n\psi_n &= \left(c_{n+1}^- c_n^+ - \lambda_n\right)\psi_n, \\ B_n\psi_n &= \left(c_{n-1}^+ c_n^- - \lambda_n\right)\psi_n. \end{aligned} \tag{43}$$

Obviously, when O_n and B_n are constants, we have the case that of the FM.

Finally, it is interesting to note that in the case of Laguerre's operator the relation of $A_n^{\pm}$ with H may be written as

$$\begin{pmatrix} D_n \\ E_n \end{pmatrix} = f(x)\,(H - \lambda_n) + \begin{pmatrix} d_n \\ e_n \end{pmatrix}, \tag{44}$$

where d_n (e_n) is the eigenvalue of D_n (E_n), and $\lambda_n = -n$, with $f(x) = -x$. But this relation might not be general.

5. Concluding remarks

The results discussed above, are summarized in the following affirmation: *Not all operator H is factored by its ladder operators*, as it happened in the former examples, but the relation (44) might establish new lines of research.

It is important to note that the above developments may be generalized when H is an N^{th} order differential operator. In this sense the above affirmation is general. Moreover, when one has a Nth order operator, it is in general sufficient with consider the ladder operators whose orders N_1, N_2 satisfy $N = N_1 + N_2$, because the other cases $(N < N_1 + N_2)$ always can be transformed into the last one.

On the other hand, our conclusion has the following interesting property: when the relation between $A_n^{\pm}$ and H is not trivial, that is, when O_n and B_n are not constants, we have the set

$$\{H,\, D_n,\, E_n\, O_n,\, B_n\} \tag{45}$$

of operators, which have the common eigenfunctions $\{\psi_n\}$ on the same index. Nevertheless, the operators in (45) do not commute, because they are depending in the index n as the relation (35) shows, since we can not separate the action of the commutator and the specific common eigenfunction. Strictly speaking, the operators in (45) share only one eigenvector by each value of n, and then, the set (45) does not generate a space of commuting operators. Hence, we must be careful in the use of them.

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References

1. P.A.M. Dirac, *Principles of Quantum Mechanics*, Clarendon Press, Oxford, Second Ed. (1935).
2. W. Pauli, *Z. Phys.* **36** (1926) 336.

3. H. Weyl, *The theory of groups and Quantum Mechanics*, 2nd Ed. E.P. Dotton and Company, Inc., New York (1931).
4. L. Infeld and T.E. Hull, *Rev. Mod. Phys.* **23** (1951) 21.
5. B. Mielnik, *J. Math. Phys.* **25** (1984) 3387.
6. M.M. Nieto, *Phys. Lett.* **145B** (1984) 208.
7. D.J. Fernández, *Lett. Math. Phys.* **8** (1984) 337.
8. There is only one case such that $A^{\pm}$ are independent on the index and factored to H. It is that which their eigenvalues satisfy $\Delta_n \equiv \lambda_{n+1} - \lambda_n = \text{const}$. It is clear from $[A^+, A^-] = \Delta_n$
9. W. Miller, Jr., *Lie Theory and Special Functions*, Academic Press, New York (1968).
10. M. Moshinsky, J. Patera and P. Winternitz, *J. Math. Phys.* **16** (1975) 82; E. Chacón, D. Levi and M. Moshinsky, *J. Math. Phys.* **17** (1976) 1919.
11. J. Morales, J.J. Peña, M. Sánchez and J. López Bonilla, *Int. J. Quant. Chem.* **S25** (1991) 155.
12. G. Arfken, *Mathematical Methods for Physicists*, 2nd Ed., Academic Press, New York (1970).