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Duality rotations in the linearized Einstein theory

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Abstract. Using the fact that the solutions of the Einstein vacuum field equations linearized about the Minkowski space-time can be expressed in terms of two real scalar potentials, the effect of a duality rotation on the metric perturbations is given. It is shown that, in the linearized Einstein theory, the field of a gravitomagnetic mass and of an accelerating mass are related by means of a duality rotation with the field of a static ordinary mass and of a rotating mass, respectively.

Resumen. Usando el hecho de que las soluciones de las ecuaciones de Einstein para el vacío linealizadas alrededor del espacio-tiempo de Minkowski pueden ser expresadas en términos de dos potenciales escalares reales, se da el efecto de una rotación de dualidad sobre las perturbaciones métricas. Se muestra que, en la teoría de Einstein linealizada, el campo de una masa gravitomagnética y de una masa acelerada están relacionadas por medio de una rotación de dualidad con el campo de una masa ordinaria estática y de una masa rotante, respectivamente.

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1. Introduction

As is well known, the source-free Maxwell equations are invariant under the transformation

$$\mathbf{E}' = \mathbf{B}, \qquad \mathbf{B}' = -\mathbf{E}, \tag{1}$$

in the sense that the fields $\mathbf{E}'$, $\mathbf{B}'$ satisfy the source-free Maxwell equations if the fields $\mathbf{E}$, $\mathbf{B}$ do. Then, the linearity of the source-free Maxwell equations implies that they are also invariant under the *duality rotations*

$$\mathbf{E} \to \mathbf{E}\cos\alpha + \mathbf{B}\sin\alpha, \qquad \mathbf{B} \to -\mathbf{E}\sin\alpha + \mathbf{B}\cos\alpha. \tag{2}$$

On the other hand, the invariance of the Maxwell equations with sources under the duality rotations (2) requires the existence of electric and magnetic charges.

In Einstein's theory of gravity, the gravitational field is represented by the curvature of the space-time. The curvature tensor of a solution of the Einstein vacuum field equations linearized about the Minkowski metric can be decomposed into "electric" and "magnetic" parts

$$E_{ij} \equiv K_{0i0j}, \qquad B_{ij} \equiv -\tfrac{1}{2}\varepsilon_{ikl}K_{kl0j}, \tag{3}$$

where $K_{\alpha\beta\gamma\delta}$ denotes the curvature tensor to first order in the metric perturbation, that satisfy the homogeneous equations (see, *e.g.*, Refs. [1,2])

$$\begin{aligned} \partial_i B_{ij} &= 0, \qquad \varepsilon_{ijk}\partial_j E_{kl} = -\frac{1}{c}\frac{\partial}{\partial t}B_{il}, \\ \partial_i E_{ij} &= 0, \qquad \varepsilon_{ijk}\partial_j B_{kl} = \frac{1}{c}\frac{\partial}{\partial t}E_{il}, \end{aligned} \tag{4}$$

which are invariant under a transformation analogous to (1):

$$E'_{ij} = B_{ij}, \qquad B'_{ij} = -E_{ij}. \tag{5}$$

In some cases (*e.g.*, in the interaction of a quantum system with the electromagnetic field), it is necessary to represent the electromagnetic field by means of the potentials ϕ, $\mathbf{A}$. Similarly, dealing with the gravitational field, the metric is usually considered as the basic field, rather than the curvature which is given in terms of second derivatives of the metric.

In this paper we give the effect of the duality rotations on the electromagnetic potentials and on the metric perturbations of the Minkowski space-time. We make use of the fact that the solution of the source-free Maxwell equations and of Eqs. (4) can be written in terms of two real scalar potentials (see, *e.g.*, Refs. [1,3,4]). As we shall show below, by contrast with Eqs. (1) and (5), the duality rotations on the electromagnetic potentials and on the metric perturbations are nonlocal. In the examples considered here, we find the effect of the duality rotations on the linearized Schwarzschild and Kerr metrics. The analysis of the metric perturbations is done by means of the so-called Taub numbers [5,6], which give the conserved quantities of the sources of the metric perturbations associated with the symmetries of the background metric. Throughout this paper, lower-case Latin indices i, j, ..., run from 1 to 3 and lower-case Greek indices α, β, ..., run from 0 to 3.

2. Debye potentials and duality rotations

The solution of the source-free Maxwell equations can be written in terms of two real scalar potentials, ψ_{E} and ψ_{M}, which satisfy the wave equation, according to

$$\begin{aligned} \phi &= -\frac{1}{r}\mathbf{r}\cdot\nabla(r\psi_{\mathrm{E}}), \\ \mathbf{A} &= \mathbf{r}\frac{1}{c}\frac{\partial\psi_{\mathrm{E}}}{\partial t} - \mathbf{r}\times\nabla\psi_{\mathrm{M}}, \end{aligned} \tag{6}$$

up to gauge transformations [3]. Hence, the electromagnetic fields are given by

$$\begin{aligned} \mathbf{E} &= \frac{1}{c}\frac{\partial}{\partial t}\mathbf{r}\times\nabla\psi_{\mathrm{M}} - \nabla\times(\mathbf{r}\times\nabla\psi_{\mathrm{E}}), \\ \mathbf{B} &= -\frac{1}{c}\frac{\partial}{\partial t}\mathbf{r}\times\nabla\psi_{\mathrm{E}} - \nabla\times(\mathbf{r}\times\nabla\psi_{\mathrm{M}}), \end{aligned} \tag{7}$$

where we have used the fact that $\psi_{\rm E}$ obeys the wave equation. From Eqs. (7) it follows that the duality rotation (1) can be obtained by making

$$\psi'_{\rm E} = \psi_{\rm M}, \qquad \psi'_{\rm M} = -\psi_{\rm E}. \tag{8}$$

Thus, by expressing the potentials, ϕ, $\mathbf{A}$, of a given electromagnetic field in the form (6), then

$$\begin{aligned} \phi' &= -\frac{1}{r}\mathbf{r}\cdot\nabla(r\psi_{\rm M}), \\ \mathbf{A}' &= \mathbf{r}\frac{1}{c}\frac{\partial\psi_{\rm M}}{\partial t} + \mathbf{r}\times\nabla\psi_{\rm E}, \end{aligned} \tag{9}$$

are potentials for the duality rotated electromagnetic fields $\mathbf{E}'$, $\mathbf{B}'$ [Eq. (1)]. Note that, while at the level of the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ the duality rotations are rather trivial [Eqs. (1) and (2)], the relation between the potentials ϕ, $\mathbf{A}$ and ϕ', $\mathbf{A}'$ is *nonlocal* (see the example below).

Even though expressions (6) are obtained looking for solutions of the source-free Maxwell equations, the electromagnetic field produced by charge and current distributions can also be expressed in the form (6) by considering solutions of the wave equation with singularities. For instance, the field produced by a static point charge q is given by the potentials

$$\phi = \frac{q}{r}, \qquad \mathbf{A} = \mathbf{0}, \tag{10}$$

which can be represented in the form (6) assuming that

$$\psi_{\rm M} = 0, \tag{11}$$

and $r\psi_{\rm E} = -q\ln r + f(\theta,\varphi)$. By requiring that $\psi_{\rm E}$ be a solution of the wave equation one finds that $L^2 f = q$, where

$$\mathbf{L} \equiv -i\mathbf{r}\times\nabla. \tag{12}$$

Choosing $f = q\ln\sin\theta$ we obtain

$$\psi_{\rm E} = \frac{q}{r}\ln\frac{\sin\theta}{r}. \tag{13}$$

Substituting Eqs. (11) and (13) into Eqs. (9) we find

$$\phi' = 0, \qquad \mathbf{A}' = \frac{q\cot\theta}{r}\,\hat{e}_\varphi, \tag{14}$$

which are potentials for the field of a magnetic monopole [*cf.* Eqs. (10)]. (Note that the most general solution of $L^2 f = q$ is given by $f = q\ln\sin\theta + g(\theta,\varphi)$, where $L^2 g = 0$, which leads to potentials equivalent to (14) up to a gauge transformation.)

As shown in Ref. [7], the solution of the Einstein vacuum field equations linearized about the Minkowski metric is given by

$$\begin{aligned} h_{00} &= -2\left(\frac{\partial^2}{\partial r^2} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) r^2 \psi_{\mathrm{E}}, \\ h_{0j} &= -4\frac{x_j}{r}\frac{\partial}{\partial r} r^2 \frac{1}{c}\frac{\partial \psi_{\mathrm{E}}}{\partial t} + 2iL_j \frac{1}{r}\frac{\partial}{\partial r} r^2 \psi_{\mathrm{M}}, \\ h_{jk} &= -2\delta_{jk}\left(\frac{\partial^2}{\partial r^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) r^2 \psi_{\mathrm{E}} - 4x_j x_k \frac{1}{c^2}\frac{\partial^2 \psi_{\mathrm{E}}}{\partial t^2} + 4i x_{(j} L_{k)} \frac{1}{c}\frac{\partial \psi_{\mathrm{M}}}{\partial t}, \end{aligned} \tag{15}$$

modulo the gauge transformations

$$h_{\alpha\beta} \to h_{\alpha\beta} - \partial_\alpha \xi_\beta - \partial_\beta \xi_\alpha, \tag{16}$$

where ψ_{E} and ψ_{M} are real solutions of the wave equation, the x_i are cartesian coordinates, L_i are the cartesian components of the operator $\mathbf{L}$ [Eq. (12)] and the parenthesis denote symmetrization on the indices enclosed. Then one finds that the electric and magnetic parts of the curvature tensor to first order in the metric perturbation are

$$\begin{aligned} E_{ij} &= \frac{1}{c}\frac{\partial}{\partial t} U_{ij}(\psi_{\mathrm{M}}) - V_{ij}(\psi_{\mathrm{E}}), \\ B_{ij} &= -\frac{1}{c}\frac{\partial}{\partial t} U_{ij}(\psi_{\mathrm{E}}) - V_{ij}(\psi_{\mathrm{M}}), \end{aligned} \tag{17}$$

[*cf.* Eqs. (7)], where [1, 4]

$$U_{ij} \equiv iL_i X_j + iL_j X_i, \qquad V_{ij} \equiv \varepsilon_{imn} \partial_m U_{nj},$$

$$\mathbf{X} \equiv i\nabla \times \mathbf{L} - \nabla.$$

(As in the case of the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$, the symmetric traceless tensor fields E_{ij} and B_{ij} are gauge-invariant.) From Eqs. (17) it is clear that the transformation

$$\psi'_{\mathrm{E}} = \psi_{\mathrm{M}}, \qquad \psi'_{\mathrm{M}} = -\psi_{\mathrm{E}} \tag{18}$$

yields the duality rotation (5). (Of course, we can also consider duality rotations analogous to that given in Eqs. (2), for an arbitrary angle α.)

The solutions to the Einstein field equations with sources linearized about the Minkowski metric can also be expressed in the form (15), modulo the gauge transformations (16), by considering solutions of the wave equation with singularities. For instance, the Schwarzschild solution linearized with respect to the mass parameter M, corresponds to the metric perturbation

$$h_{00} = \frac{2GM}{c^2 r}, \qquad h_{0i} = 0, \qquad h_{ij} = \frac{2GM}{c^2 r^3} x_i x_j,$$

which, by means of a gauge transformation (16) with $\xi_0 = 0$, $\xi_i = -GMx_i/c^2r$, is transformed into

$$h_{00} = \frac{2GM}{c^2r}, \qquad h_{0i} = 0, \qquad h_{ij} = \frac{2GM}{c^2r}\,\delta_{ij}. \tag{19}$$

This metric perturbation can be written in the form (15) with

$$\psi_{\rm E} = \frac{GM}{c^2r}\ln\frac{\sin\theta}{r}, \qquad \psi_{\rm M} = 0, \tag{20}$$

[*cf.* Eq. (13)]. Then, one easily finds that the metric perturbation generated by the potentials

$$\psi'_{\rm E} = 0, \qquad \psi'_{\rm M} = -\frac{GM}{c^2r}\ln\frac{\sin\theta}{r}, \tag{21}$$

obtained from Eqs. (20) by means of the duality rotation (18), is given by

$$h'_{00} = 0, \qquad h'_{0j} = -\frac{2GM}{c^2r}\cot\theta\, iL_j\theta, \qquad h'_{jk} = 0,$$

i.e.,

$$ds^2 = -c^2\,dt^2 + dr^2 + r^2\,(d\theta^2 + \sin^2\theta\,d\varphi^2) - \frac{4GM}{c}\cos\theta\,d\varphi\,dt. \tag{22}$$

The metric (22) is the Taub-NUT solution, which represents the gravitational field of a spherically symmetric source with ordinary and gravitomagnetic mass (see, *e.g.*, Ref. [8]),

$$ds^2 = -U^{-1}\,dr^2 + (2l)^2U\,(d\psi + \cos\theta\,d\varphi)^2 + (r^2 + l^2)(d\theta^2 + \sin^2\theta\,d\varphi^2),$$

where

$$U \equiv -1 + \frac{2(mr + l^2)}{r^2 + l^2},$$

to first order in l, when $m = 0$, making the identifications $c\,dt = 2l\,d\psi$ and $l = GM/c^2$. (The parameters m and l are related to the mass and the gravitomagnetic mass, respectively.)

Similarly, starting from the Kerr solution, one finds that in the case of a rotating mass, in addition to (19), there is a contribution

$$h_{00} = 0, \qquad h_{0i} = \frac{2G}{c^3r^2}\varepsilon_{ijk}S_jn_k, \qquad h_{ij} = 0, \tag{23}$$

where S_j are the components of the angular momentum of the rotating mass and $n_k \equiv x_k/r$. The metric perturbation (23) is of the form (15) with

$$\psi_{\rm E} = 0, \qquad \psi_{\rm M} = -\frac{G}{c^3r^2}S_jn_j\ln r + \frac{H(\theta,\varphi)}{r^2}, \tag{24}$$

where, in order that $\psi_{\rm M}$ be a solution of the wave equation, $H(\theta,\varphi)$ must satisfy

$$L^2H - 2H = \frac{3G}{c^3}S_jn_j$$

(the explicit form of H will not be required in what follows). Then, the metric perturbation generated by

$$\psi'_{\rm E} = -\frac{G}{c^3 r^2} S_j n_j \ln r + \frac{H(\theta,\varphi)}{r^2}, \qquad \psi'_{\rm M} = 0 \tag{25}$$

is

$$h'_{00} = -\frac{2G}{c^3 r^2} S_j n_j, \qquad h'_{0i} = 0, \qquad h'_{ij} = -\frac{2G}{c^3 r^2} S_k n_k \delta_{ij}. \tag{26}$$

In order to give an interpretation of the metric perturbation (26), in the next section we shall make use of the Taub numbers [5, 6].

3. Taub numbers of metric perturbations

The Einstein field equations linearized about the Minkowski metric are given by

$$G_{\alpha\beta} = -\frac{8\pi G}{c^4} T_{\alpha\beta}, \tag{27}$$

where

$$G_{\alpha\beta} \equiv \tfrac{1}{2}\{\partial^\gamma \partial_\gamma h_{\alpha\beta} - \partial_\alpha \partial^\gamma h_{\gamma\beta} - \partial_\beta \partial^\gamma h_{\gamma\alpha} + \partial_\alpha \partial_\beta h + \eta_{\alpha\beta}(\partial^\gamma \partial^\delta h_{\gamma\delta} - \partial^\gamma \partial_\gamma h)\}, \tag{28}$$

$$h \equiv h_\alpha{}^\alpha, \tag{29}$$

$T_{\alpha\beta}$ is the energy-momentum tensor of the sources to first order in the metric perturbation and the indices are raised and lowered by means of $(\eta_{\alpha\beta}) = {\rm diag}(-1,1,1,1) = (\eta^{\alpha\beta})$. The tensor field $G_{\alpha\beta}$ satisfies $\partial^\alpha G_{\alpha\beta} = 0$, identically (*i.e.*, for any $h_{\alpha\beta}$) therefore, if K^α is a Killing vector of the Minkowski metric,

$$\partial_\alpha K_\beta + \partial_\beta K_\alpha = 0, \tag{30}$$

then the contraction $G^{\alpha\beta} K_\beta$ satisfies

$$\partial_\alpha (G^{\alpha\beta} K_\beta) = 0. \tag{31}$$

The continuity equation (31) implies that

$$\tau(K) \equiv \frac{c^3}{8\pi G} \int G^{0\alpha} K_\alpha \, dv = -\frac{1}{c} \int T^{0\alpha} K_\alpha \, dv, \tag{32}$$

where the integral is taken over a hypersurface $t = $ const., is a constant, which represents the component along K^α of the *four-momentum of the sources* of the metric perturbation. The constant $\tau(K)$ is referred to as the Taub number of the metric perturbation associated with the Killing vector K^α.

A straightforward computation shows that

$$\partial_\alpha U^{\alpha\beta} = -G^{\alpha\beta} K_\alpha \tag{33}$$

where [5, 6]

$$U^{\alpha\beta} \equiv K^{[\alpha}\partial_\gamma h^{\beta]\gamma} - K^{[\alpha}\partial^{\beta]} h - \tfrac{1}{2} h \partial^\alpha K^\beta - K^\gamma \partial^{[\alpha} h^{\beta]}{}_\gamma - h_\gamma{}^{[\alpha}\partial^{\beta]} K^\gamma, \tag{34}$$

and the square brackets denote antisymmetrization on the indices enclosed. Then, from Eqs. (32) and (33), making use of the Gauss theorem one finds that

$$\tau(K) = -\frac{c^3}{8\pi G}\int \partial_\alpha U^{\alpha 0}\, dv = \frac{c^3}{8\pi G}\lim_{r\to\infty}\int U^{0i} n_i\, da, \tag{35}$$

where the last integral is taken over a sphere $r = \text{const.}$

For instance, substituting Eqs. (23) and $K_i = \varepsilon_{imj} x_j$, $K_0 = 0$ (which are the components of a Killing vector corresponding to spatial rotations about the x_m-axis) into Eq. (34) one obtains $U^{0i} n_i = 3G(S_m - S_i n_i n_m)/(c^3 r^2)$. Hence, Eq. (35) gives $\tau(K) = S_m$, in accordance with the meaning assigned to S_i. On the other hand, the Taub numbers for the metric perturbation (23) associated with translations and boosts turn out to be zero.

In a similar manner, one finds that the Taub numbers of the metric perturbation (26) are zero for the Killing vectors corresponding to translations and spatial rotations and that $\tau(K) = -S_m$, when K corresponds to boosts along the x_m-axis. Thus, we conclude that the metric perturbation (26) is produced by an accelerated mass, with the acceleration being parallel to $\mathbf{S}$. (It may also be noticed that the components of the curvature with spin weight ± 2, which correspond to Ψ_0 and Ψ_4 in the Newman-Penrose notation, of the metric perturbations (19), (22), (23) and (26) vanish, hence, according to Wald's results [9], these perturbations can only contain ordinary mass, gravitomagnetic mass, rotation and acceleration parameters.)

It would be desirable to relate the metric perturbation (26) with an exact solution of Einstein's equations, as in the case of Eq. (22); however, although there exist exact solutions of the Einstein field equations that represent the field of a uniformly accelerating point mass (see, *e.g.*, Refs. [10, 11] and the references cited therein), it is rather difficult to find the corresponding linearized solution (note that the magnitude of the vector $\mathbf{S}$ appearing in Eq. (26) is related to the inverse of the acceleration).

4. Concluding remarks

The Plebański-Demiański solution of the Einstein-Maxwell equations [12] contains, besides the cosmological constant, six parameters that group in a natural way into three complex combinations, $m + in$, $a + ib$, $e + ig$, which correspond to mass, gravitomagnetic mass, angular momentum per unit mass, acceleration, and electric and magnetic charge. The examples considered here show that, in the linearized Einstein theory, the real and imaginary parts of each of these complex combinations are mixed by the duality rotations.

It may be noticed that the potentials [Eq. (24)] for the metric perturbation (23), which has a magnetic-dipole character, can be obtained by differentiating the potentials (21), which generate the gravitational field of the analogue of a magnetic monopole; this means that the linearized Kerr solution is equivalent to the field of two gravitomagnetic masses of the same magnitude and opposite signs placed at opposite sides of an ordinary point mass.

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