Revista Mexicana de Física **43**, *No. 1 (1997) 93–107*

Parametric noise induced birefringence in nematic liquid crystals

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Recibido el 28 de marzo de 1996; aceptado el 11 de julio de 1996

ABSTRACT. The effects of external fluctuations in a laser beam interacting with a nematic liquid crystal film are studied. We model the fluctuations in the intensity and amplitude of the incident optical field and derive the associated stochastic dynamics of the director's reorientation. When the parametric noise is in the intensity, the dynamics is described by a linear Langevin-like equation with gaussian white or colored noise. In contrast, the case of a fluctuating amplitude leads to a stochastic equation with nonlinear (quadratic) colored noise. By using a systematic approximation method we calculate the noise induced birefringence for both cases in the final stages of the reorientation process in the absence of hydrodynamical backflows. Numerical estimates of this quantity for 5CB indicate that it may be quite large and that it should be readily measurable. Actually, we show that this noise induced birefringence may be comparable, and even larger, than its value in the purely deterministic case. In this sense our work suggests a new effect and new experiments to be performed. The limitations and some generalizations of our model are also discussed.

RESUMEN. Se estudian los efectos producidos por fluctuaciones externas en un haz laser que interacciona con un cristal líquido nemático. Presentamos un modelo que describe las fluctuaciones en la intensidad o en la amplitud del campo óptico incidente y describimos la dinámica estocástica de la reorientación del director. Mostramos que cuando el ruido está presente en la intensidad, la dinámica obedece una ecuación tipo Langevin con ruido blanco gausiano o de color. En cambio, el caso en que la amplitud fluctúa, se describe por una ecuación estocástica con ruido no lineal (cuadrático) de color. Utilizando un método de aproximación sistemático calculamos la birrefringencia inducida por el ruido en ambos casos en las etapas finales del proceso de reorientación, cuando los contraflujos hidrodinámicos son despreciables. Las estimaciones del valor de la birrefringencia inducida para 5CB muestran que ésta puede ser apreciable y por lo tanto medible. De hecho, ésta cantidad puede ser comparable y aún mayor que la inducida en el caso determinista puro. En este sentido nuestro trabajo sugiere un nuevo efecto físico y nuevos experimentos.

PACS: 42.65.Jx; 61.30.Gd; 64.70.Md

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1. INTRODUCTION

The interaction of laser light with liquid crystals has given rise to many interesting electro and magneto-optical effects [1, 2]. Among these, optical self-focusing stands out as a phenomenon that has revealed the very strong optical nonlinearities of liquid crystals [3, 4]; an extreme nonlinearity that permits the investigation of nonlinear optical effects with low power cw lasers. For instance, it is experimentally well established that under low optical fields the response times of the reorientation of the director depend on characteristic lengths such as thickness or grating constants, and are typically in the milliseconds to seconds range [5]. However, recent experiments have shown that liquid crystals can also respond to sufficiently intense ($\cong$ 100 MW/cm^2) nanosecond or even picoseconds laser pulses and that this response is also in the nanoseconds regime [6].

But although these phenomena have been well studied for deterministic light pulses where their internal fluctuations are completely ignored, much less is known about the effects that a noisy incident signal might produce on the reorientation of the nematic's director and therefore, on some of its optical properties such as birefringence. Actually, a proper characterization of laser light should take into account fluctuations. They are needed to describe statistical properties such as intensity or amplitude fluctuations, correlation functions or dynamical transient processes triggered by the fluctuations themselves, like the decay of unstable or metastable states. Laser fluctuations have their origin in different sources of noise which may be classified as internal or external noise.

In fact, in the study of open systems it is convenient to distinguish between internal and external fluctuations. This difference depends, of course, in how the system is defined. In practice, however, the difference between the system and the external parameters acting on it should be clearly established in each particular case. The external parameters are determined by the environment of the system, that is, by boundary conditions or applied fields.

Internal fluctuations are self-generated by the system and have their origin in the large number of degrees of freedom that have been averaged out in its statistical description. Their study is a well known part of statistical mechanics in equilibrium and far from equilibrium [7] and an important feature of them is that they scale with an inverse power of the size of the system. Therefore, they vanish in the thermodynamic limit, except at the critical point where long range order is established [8]. In the case of laser signals these fluctuations are related, for instance, with spontaneous emission or quantum noise. From a fundamental point of view this is, perhaps, the most interesting source of internal fluctuations. However, in practical devices such as memory elements or optical transistors, quantum fluctuations are overshadowed by other more standard sources of noise [9].

In contrast, external fluctuations are those present in the system when it is subject to an external noise, that is, when the external parameters do not take on fixed deterministic but random values. This noise has its origin outside the system in an intrinsic natural randomness of the environment. It may also be imposed on a given experimental setup by forcing a control parameter to take random values with a well defined prescription. In this case it may be regarded as an external field driving the system which is controlled independently and that does not scale with the system's size [10]. This type of noise occurs in a variety of systems that have been studied in the literature and includes photochemical reactions [11], hydrodynamic systems near the Rayleigh-Benard instability [12] or liquid crystals in the vicinity of the electrohydrodynamic instability [13].

It is in connection with this latter situation where the motivation for this paper lies. The main objective is to analyze a simple model for the reorientation produced in a nematic cell by a noisy laser beam and to calculate the induced stochastic electric birefringence (Kerr effect). This effect occurs when the medium is optically anisotropic and the refractive indices differ for different directions of polarization of the light.

There are different ways in which the presence of external noise can be taken into account. We first consider the case where the intensity of the incident beam is the fluctuating parameter. As will be shown below, this leads to linear stochastic equations with white or color noise and to the unphysical result that the induced birefringence does not depend on the field's intensity. For this reason we then analyze the situation where the external noise is in the amplitude of the electric field of the incident beam. In this case the reorientation dynamics is described in terms of a stochastic equation with nonlinear quadratic external noise. We find that for the final stages of the reorientation where hydrodynamic flows can be neglected, the stochastically induced birefringence does depends on the optical's field intensity and that it may be as large as the usual deterministic birefringence associated with the intrinsic asymmetry of nematic liquid crystals. In principle, such a large effect should be measurable with high precision and in this sense our model predicts a new effect and suggests new experiments to be performed.

To this end the paper is organized as follows. In the next section we define the model and write down the basic dynamic equations for the particular geometry under consideration. Then in Sect. 3 we define the stationary nonequilibrium state to be studied and from the dynamical equations we derive an amplitude equation for the Fourier modes of the components of the director field. In Sect. 4 we introduce noise effects into this equation and in Sect. 5 we calculate the induced birefringence for the different situations mentioned above and estimate its value for 5CB and for different noise parameters. We also discuss the limitations of the model as well as possible generalizations where the results of this work could be used as a starting point and we close the paper by making some further physical remarks.

2. MODEL AND BASIC EQUATIONS

Consider a nematic liquid crystal placed between two glass plates perpendicular to the z-axis and separated a distance d, as shown in Fig. 1. Along the transverse directions x and y the cell has infinite dimensions. Boundary conditions corresponding to strong anchoring of the director are assumed on the plates at $z = 0$, d and are such that the director's initial orientation is arbitrarily set along the z direction, $\mathbf{n}^0 = (0, 0, 1)$ (homeotropic configuration). At some initial time an optical field (pump beam) with a linear and constant polarization incides obliquely into the cell with an angle $\Gamma = (\pi/2) - \beta$ with respect to the z-axis. If the laser field is intense enough, an orientational transition, the so called optical Freedericksz transition (OFT), is induced in the nematic film [14]. Above the transition threshold, E_c, the optical field distorts the initial homeotropic alignment

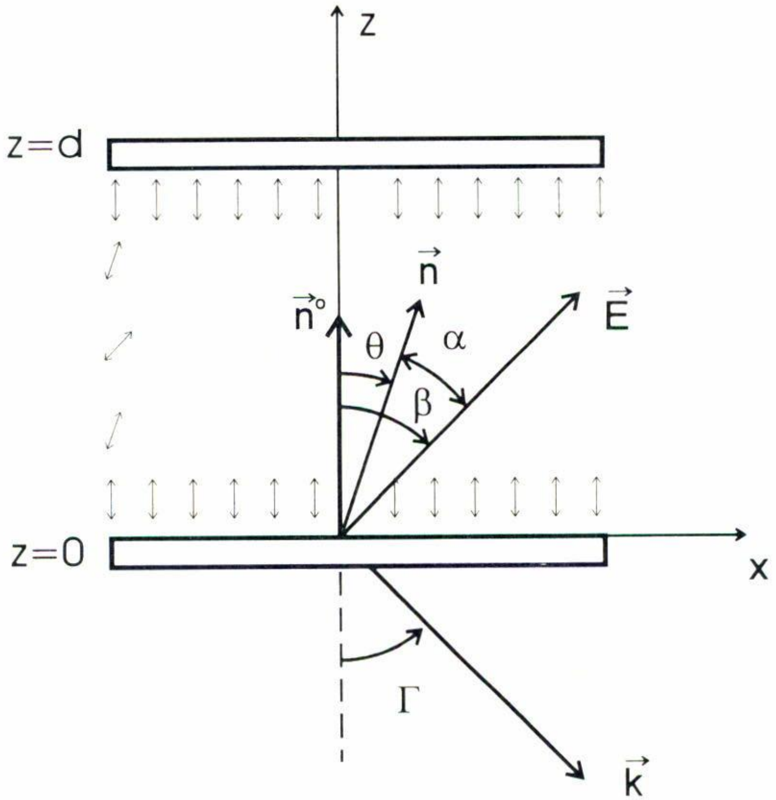

FIGURE 1. Schematics of a linearly polarized optical field acting on a momeotropically aligned nematic crystal film.

of the film by reorienting the molecules against the elastic torques. This reorientation is described by the director field $\mathbf{n}(\mathbf{r}, t)$ and if the polarization always remains in the x–z plane of incidence, it occurs in the same plane. Furthermore, if the aspect ratio of the cell is chosen in such a way that the transverse direction z is small compared to x, we may further assume that $\mathbf{n}(z, t) = [\sin\theta(z, t),\ 0,\ \cos\theta(z, t)]$, where θ is the reorientation angle in the x–z plane.

Although the reorientation of the director is usually coupled to hydrodynamic backflows generated by the motion of the molecules of the nematic, for the stationary state that we shall consider later on, they are small and we shall ignore them completely.

To describe the OFT we consider the Helmholtz free energy functional

$$F[\mathbf{n}] = \int_V d\mathbf{r}\,(f_{\mathrm{el}} + f_{\mathrm{em}}), \tag{1}$$

where V is the volume of the liquid crystal. Here f_{el} is the Oseen-Frank energy density of elastic deformations [15], which in the equal elastic constants approximation $K \equiv K_1 = K_2 = K_3$ and for the above mentioned geometry, it is given by

$$f_{\mathrm{el}} = K\left[\left(\partial_z n_x(z, t)\right)^2 + \left(\partial_z n_z(z, t)\right)^2\right], \tag{2}$$

where $\partial_z \equiv \partial/\partial z$. Since usually the magnetic susceptibility of nematics is much smaller than their dielectric susceptibility, we assume that the liquid crystal is nonmagnetic. In this case the electromagnetic free energy density can be written as

$$f_{\text{em}} = -(\mathbf{E} \cdot \mathbf{D})/2 = -\varepsilon_a \varepsilon_0 (\mathbf{n} \cdot \mathbf{E})^2/2, \tag{3}$$

where ε_0 is the electric permittivity of vacuum and where we have used the usual constitutive relation between the nematic and the optical field, namely

$$\mathbf{D} = \underline{\underline{\varepsilon}} \cdot \mathbf{E}. \tag{4}$$

The dielectric tensor $\underline{\underline{\varepsilon}}$ is given by

$$\varepsilon_{ij}/\varepsilon_0 = \varepsilon_\perp \delta_{ij} + \varepsilon_a n_i n_j, \tag{5}$$

where $\varepsilon_a \equiv \varepsilon_{\|} - \varepsilon_\perp$ is the dielectric anisotropy; $\varepsilon_{\|} - \varepsilon_\perp$ denote, respectively, the nematic's dielectric constants along the parallel and perpendicular directions to $\mathbf{n}$.

Using the formulation of the nematodynamic equations developed by San Miguel and Sagués [16], from the free energy functional obtained by substitution of Eqs. (2) and (3) into Eq. (1), we arrive at the following equations of motion for the components of the director field:

$$\partial_t n_x(z,t) = \frac{K}{\gamma_1} \partial_z^2 n_x(z,t) + \frac{\varepsilon_a \varepsilon_0}{2\gamma_1} \left(n_x E_x^2 + n_z E_x E_z \right), \tag{6}$$

$$\partial_t n_z(z,t) = \frac{K}{\gamma_1} \partial_z^2 n_z(z,t) + \frac{\varepsilon_a \varepsilon_0}{2\gamma_1} \left(n_x E_x E_z + n_z E_z^2 \right), \tag{7}$$

where γ_1 is the reorientational viscosity. Note that the electric field enters quadratically, that is, as the intensity of the beam.

3. Amplitude equation

The aim of this section is to introduce noise into the above equations. For this purpose we restrict our analysis to the final stages of the reorientation around the final stationary state where $\mathbf{n}$ is almost oriented along the polarization of the optical field. In this state the induced backflows are negligible since reorientation is almost complete. We specialize the general Eqs. (6)–(7) to this situation by linearizing them around this final orientation and expressing them in terms of the angle α between $\mathbf{E}$ and $\mathbf{n}$ when $\alpha \to 0$ (see Fig. 1). The corresponding dynamical equations then reduce to

$$\partial_t n_x(z,t) = \frac{K}{\gamma_1} \partial_z^2 n_x(z,t) + \frac{\varepsilon_a \varepsilon_0}{2\gamma_1} E^2 \sin\beta, \tag{8}$$

$$\partial_t n_z(z,t) = \frac{K}{\gamma_1} \partial_z^2 n_z(z,t) + \frac{\varepsilon_a \varepsilon_0}{2\gamma_1} E^2 \cos\beta, \tag{9}$$

with $\partial_t \equiv \partial/\partial t$. Note that while in Eqs. (6)–(7) the field's amplitude enters in a multiplicative way, with these approximations it is additive in Eqs. (8)–(9). This fact will be important for introducing parametric noise, as will be discussed below. Before doing this, however, it is important to remark that although these equations are linear, they are partial differential equations and to convert them into stochastic partial differential equations by including noise in the parameters E or $I \sim E^2$, is a difficult problem to deal with. To proceed further it is more convenient to derive an amplitude equation for the Fourier's amplitude of the most unstable mode of $n_x(z,t)$ with the inclusion of noise terms. In this way one reduces the problem posed by a stochastic partial differential equation for a vector variable to an ordinary stochastic differential equation for a scalar variable [17]. The method is of general validity and can be applied to a variety of similar problems.

This is conveniently carried out by introducing the following Fourier's transform compatible with the imposed boundary conditions on $\mathbf{n}$

$$n_x(z,t) = \sum_{m=0}^{\infty} n_{q_x,m}(t) \sin\left[(2m+1)(\pi z/d)\right], \tag{10}$$

where the index m identifies the discrete modes in the z-direction and $\mathbf{q}$ is the corresponding wave vector. It should be pointed out that since for a nematic $\mathbf{n}^2 = n_x^2 + n_z^2 = 1$, for every z and t, it is sufficient to describe the dynamics of one of the components of $\mathbf{n}$ only. Moreover, at this point we assume that the energy of the incident beam is such that its amplitude is slightly greater than E_c. Therefore, only the dominant mode $m = 0$ is excited and only its dynamics will be described. With these approximations the resulting equation for the amplitude $n_{q_x}(t)$ is derived from Eqs. (8–10) and turns out to be the following linear stochastic amplitude equation:

$$d_t n_{q_x}(t) = -A n_{q_x}(t) + BE^2, \tag{11}$$

where $d_t \equiv d/dt$, $A \equiv (K/\gamma_1)(\pi/d)^2$ and $B \equiv (\varepsilon_a \varepsilon_0 / 4\gamma_1) \sin\beta$. Note that the constant A has the dimensions of inverse time and denotes a characteristic (relaxation) time of the dominant mode.

Next we consider parametric noise which can be imposed on this equation in different ways. For instance, we can assume that either the intensity $I \propto E^2$ or the amplitude E itself are not constants in time but fluctuate. In the former case the noise is additive and enters linearly in the amplitude equation, whereas for the latter one it is quadratic. Of course, it is necessary to specify the statistics of the noise in each case and this will be done below.

4. Parametric noise effects

4.1. Linear noise

We first consider the case where the intensity $I = c\varepsilon_0 E^2/2$ (c is the speed of light in vacuum) of the incident signal fluctuates with white, gaussian noise. It should be stressed, however, that although this is perhaps the simplest model of noise one could think of, and will formally cast Eq. (11) in a Langevin form, there is no physical basis to justify this choice. Furthermore, as the above given definition of I shows, the noise in I originates in E and therefore it really should be nonlinear. However, let us consider this possibility because in some way it is intuitive and easy to analyze, but we will show that it leads to unphysical predictions. Thus we assume $I(t)$ to be a stochastic process with zero average, $\langle I(t)\rangle = 0$, and correlation

$$\langle I(t)\, I(t')\rangle = D\delta(t-t'), \tag{12}$$

where D is the intensity of the noise. Since in the next section we will express the birefringence in terms of the second moments of the amplitude $n_{q_x}(t)$, from Eqs. (11) and (12) it is easy to show that

$$\langle n_{q_x}^2(t)\rangle = n_{q_x}^2(0)\exp(-2At) + \frac{D}{8K\gamma_1}\left(\frac{\varepsilon_a d\sin\beta}{c\pi}\right)^2 [1-\exp(-2At)], \tag{13}$$

which for the final stationary state, $t\to\infty$, reduces to

$$\langle n_{q_x}^2\rangle^{\rm st} = \frac{D}{8K\gamma_1}\left(\frac{\varepsilon_a d\sin\beta}{c\pi}\right)^2. \tag{14}$$

Since white noise is always an idealization, a more realistic choice for the fluctuations of I is a colored, gaussian noise. This has the advantage of introducing the correlation time τ of the noise as a second parameter, apart from the noise intensity D. Actually, many experimental results are given in terms of the parameter D/τ, which is not present in the above given white noise description. Since we want to deal with a stationary state, let us model the linear parametric noise in I as an Ornstein-Uhlenbeck process (O-U) [10] with zero mean and time correlation function

$$\langle I(t)\, I(t')\rangle = \frac{D}{\tau}\exp[-|t-t'|/\tau]. \tag{15}$$

If as before we calculate the stationary second moment $\langle n_{q_x}^2\rangle^{\rm st}$, after some tedious but straightforward algebra from Eqs. (11) and (15), we arrive at

$$\langle n_{q_x}^2\rangle^{\rm st} = \frac{1}{4}\frac{D}{\tau}\left(\frac{\varepsilon_a \sin\beta}{c\gamma_1}\right)^2 \frac{\tau}{A}[1+A\tau]^{-1}. \tag{16}$$

4.2. Nonlinear noise

We now consider the amplitude equation (11) under the influence of quadratic parametric noise in the field's amplitude. As before, we assume that the amplitude E fluctuates with gaussian statistics and mean value $\overline{E}$

$$E = \overline{E} + \zeta(t), \tag{17}$$

where $\zeta(t)$ is a stationary Markov process. Then Doob's theorem [18] assures that up to changes in scale, $\zeta(t)$ is the Ornstein-Uhlenbeck process of mean value zero and correlation function like in (15), but with a different noise intensity D^*

$$\gamma(t-t') \equiv \langle\zeta(t)\,\zeta(t')\rangle = \frac{D^*}{\tau}\exp[-|t-t'|/\tau]. \tag{18}$$

We are then concerned with a stochastic amplitude equation in which the Ornstein-Uhlenbeck noise enters nonlinearly, that is,

$$d_t n_{q_x}(t) = -An_{q_x} + \left[\frac{\varepsilon_a \varepsilon_0}{4\gamma_1} \sin\beta\right] \left[\overline{E}^2 + 2\overline{E}\zeta(t) + \zeta^2(t)\right]. \tag{19}$$

The problem posed by this equation is much more involved than in the previous cases due to the nonlinearity of the noise. However, there is a systematic approach introduced by San Miguel and Sancho [19] to obtain the probability density associated with Eq. (19) in the limit of small intensity D^* and small correlation time τ. More specifically, in Ref. [18] is shown that to first order in D^* and τ, being D^*/τ finite, and neglecting transient times of order $\exp(-t/\tau)$, the probability density associated with Eq. (19) satisfies the following Fokker-Planck equation:

$$\partial_s P(x,s) = -\partial_x \left(1 - x + \frac{D^*}{\tau}\right) P(x,s) + D^* \left(4 + \frac{D^*}{\tau}\right) \partial_x^2 P(x,s), \tag{20}$$

where we have introduced the dimensionless variables $x \equiv n_{q_x}(t)A/B\overline{E}^2$ and $s \equiv At$.

Although the details of the method can be found in Ref. [19], at this point it is worth emphasizing some of its essential features. It is obviously based on the requirement that $D^* \ll 1$ when $\tau \ll 1$, since otherwise the fluctuating part $\zeta(t)$, whose order of magnitude is $(D^*/\tau)^{1/2}$, would become very large. However, it does not require that the fluctuations are small, as in equilibrium states. Actually, it is only necessary that they do not become too large with respect to the mean value $\overline{E}$; a situation compatible with the stationary nature of the nonequilibrium stationary state we are considering. Another important aspect of the method to be used is that the truncation performed respects the boundary conditions of the problem. Therefore the stationary solutions obtained from the truncated equation Eq. (20) remain valid.

Using the stochastic properties of $\zeta(t)$, from Eq. (20) we can derive the contribution to $\langle n_{q_x}^2\rangle^{\rm st}$ that originates in the parametric noise. That is, keeping only the terms that involve D^* or τ, we find that

$$\langle n_{q_x}^2\rangle^{\rm st} = \left[\left(\frac{\varepsilon_a \sin\beta}{2cK}\right)\left(\frac{d}{\pi}\right)^2\right]^2 \frac{D^*}{\tau}\left[2I\,\frac{1+3\tau A}{(1+\tau A)} + \frac{D^*}{\tau}\,\frac{2+3\tau A}{2+\tau A}\right]. \tag{21}$$

Note that in contrast to Eqs. (14) and (16), this expression depends explicitly on the intensity I and contains quadratic terms in D^*/τ.

5. Results and discussion

In the Appendix we show that a normal to the cell plates incident probe beam with a wavelength $\lambda_{\rm p}$, traversing the liquid crystal in the final stationary state under consideration, should have its extraordinary component experience an induced phase shift given by

$$\phi = \frac{2\pi}{\lambda_{\rm p}} \int_0^d dz \left\{ N_{\rm o} N_{\rm e} \left[N_e^2 - \langle n_x^2\rangle^{\rm st} \left(N_{\rm e}^2 - N_{\rm o}^2\right)\right]^{-1/2} - N_{\rm o} \right\}, \tag{22}$$

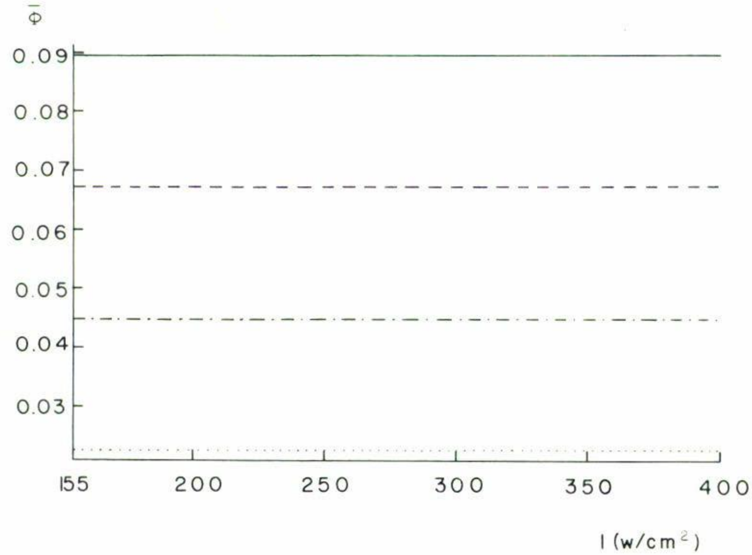

FIGURE 2. Noise induced birefringence $\overline{\phi}$ (arbitrary units) as a function of the beam's intensity I above its critical value $I_c = 155$ w/cm^2, for linear white noise in 5CB ($\beta = \pi/2$). (—) $D = 2D_c$; (- - -) $D = 1.5D_c$; (-·-) $D = D_c$; (· · ·) $D = 0.5D_c$.

where N_o and N_e are, respectively, the ordinary and maximum extraordinary refractive indices of the nematic. The expressions for the second moments $\langle n_x^2 \rangle^{st}$ to be used in Eq. (22) are given by Eqs. (14), (16) and (21) for the different noise models considered previously. By defining

$$M(\beta) \equiv \left[1 - \left(\frac{N_o}{N_e}\right)^2\right] \langle n_{q_x}^2 \rangle^{st}, \tag{23}$$

where M is such that $0 < M < 1$, the phase shift Eq. (22) may be rewritten in the more convenient form

$$\phi = \frac{2N_o d}{\lambda_p}[2K(M) - \pi]. \tag{24}$$

Here $K(M)$ is the first order elliptic integral defined by [20]

$$K(M) \equiv \int_0^{\pi/2} du\, (1 - M \sin^2 u)^{-1/2}, \tag{25}$$

with $u \equiv \pi z/d$. Thus, the noise induced birefringence, $\overline{\phi} \equiv \phi/2\pi$, will be calculated by substituting Eqs. (14), (16) and (21) into Eq. (24) for the material parameters of 5CB, namely, $\varepsilon_a = 0.62$, $\gamma_1 = 0.01$ kg/m s, $K = 7 \times 10^{-12}$ N, $N_o = 1.54$ and $N_e = 1.73$. On the other hand, for the separation between the cell plates we take $d = 2.5 \times 10^{-4}$ m and the incident signal is assumed to be a He-Ne beam with $\lambda_p = 632.8 \times 10^{-9}$ m and $\beta \cong \pi/2$, which corresponds to an incident pump beam almost normal to the plates.

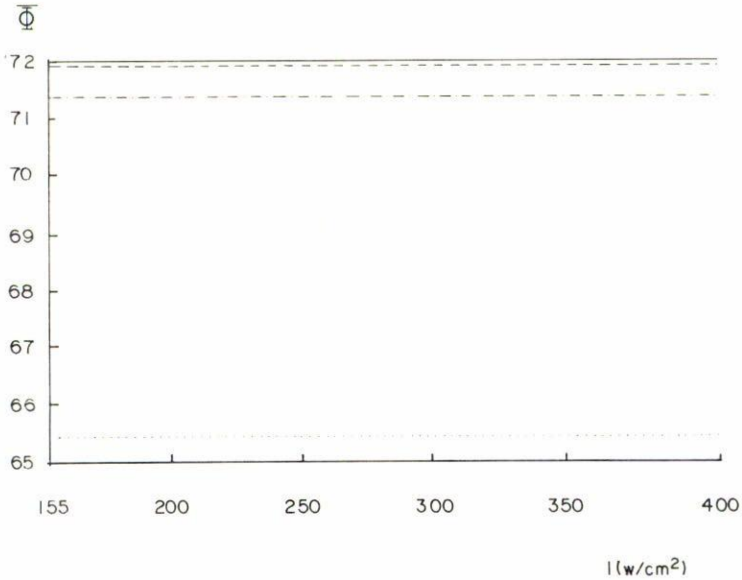

FIGURE 3a. $\overline{\phi}$ *vs.* $I > I_c$ for linear colored noise with fixed $D = 1.5D_c$. (—) $\tau = 10^{-4}t_c$; (- - -) $\tau = 10^{-3}t_c$; (- · -) $\tau = 10^{-2}t_c$; (· · ·) $\tau = 10^{-1}t_c$. Recall that $t_c \equiv A^{-1} = \gamma_1/K(d/\pi)^2 \cong 10$ s.

For the linear white noise case the values of the noise intensity D, with dimensions $[D] = W^2$ s/cm^4, are unrestricted since the noise is a free external parameter. On the other hand, since the critical field can be estimated to be $E_c = (\pi/d)(4\pi K/\varepsilon_a)^{1/2}$ [1], then $I_c = c\varepsilon_0 E_c^2/2$. It will be convenient to define D_c as $D_c \equiv I_c^2$ and from Eq. (12) express the values of D in terms of D_c. Consequently, by choosing $D_1 = 2D_c$ and $D_2 = \frac{3}{2}D_c$, from Eqs. (14), (23) and (24) we calculate $\overline{\phi}$. This yields the plots in Fig. 2 of $\overline{\phi}$ *versus* the intensity I above I_c. The first feature to note is that consistently with the fact that $\langle n_{q_x}^2\rangle^{\text{st}}$, as given by Eq. (14), does not depend on the intensity I of the incident beam I for this type of noise, $\overline{\phi}$ is given by straight lines parallel to the I-axis. Clearly, this is an unphysical result since, as mentioned in Sect. 1, it is experimentally well established that the response of the nematic varies with the type and intensity of the laser beam. Since for the other models we shall use the same units to express $\overline{\phi}$, this figure can be easily compared with the corresponding curves for the other types of noise. This shows that the magnitude of $\overline{\phi}$ is quite low for all the considered values of the only available parameter D, as compared with its value for the other cases.

A similar result is obtained for the linear colored noise with $\langle n_{q_x}^2\rangle^{\text{st}}$ given by Eq. (16). However, in this case we have an additional free parameter, the correlation time τ of the noise. Actually, we shall consider two situations. We first fix the value of D and calculate $\overline{\phi}$ for different values of τ and then for a constant τ we vary D. In this case both parameters are free, except from the restrictions imposed by the approximation method and by the fact that their values should be such that the corresponding value of $M(\beta)$ is contained in $0 < M < 1$. The results are shown, respectively, in Figs. 3a and 3b for the same values of D and the same material and geometric parameters of Fig. 2. Notice that, as for the linear white noise case, we get the same unphysical result that $\overline{\phi}$ is independent

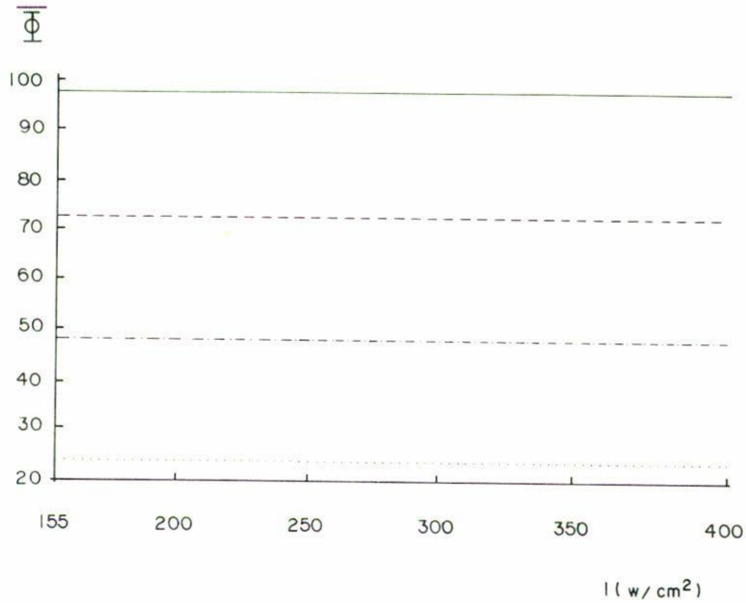

FIGURE 3b. Same as Fig. 3a for fixed $\tau = 10^{-3}t_c$. (—) $D = 2D_c$; (- - -) $D = 1.5D_c$; (- · · -) $D = D_c$; (· · ·) $D = 0.5D_c$.

of I. However, note that for the situation in Fig. 3b, the choice of τ may produce a one order of magnitude larger induced birefringence than for the case in Fig. 3a.

In contrast, for the nonlinear case $\overline{\phi}$ depends explicitly on I, as expected from Eq. (21) and as shown in Figs. 4. In Fig. 4a we first fix the value of D^* and plot $\overline{\phi}$ as a function of I, for different values of τ. Then D^* is varied for a fixed value of τ yielding a $\overline{\phi}$ that also depends on I. However, the magnitude of the induced birefringence may be considerably larger in the former case than for the latter one. Furthermore, note that the value of $\overline{\phi}$ in the case of Fig. 4a is not only much larger than for the two previous cases, but that it is comparable and even larger than its value in the purely deterministic case [14]. Typical experimental values of $\overline{\phi}$ for this latter case are taken from Ref. 14 and are represented by the points R and Q in Fig. 4b. The former corresponds to $\overline{\phi}$ for $I = 306$ w/cm^2, while the latter one implies that $\overline{\phi} = 50$ for $I = 306$ w/cm^2. However, a word of caution must be said in connection with the above results: the values of $\overline{\phi}$ are limited to the maximum value obtained from Eq. (22) when $\phi = \pi/2$ and $n_{q_x}^2$. In this case

$$\overline{\phi}_{\max} = \frac{(N_e - N_o)d}{\lambda_p}, \tag{26}$$

where we have assumed a constant maximum reorientation of the nematic throughout the cell. For the 5CB cored cell used above we find $\overline{\phi}_{\max} = 75.06$. This result restricts the validity of the results shown in Figs. 4. Indeed, they are only correct for small values of the irradiance of the pumping laser or for small noise correlation times. Nontheless, note that these allowed values are still comparable or even larger than the values of $\overline{\phi}$ for the deterministic case, points R and Q in Fig. 4b.

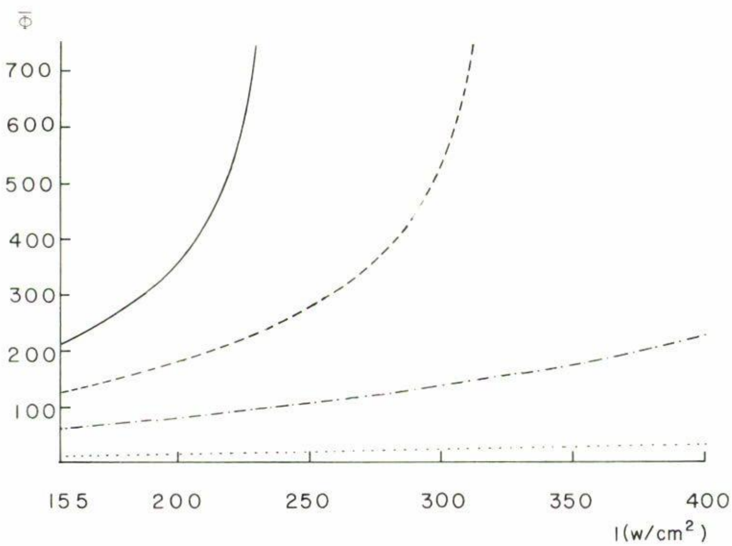

FIGURE 4a. $\overline{\phi}$ *vs.* $I > I_c$ for quadratic O-U noise with fixed $D^* = 5 \times 10^{-5} (I_c t_c)$. (—) $\tau = 10^{-4.1} t_c$; (- - -) $\tau = 10^{-4} t_c$; (-·-) $\tau = 10^{-3.8} t_c$; (···) $\tau = 10^{-3.2} t_c$.

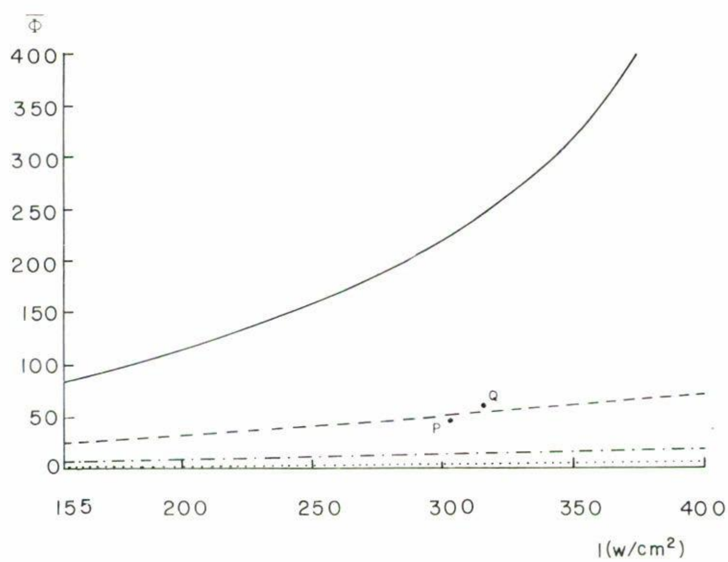

FIGURE 4b. Same as Fig. 4a for fixed $\tau = 10^{-5} t_c$. (—) $D^* = 5 \times 10^{-6.1} (I_c t_c)$; (- - -) $D^* = 5 \times 10^{-6.5} (I_c t_c)$; (-·-) $D^* = 5 \times 10^{-7} (I_c t_c)$; (···) $D^* = 5 \times 10^{-7.5} (I_c t_c)$. The points R and Q denote deterministic values of $\overline{\phi}$ according to Ref. [14].

At this point it should be recalled once more that we have only considered the final stationary state where hydrodynamic backflows have almost vanished. It would be of interest to study the initial stages of the reorientation near the OFT transition point as well. In this case hydrodynamic flows would be important and could affect in a significant way the optical properties [21]. Also, in these nonequilibrium transient states internal fluctuations are known to grow and to be anomalously large, so that they are an essential part of the dynamics. On the other hand, whether external fluctuations may also induce important effects on the optical properties for these states remains to be assessed.

Although the consideration of general transient nonequilibrium states with coupled internal or externally imposed hydrodynamic flows may have an important effect on various optical properties of the nematic [9], it should be stressed that for these states even the description of the deterministic orientational dynamics is intrinsically nonlinear. Moreover, with the inclusion of fluctuations the analysis becomes much more involved [22] and the simplicity of models such as the one discussed here is lost. Presumably, one will have to restore to numerical methods in this case, although analytical approaches are valid to some extent [22].

Another feature of the model that should be pointed out is that the dynamics of the optical field is strictly speaking coupled to the director's dynamics and the field's modes should change during reorientation. However, this generalization has been entirely neglected in our discussion but it may be considered explicitly [23].

In summary, in spite of the restrictions on the model already mentioned, our analytical results show that the external noise induced birefringence may be a large effect. Since a difference in refractive indices can be measured very accurately (10^{-15}) for optically nonlinear polar gases like CO_2 [24], it is to be expected that this effect might be measured also with high accuracy for liquid crystals, where the nonlinearity in the index of refraction is much larger. However, to our knowledge this has not been measured so far. In this sense our work suggests new experiments to be performed.

ACKNOWLEDGEMENTS

We acknowledge partial financial support from Grant DGAPA-UNAM IN100494, México.

APPENDIX

The purpose of this appendix is to derive Eq. (22) used in the main text. The phase difference, $\Delta\phi$, between the transmitted wave when reorientation occurs and when it propagates without reorientation,

$$\Delta\phi = \phi_{\rm re} - \phi_0, \tag{27}$$

is proportional to the optical path difference

$$\Delta\phi = k_0 \int_{z=0}^{z=d} \Delta N \, ds, \tag{28}$$

where $k_0 = \omega/c$ is the wave number in vacuum and $\Delta N \equiv N(\theta_{\rm re}) - N(\theta_0)$ is the corresponding change in the refractive index. Since the nematic behaves as an uniaxial crystal, the refractive index $N_{\rm e}$ for the extraordinary ray, which is the case under consideration, depends on the relative orientation, θ, between the direction of propagation of the probe's beam and the optical axis of the liquid crystal. As is well known [25], $N(\theta)$ may be written as

$$N(\theta) = N_e N_o \left[N_e^2 \cos^2\theta + N_o^2 \sin^2\theta\right]^{-1/2}, \tag{29}$$

where N_o and N_e denote, respectively, the ordinary and extraordinary refractive indices.

As we pointed out in Sect. 2, $\mathbf{n}(z,t) = [\sin\theta(z,t),\, 0,\, \cos\theta(z,t)]$, and therefore we can express $\cos\theta$ as $n_z = (1 - n_x^2)^{1/2}$ and $\sin\theta = n_x$. Then using Eq. (10) for the dominant mode $m = 0$, we may rewrite $N(\theta)$ in the form

$$N(\theta) = N_e N_o \left[N_e^2 - n_{q_x}^2 (N_e^2 - N_o^2) \sin^2(\pi z/d)\right]^{-1/2}. \tag{30}$$

In this expression we have considered that the director has reached its final stationary state. Furthermore, if we assume that the time needed to measure $\Delta\phi$ is large enough as compared with the relaxation time t_c of the dominant mode, the director will reach the stationary state associated with the stochastic amplitude equation for the corresponding noise. Therefore, it is reasonable to replace $n_{q_x}^2$ in Eq. (30) by its average value in the final stationary state $\langle n_{q_x}^2 \rangle^{st}$, given, repectively, by Eqs. (14), (16) and (21). This then yields Eq. (22). However, it should be pointed out that this replacement is indeed an approximation, since Eq. (29) is strictly valid only for the complete and deterministic reorientation process. Nontheless, the above arguments make plausible the use of Eq. (29) also in the stochastic case discussed here.

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