Revista Mexicana de Física **43**, *No. 2 (1997) 190–204*

Field theory in condensed matter physics*

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Recibido el 27 de enero 1997; aceptado el 5 de marzo de 1997

ABSTRACT. Field theory, born as a description of high energy physics, is also used at much lower energies, in condensed matter physics and statistical mechanics. We make a historical survey of how this usage evolved, from the Dirac equation to the present.

RESUMEN. La teoría de campo, inicialmente utilizada en la física de altas energías, se usa también a mucho más bajas energías, en la física del estado sólido y en mecánica estadística. En este trabajo se hace una revisión histórica de como evolucionó el uso de la teoría de campo, desde la ecuación de Dirac hasta nuestros días.

PACS: 11.10.-z

1. INTRODUCTION

The first two decades of the twentieth century witnessed the development of relativity and quantum mechanics. Both the speed of light, c, and Planck's constant, $\hbar$, were firmly established as fundamental quantities of Nature. The stage was, then, set for an attempt at a quantum description of relativistic electrons, which could not be accomplished with the nonrelativistic formulation of Heisenberg and Schrödinger.

Dirac [1] provided the solution to the problem by writing down his differential equation for free relativistic electrons:

$$(i\hbar\gamma^\mu\partial_\mu - mcI)\psi = 0. \tag{1}$$

We use the notation: $x^\mu = (ct, \vec{x})$, with $\partial_\mu \equiv \partial/\partial x^\mu$; the γ^μ's are 4×4 Dirac matrices, which satisfy the anticommutation rule, $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}I$; $\eta^{\mu\nu}$ is the metric tensor: $\eta^{00} = -\eta^{ii} = 1$, $\eta^{ij} = 0$ for $i \neq j$; I stands for the identity matrix. $\psi(ct, \vec{x})$, a Dirac spinor, was initially regarded as a four-component wavefunction.

*Invited talk at the XXXIX meeting of the Mexican Physical Society, Oaxaca, October, 1996

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The equation admitted a conserved quantity, $j^\mu = \psi^\dagger \gamma^0 \gamma^\mu \psi$ ($\psi^\dagger$ is the Hermitean conjugate of ψ), satisfying a continuity equation, $\partial_\mu j^\mu = 0$. Its zeroth component, $j^0 = \psi^\dagger \psi$, for ψ's defined as (four-component) c-number functions, was clearly positive definite. This led to its interpretation as a probability density [2].

Equation (1), written in the form $i\hbar(\partial\psi/\partial t) = H_{\text{D}}\psi$, allows for the identification of the Dirac Hamiltonian

$$\hat{H}_{\text{D}} = \vec{\alpha} \cdot \vec{p} + mcI, \tag{2}$$

with $\vec{\alpha} \equiv \gamma^0 \vec{\gamma}$. The equation admits plane-wave solutions, $\psi_{\vec{p}}$, with energy eigenvalues which may be either positive ($E \geq mc^2$) or negative ($E \leq -mc^2$). It follows that the expectation values for such plane waves;

$$E = \int d^3x \, \psi^\dagger_{\vec{p}} H_{\text{D}} \, \psi_{\vec{p}} = \pm\sqrt{\vec{p}^{\,2}c^2 + m^2c^4}, \tag{3}$$

are unbounded below, indicating an instability of the system.

In order to cope with that difficulty, Dirac postulated that the ground-state of the system would have all the negative energy eigenvalues populated, respecting Pauli's exclusion principle. This lowest energy state, named the "Dirac sea", would, then, have many degrees of freedom. Excited states would correspond to populating one or more positive energy eigenvalues and would, thus, describe systems with one or more (free) electrons (particles).

Dirac's hypothesis opened the way for the prediction of a remarkable mechanism: as soon as one allowed the electrons to interact with an electromagnetic field [formally, by letting $\partial_\mu \to \partial_\mu + (ie/\hbar c)A_\mu$], one had the possibility of exciting an electron belonging to the sea to a positive energy eigenvalue; for this to occur without depleting the sea (assumed to be the ground-state), required the creation of another particle, with the same energy of the excited electron and opposite charge. The would-be "hole" in the sea would materialize as an antiparticle of the electron, the positron, in a process of pair creation which conserved charge and energy.

The experimental discovery of the positron [3], in 1932, was a great triumph. Nevertheless, pair creation demanded a reformulation of some of the ideas that had led to its very prediction: particle number was not conserved and, therefore, the conserved quantity obtained from the integral of j^0 over all space had to be reinterpreted. Relativistic quantum mechanics led to the birth of quantum field theory.

2. QUANTUM FIELD THEORY: A SECOND QUANTIZATION

The key to the correct interpretation of the conserved current, j^μ, was to consider ψ not as a (four-component) c-number function, but as an operator acting on an underlying space, called Fock space.

First quantization, which led to quantum mechanics, promoted physical quantities, such as positions and momenta, to operators acting on a Hilbert space, such as the space of wavefunctions. Second quantization promoted Dirac spinors, initially viewed as

relativistic wavefunctions, to field operators of the type

$$\hat{\psi}(ct,\vec{x}) = \sum_{\vec{p}} \left[\psi^{(+)}_{\vec{p}}(ct,\vec{p})\,\hat{b}_{\vec{p}} + \psi^{(-)}_{\vec{p}}(ct,\vec{p})\,\hat{d}^{\dagger}_{\vec{p}}\right], \tag{4}$$

where the sum, whenever performed over a continuum of states, is shorthand for an integral; $\psi^{(\pm)}_{\vec{p}}$ denotes positive and negative energy c-number solutions of Dirac's equation, whereas $\hat{b}^{\dagger}_{\vec{p}}$ and $\hat{d}^{\dagger}_{\vec{p}}$ create electrons and positrons, respectively, by acting on the lowest energy state of Fock space. The latter will be denoted $|0\rangle$ and, heretofore, called the vacuum, as it is annihilated by both particle and antiparticle operators, if a normal ordering instruction is adopted [4]. Successive application of the creation operators generates Fock space.

Going back to the conserved current, we can write

$$:\hat{\mathcal{N}}: = \int d^3x\, :j^0(ct,\vec{x}): = \sum_{\vec{p}} :\hat{b}^{\dagger}_{\vec{p}}\,\hat{b}_{\vec{p}} - \hat{d}^{\dagger}_{\vec{p}}\,\hat{b}_{\vec{p}}:, \tag{5}$$

which has an obvious interpretation as the fermion number operator, a quantity that assigns $(+1)$ to particles and (-1) to antiparticles. $\hat{\mathcal{N}}$ is conserved, but not positive definite; the double-dots denote the normal ordering prescription which ensures that $\langle 0|:\hat{\mathcal{N}}:|0\rangle = 0$.

The vacuum of a field theory is the natural descendant of the "Dirac sea" idea. It may be thought of as an immense reservoir of dormant particles and antiparticles, ready to be excited, which has many degrees of freedom.

Many-body theory [5] is nothing but the condensed matter version of the view just described. Instead of dealing with highly energetic electrons, it treats systems with a very large ($N \sim 10^{24}$) number of particles, at much lower energies (down to eV, for example).

Clearly, many-body systems of N particles of mass M can be treated in first quantization, by using a Hamiltonian

$$\hat{H}_N = \sum_{a=1}^{N}\left[-\frac{\hbar^2\vec{\nabla}_a^2}{2M} + U^{(1)}(\vec{x}_a)\right] + \sum_{a\neq b} U^{(2)}(\vec{x}_a,\vec{x}_b) + \cdots, \tag{6}$$

which is a sum of a one-body term, $\hat{H}^{(1)}_a$ (the term inside the brackets), with two, $\hat{H}^{(2)}_{ab}$ (*i.e.*, the next term), and higher-body interactions; one may, then, construct a basis for the many-particle wavefunction from the products of one-body wavefunctions, ψ_i:

$$\hat{H}^{(1)}_a\psi_i = E_i\psi_i. \tag{7}$$

This is accomplished by using products which respect the appropriate statistics. If n_{i_j} denotes the number of particles in state i_j ($\sum_{j=1}^{N} n_{i_j} = N$),

$$\Psi_{i_1,\ldots,i_N}(\vec{x}_1,\ldots,\vec{x}_N,t) = \sum_{P(i)} C_{P(i)}\,\psi_{i_1}(\vec{x}_1,t)\ldots\psi_{i_N}(\vec{x}_N,t), \tag{8}$$

$P(i)$ stands for permutations of the indices; $C_{P(i)}$ is given by $(n_{i_1}!\,n_{i_2}!\ldots/N!)^{1/2}$, for bosons, and by $[(-1)^{P(i)}/(N!)]^{1/2}$, for fermions. The collection, $\{\Psi_{i_1,\ldots,i_N}\}$, for all values of the i_a's, defines a basis. Eigenfunctions of $\hat{H}_N$ can be expanded in this basis.

Alternatively, one may adopt a second-quantized approach by defining a Fock-type basis from one-particle creation operators acting on a vacuum state (the many-body ground-state):

$$|n_{i_1}, \ldots, n_{i_N}\rangle \equiv (\hat{a}^\dagger_{i_N})^n_{i_N} \ldots (\hat{a}^\dagger_{i_1})^n_{i_1} |0\rangle. \tag{9}$$

Since the expectation values of physical quantities are invariant under the change of basis, $\{\Psi_{i_1,\ldots,i_N}\} \to \{|n_{i_1}, \ldots, n_{i_N}\rangle\}$, *i.e.*,

$$\langle n_{i_1}, \ldots, n_{i_N}|\hat{\mathcal{H}}|n_{i_1}, \ldots, n_{i_N}\rangle = \int d^3x_1 \ldots d^3x_N \; \Psi^\dagger_{i_1,\ldots,i_N} \hat{H}_N \Psi_{i_1,\ldots,i_N} \tag{10}$$

with

$$\hat{\mathcal{H}} = \sum_{ij} \mathcal{H}^{(1)}_{ij} \hat{a}^\dagger_i \hat{a}_j + \sum_{ijkl} \mathcal{H}^{(2)}_{ijkl} \hat{a}^\dagger_i \hat{a}^\dagger_j \hat{a}_k \hat{a}_l + \cdots \tag{11}$$

Field operators may be defined using one-particle wavefunctions:

$$\hat{\psi}(\vec{x}, t) \equiv \sum_i \psi_i(\vec{x}, t)\hat{a}_i, \qquad \hat{\psi}^\dagger(\vec{x}, t) \equiv \sum_i \psi^\dagger_i(\vec{x}, t)\hat{a}^\dagger_i \tag{12}$$

so that the second-quantized Hamiltonian reads:

$$\begin{aligned}\hat{\mathcal{H}} = &\int d^3x \left(-\frac{\hbar^2}{2M} \vec{\nabla}\hat{\psi}^\dagger \cdot \vec{\nabla}\hat{\psi} + \hat{\psi}^\dagger U^{(1)} \hat{\psi} \right) \\ &+ \int d^3x_1 \, d^3x_2 \, \hat{\psi}^\dagger(\vec{x}_1)\hat{\psi}^\dagger(\vec{x}_2) U^2(\vec{x}_1, \vec{x}_2)\hat{\psi}(\vec{x}_2)\hat{\psi}(\vec{x}_1) + \cdots \end{aligned} \tag{13}$$

The field-theoretic description of many-body systems, widely used at very low energies, only differs from the field theories used at very high energies in that its "oscillators" are labelled by discrete, $\{i_a\}$, rather than continuous, $\{\vec{p}\}$, indices. This reflects itself in the need for a highly nontrivial continuum limit in field theories, which led to the development of renormalization theory [6].

Nowadays, rigorous results in quantum field theory are often obtained by studying the theory on a (hyper)cubic discrete lattice of space-time points (a many-body system) and taking, besides the usual (large-distance, infrared) thermodynamic limit, a (small-distance, ultraviolet) continuum limit.

3. PATH INTEGRALS

The close relationship between field theory and the many-body treatment of condensed matter physics is also evident if one adopts, instead of the operator formalism of the previous section, a description in terms of path integrals.

Path integrals were introduced, in 1947, by Feynman and Wheeler [7], as an alternative way of treating problems in quantum mechanics. Indeed, a one-dimensional quantum

mechanical probability amplitude to go from position q_0, at time t_0, to position q_1, at time t_1, can be written as

$$\langle q(t_1)|e^{-i\hat{H}\Delta t/\hbar}|q(t_0)\rangle = \int [Dp(t)] \int_{q_0}^{q_1} [Dq(t)] e^{iS[p,q]/\hbar}, \tag{14}$$

$\Delta t \equiv t_1 - t_0$; the integrals on the right-hand side are over all possible paths, $q(t)$, which satisfy the boundary conditions, $q(t_1) = q_1$ and $q(t_0) = q_0$, with all possible momenta $p(t)$, regarded as independent, weighted with an exponential of the corresponding action

$$S[p,q] = \int_{t_0}^{t_1} dt\,(p\dot{q} - H[p,q])\,. \tag{15}$$

Although a rigorous mathematical definition of such objects is only available for a limited class of Hamiltonians [8], various perturbative and nonperturbative computational techniques exist which lead to approximate evaluations of the sums. In fact, this formulation reduces a quantum mechanical problem to a classical sum over paths.

The same techniques can be used in a path integral formulation of equilibrium quantum statistical mechanics. The partition function for a system described by a Hamiltonan operator, $\hat{H}$, is

$$Z(\beta) = \mathrm{Tr}\left[e^{-\beta\hat{H}}\right] = \int_{-\infty}^{+\infty} dq_0\, \langle q_0|e^{-\beta\hat{H}}|q_0\rangle, \tag{16}$$

where the amplitude can be thought of as that for starting from, and returning to, q_0, within an imaginary time interval, Δt, such that $i\Delta t = \beta\hbar$. This amplitude can be obtained from (14) by analytically continuing to imaginary time, *i.e.*, $it \to \tau$, τ being a real (Euclidean) variable. Such a procedure is known, in the jargon of field theorists, as a Wick rotation (after Giancarlo Wick). Expression (16), then, becomes

$$Z(\beta) = \int_{-\infty}^{+\infty} dq_0 \int [Dp(\tau)] \int_{q_0}^{q_0} [Dq(\tau)]\, e^{-S_E[p,q]/\hbar}\,, \tag{17}$$

where the Euclidean action, S_E, is defined as

$$S_E[p,q] = \int_0^{\beta\hbar} d\tau \left(p\frac{\partial q}{\partial \tau} + H[p,q]\right). \tag{18}$$

The generalization of (14) to field theory can be accomplished, formally, by first extending it to many degrees of freedom, $q(t) \to q_j(t)$, and, finally, letting the discrete index, j, become continuous [9]. Defining a field opeerator, $\hat{\phi}(\vec{x})$, and its eigenvectors, $|\phi(\vec{x})\rangle$, and eigenvalues, $\phi(\vec{x})$, such that

$$\hat{\phi}(\vec{x})|\phi(\vec{x})\rangle = \phi(\vec{x})|\phi(\vec{x})\rangle, \tag{19}$$

one may write

$$\langle \phi(\vec{x},t_1)|e^{-i\hat{H}\Delta t/\hbar}|\phi(\vec{x},t_0)\rangle = \int [D\pi(\vec{x},t)] \int_{\phi_0(\vec{x})}^{\phi_1(\vec{x})} [D\phi(\vec{x},t)]\, e^{iS[\pi,\phi]/\hbar}, \tag{20}$$

$\pi(\vec{x},t)$ being the canonically conjugated momentum density, and the action given as

$$S[\pi,\phi] = \int_{t_0}^{t_1} dt \int d^D x \left(\pi\dot{\phi} - H[\pi,\phi]\right). \tag{21}$$

Continuing the formulae of the previous paragraph to Euclidean times allows one to write down a path integral formalism for equilibrium quantum statistical field theory, *i.e.*, the statistical mechanics of field theories. Indeed, the partition function(al) for such theories may be expressed as

$$Z[\beta, J(\vec{x})] = \int [D\pi(\vec{x},\tau)] \oint [D\phi(\vec{x},\tau)]\, e^{-S_E[\pi,\phi;J]/\hbar}, \tag{22}$$

where the Euclidean action is given by

$$S_E[\pi,\phi;J] = \int_0^{\beta\hbar} d\tau \int d^D x \left(\pi\frac{\partial\phi}{\partial\tau} - H[\pi,\phi] - J\phi\right). \tag{23}$$

We have included an external local source, $J(\vec{x})$, coupled to the field, in the action. The trace in Z leads to the instruction, encoded in the symbol $\oint$, that the sum is to be performed over all fields which satisfy the boundary conditions

$$\phi(\vec{x},\beta\hbar) = \pm\phi(\vec{x},0) = \pm\phi(\vec{x}); \tag{24}$$

the upper (lower) sign is for bosons (fermions), to account for the appropriate statistics. Furthermore, one has to sum over all boundary fields, $\phi(\vec{x})$. Thus, formally

$$\oint [D\phi(\vec{x},\tau] \equiv \int [\phi(\vec{x})] \int_{\phi(\vec{x})}^{\pm\phi(\vec{x})} [D\phi(\vec{x},\tau)]. \tag{25}$$

Quantum statistical field theory enables us to compute correlation functions for systems which admit a field-theoretic treatment, whether they describe high-energy (continuum) field theories or low-energy many-body systems. Indeed, one may use the operator formalism discussed in the previous section, or the path integral formulation, to compute the partition function(al) for a field theory Hamiltonian and, from it, obtain other generating function(al)s, as well as correlations, which are associated to physical quantities.

Expectation values of the fields can be calculated from the functional derivatives [9] of $Z[\beta, J(\vec{x})]$ with respect to the external current, $J(\vec{x})$,

$$\langle\phi(\vec{x}_1)\ldots\phi(\vec{x}_N)\rangle_{\beta,J(\vec{x})} = Z^{-1}\int [D\pi] \oint [D\phi]\, \phi(\vec{x}_1)\ldots\phi(\vec{x}_N) e^{-S_E/\hbar}; \tag{26}$$

whenever the expression above is evaluated at $J = 0$, we shall denote it $G^{(N)}(\vec{x}_1,\ldots,\vec{x}_N;\beta)$. Clearly

$$\beta^N G^{(N)}(\vec{x}_1,\ldots,\vec{x}_N;\beta) = Z^{-1}\left(\frac{\partial^N Z}{\partial J(\vec{x}_1)\ldots\partial J(\vec{x}_N)}\right)_{J=0}. \tag{27}$$

We may define a Helmholtz free energy, $\mathcal{F}[\beta, J(\vec{x})]$, through a functional, $W[\beta, J(\vec{x})]$, by writing

$$Z \equiv e^{-\beta\mathcal{F}} \equiv e^{-W}. \tag{28}$$

Functional derivatives of W, at $J = 0$, yield the, so-called, connected (zero field) correlation functions

$$\beta^N G_c^{(N)}(\vec{x}_1, \ldots, \vec{x}_N; \beta) = \left(\frac{\partial^N W}{\partial J(\vec{x}_1) \ldots \partial J(\vec{x}_N)} \right)_{J=0} \tag{29}$$

As a consequence, W admits a functional Taylor expansion around $J = 0$:

$$W = \sum_N \frac{\beta^N}{N!} G_c^{(N)}(\vec{x}_1, \ldots, \vec{x}_N; \beta)\, J(\vec{x}_1) \ldots J(\vec{x}_N). \tag{30}$$

A simple Legendre transform will define a Gibbs free energy functional, $\mathcal{G}[\beta, \langle\phi(\vec{x})\rangle]$, through a functional, $\Gamma[\beta, \langle\phi(\vec{x})\rangle]$, by

$$\Gamma \equiv \beta\mathcal{G} \equiv \beta\mathcal{F} + \beta \int d^D x\, J(\vec{x}) \langle\phi(\vec{x})\rangle, \tag{31}$$

which admits a functional Taylor expansion around $\langle\phi\rangle = 0$ of the type

$$\Gamma = \sum_N \frac{\beta^N}{N!} \Gamma^{(N)}(\vec{x}_1, \ldots, \vec{x}_N; \beta)\, \langle\phi(\vec{x}_1)\rangle \ldots \langle\phi(\vec{x}_N)\rangle. \tag{32}$$

The $\Gamma^{(N)}$'s are called one-particle irreducible vertex functions in the field theory jargon.

Use of the previous formulae, for $N = 1$ and $N = 2$, yields

$$\langle\phi(\vec{x})\rangle = -\beta^{-1} \frac{\partial W}{\partial J(\vec{x})}; \qquad J(\vec{x}) = \beta^{-1} \frac{\partial \Gamma}{\partial \langle\phi(\vec{x})\rangle}, \tag{33}$$

as well as

$$G_c^{(2)}(\vec{x}_1, \vec{x}_2; \beta) = \langle\phi(\vec{x}_1)\phi(\vec{x}_2)\rangle_{\beta, J=0} - \langle\phi(\vec{x}_1)\rangle_{\beta, J=0} \langle\phi(\vec{x}_2)\rangle_{\beta, J=0}. \tag{34}$$

It can be shown that $\Gamma^{(2)}$ is the inverse (in the matrix sense) of G_c^2. These correlations will play an important role in our discussion of phase transitions in the next section.

4. PHASE TRANSITIONS

The use of field-theoretic methods in condensed matter theory, outlined in the two preceding sections, led to a major breakthrough in the study of phase transitions which goes back to the late 60's, early 70's, when the ideas of Leo Kadanoff and Michael Fisher were formalized in the Nobel Prize winning renormalization group studies of Kenneth Wilson [10].

Phase transitions are quite common in Nature and occur in a variety of systems. To make contact with the previous discussion, imagine a magnetic system which has a certain scalar magnetic dynamic variable, ϕ, such as a spin projection along an axis, defined at every point $\vec{x}$ (if the variable can only assume values ± 1 and if the points belong to a square or cubic lattice, one would have the familiar Ising model). Then, the external source, $J(\vec{x})$, would have to be an external magnetic field along the axis, whereas the expectation value of the field, $\langle\phi\rangle$, we should identify with the magnetization per unit volume along the axis.

Equations (33) and (34), in this example, can be interpreted as the usual statements that relate the Helmholtz free energy to the magnetization or the Gibbs free energy to the magnetic field. Furthermore, the two-point (connected) correlation function measures the net influence of the magnetic variable at point $\vec{x}_1$ on that at point $\vec{x}_2$. The integral of that quantity over all space defines the magnetic susceptibility

$$\chi(\beta) \equiv \int d^D r\, G_c^{(2)}(r;\beta); \tag{35}$$

we have assumed that our system is translationally invariant, so that $G_c^{(2)}$ depends only on the relative coordinate, $r \equiv |\vec{x}_1 - \vec{x}_2|$.

A continuum (2nd order) phase transition [11] will occur whenever the expectation value of the field, $\langle\phi\rangle$, has finite values, at low temperatures, and drops *continuously* down to zero as we raise the temperature of the system. Once it vanishes, at a given temperature T_c, it remains zero for $T \geq T_c$. This means that we go from an *ordered* phase, at low temperatures, which has a residual magnetization at zero external field, to a *disordered* phase, above T_c, where there is no residual magnetization. Near T_c, the two-point correlator will transition from an ordered behavior

$$G_c^{(2)}(r;\beta) \sim \frac{1}{r^{D-2+\eta}} \qquad T \to T_c^-, \tag{36}$$

below T_c, characterized by long-range (power-law) correlations, to a disordered behavior

$$G_c^{(2)}(r;\beta) \sim \frac{e^{-r/\xi}}{r^{D-2+\eta}} \qquad T \to T_c^+, \tag{37}$$

above T_c, characterized by short-range (exponentially suppressed) correlations.

At the critical temperature, the correlation length, $\xi(\beta)$, appearing in (37) will diverge. The susceptibility also diverges as we approach T_c. These singularities are the signature of a 2nd order phase transition; the leading behavior of $\xi(\beta)$ and $\chi(\beta)$ near (and below) T_c can be described by:

$$\xi(T) \sim \left(\frac{T - T_c}{T_c}\right)^{-\nu} \tag{38}$$

$$\chi(T) \sim \left(\frac{T - T_c}{T_c}\right)^{-\gamma}. \tag{39}$$

This defines the, so-called, critical exponents, ν and γ, which characterize the singularities.

4.1. THE RENORMALIZATION GROUP AND CRITICAL EXPONENTS

Up until the work of Kadanoff, Fisher and Wilson, the treatment of phase transitions made use of Landau theory [12], which gave a good overall qualitative description of the phenomenon, but failed to yield theoretical values in agreement with experimental data for critical exponents in two and three dimensions.

Kadanoff and Fisher, in the late 60's, recognized that critical exponents were rather universal. Their values seemed to be the same for a large class of phase transitions. In fact, transitions were soon catalogued under different universality classes, each class containing many examples of rather different systems whose critical behavior was, nonetheless, characterized by the same values of critical exponents. Those authors explained this feature by resorting to a scaling theory which, essentially, attributed universality to the independence of collective behavior on the microscopic details of the Hamiltonian of the system. Scaling implied that, near criticality, various systems, when viewed at larger and larger scales, could be described by the same type of effective (critical) Hamiltonian. Their large scale properties would be, for some quantities, the same, thus leading to the universality of the critical exponents.

Field theorists were quite familiar with scaling ideas, although in a different context. Ever since the days of the Gell-Mann-Low equation [13], they had been interested in the scaling behavior of correlations—only at the other limit, that is, the short-scale or high-energy limit (the ultraviolet or continuum limit). At about the same time of Kadanoff and Fisher's work, the renormalization program of field theories had developed into a much better understood discipline, as a result of the contributions of Callan [14] and Symanzik [15]. Thanks to Wilson, these developments were fused with the advances in the study of phase transitions to yield what is now known as renormalization group theory [10].

Suppose that we have a four-dimensional system characterized by a Lagrangian density such as

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \phi_0)^2 - \tfrac{1}{2} m_0^2 \phi_0^2 - \tfrac{1}{4!}\lambda_0 \phi_0^4. \tag{40}$$

The subscripts are there to remind the reader that quantities in the Lagrangian density, which are called "bare", are devoid of physical meaning; in field theory, what matters are the correlation functions, which have to be computed in an unambiguous way. To do so, one has to cope with the singularities that appear whenever we have correlations involving two (or more) fields at the same space-time point. That is where the renormalization procedure comes in: the field theory is first defined in a regularized fashion (*i.e.*, by defining it on a lattice whose points are separated by a finite lattice parameter, Λ^{-1}); one can, then, compute finite correlation functions and, from them, parameters; by imposing renormalization conditions, one can relate these parameters to physical values, extracted from experiment, in the continuum limit of interest. These physical parameters are defined at an arbitrary scale, μ: if their number is finite and if they allow one to obtain physical quantities at any other scale, the program is accomplished and the theory is said to be renormalizable.

Coming back to our problem, one may relate renormalized (physical) quantities to bare ones by introducing three dimensionless renormalization constants, Z_ϕ, Z_m and Z_λ,

such that

$$\phi_0 \equiv Z_\phi^{1/2}\phi \;\; m_0 \equiv Z_m^{1/2} m \;\; \lambda_0 \equiv Z_\lambda^{1/2}\lambda; \tag{41}$$

in particular, an N-point correlation, which involves the product of N ϕ-fields, will be written as

$$\Gamma^{(N)}(\lambda, m; \mu) = Z_\phi^{N/2}(\lambda_0 \mu^{4-D}, m_0\mu^{-1}, \Lambda\mu^{-1})\Gamma_0^{(N)}(\lambda_0, m_0; \Lambda), \tag{42}$$

where both (41) and (42) are supposed to hold for $\Lambda \to \infty$ (the lattice space vanishing in the continuum limit) and μ is the arbitrary reference scale, called a renormalization point.

The physical requirement that the theory (*i.e.*, the physical correlations) should be the same whatever the choice of reference scale translates into

$$\left[\mu \frac{d}{d\mu}\right]_{\lambda_0, m_0, \Lambda} \Gamma_0^{(N)} = 0, \tag{43}$$

leading to the following equation, due to Weinberg [16]:

$$\left[\mu\frac{\partial}{\partial\mu} + \tilde{\beta}(\lambda)\frac{\partial}{\partial\lambda} - N\tilde{\gamma}(\lambda) + \tilde{\gamma}_m m \frac{\partial}{\partial m}\right]\Gamma^{(N)}(\lambda, m; \mu) = 0, \tag{44}$$

which used $\Gamma_0^{(N)} = Z_\phi^{-N/2}\Gamma^{(N)}$. Defining $\mu(t) \equiv \bar{\mu}e^t$, we have

$$\frac{d\lambda}{dt} = \tilde{\beta}[\lambda(t)] \tag{45}$$

$$\frac{dm}{dt} = m\tilde{\gamma}_m[\lambda(t)], \tag{46}$$

which define a running effective coupling constant and an effective mass parameter, as we change the reference scale. Physical quantities, which should be invariant under changes of reference scale, such as the physical mass (the inverse physical correlation length), obey

$$P(\lambda, m; \mu) = P(\lambda(t),\, m(t);\, \mu(t)). \tag{47}$$

Equation (44) is one of the forms of a renormalization group equation. It can be solved to yield the behavior of correlations, in momentum space, either under changes of reference scale or, alternatively, under changes of momenta (keeping the scale fixed):

$$\Gamma^{(N)}(e^t p_i, \lambda, m; \mu) = \exp\left\{(4-N) - N\int_0^t \tilde{\gamma}[\lambda(t')]\,dt'\right\}\Gamma^N(p_i,\, \lambda(t)\,, e^{-t}m(t);\, \mu). \tag{48}$$

Whenever $\tilde{\beta}(\lambda^*) = 0$, λ will be fixed at the critical value λ^*. Clearly, if $d\tilde{\beta}/d\lambda < 0$, it will be stable for $t \to \infty$ (ultraviolet stable), whereas, if $d\tilde{\beta}/d\lambda > 0$, it will be stable for $t \to -\infty$ (infrared stable). It is the latter situation that is relevant for the study of criticality in phase transitions. Critical points correspond to infrared stable fixed points

in renormalization group language. In fact, the critical exponents defined previously can be related to the values of $\tilde{\gamma}_m(\lambda^*_{\rm IR})$ and $\tilde{\gamma}(\lambda^*_{\rm IR})$, at an infrared fixed point, $\lambda^*_{\rm IR}$.

The previous formulae apply to $D = 4$. In that case, $\lambda^*_{\rm IR} = 0$ and the critical exponents thus obtained coincide with the values of Landau theory. However, for $D = 4 - \epsilon$, $\epsilon \ll 1$, Wilson was able to show that there existed a $\lambda^*_{\rm IR} \neq 0$, for $\epsilon > 0$, in a perturbative expansion in ϵ. Amazingly enough, computing the corresponding exponents in an ϵ-expansion and setting $\epsilon = 1$ yields the critical exponents for $D = 3$ observed experimentally to an accuracy of 1% or better.

4.2. CONFORMAL THEORIES IN TWO DIMENSIONS

The ideas underlying the renormalization group analysis of phase transitions can be illustratred, in a remarkable fashion, in two-dimensional systems described by conformally invariant field theories [17].

Infinitesimal conformal transformations change coordinates in the following way:

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x). \tag{49}$$

Conformal transformations multiply the metric by a conformal factor, $\Omega(x)$:

$$g_{\mu\nu} \to g'^{\mu\nu}(x') = \Omega(x) g^{\mu\nu}(x). \tag{50}$$

These transformations form a continuous Lie group, the conformal group, which, in four dimensions, has fifteen generators: ten that generate the Poincaré group, four that generate the, so-called, special conformal transforms and one that generates dilatations.

In two dimensions, however, the conformal group has an infinite number of generators. In fact, if we represent a point on the plane by a complex number, $z = x + iy$, conformal transformations will map

$$z \to z' = f(z), \tag{51}$$

$$\bar{z} \to \bar{z}' = \bar{f}(\bar{z}), \tag{52}$$

with $\bar{z} \equiv x - iy$. f and $\bar{f}$ are arbitrary holomorphic and antiholomorphic functions, respectively. The infinite generators in two dimensions can be represented as operators of the type

$$l_n \equiv -z^{n+1}\partial_z, \tag{53}$$

$$\bar{l}_n \equiv -\bar{z}^{n+1}\partial_{\bar{z}}, \tag{54}$$

which satisfy a local algebra of commutators:

$$[l_n, l_m] = (m - n) l_{m+n}, \tag{55}$$

$$[\bar{l}_n, \bar{l}_m] = (m - n) \bar{l}_{m+n}, \tag{56}$$

$$[l_n, \bar{l}_m] = 0. \tag{57}$$

In order to implement conformal transformations at the quantum level, we start from the current that generates the local coordinate transformations of (49), *i.e.*, $J^\mu \equiv T^{\mu\nu}\epsilon_\nu$, where $T^{\mu\nu}$ is the stress-energy tensor of the system. Changing to complex notation and introducing $T_{zz} \equiv T(z)$, $T_{\bar{z}\bar{z}} \equiv \bar{T}(\bar{z})$ ($T_{z\bar{z}} = T_{\bar{z}z} = 0$), we may define the associated charge

$$Q = \frac{1}{2\pi i}\left[\oint dz\, T(z)\epsilon(z) + \oint d\bar{z}\, \bar{T}(\bar{z})\bar{\epsilon}(\bar{z})\right], \tag{58}$$

which leads to infinitesimal transformations of the fields, $\Phi(w, \bar{w})$:

$$\delta\Phi(w, \bar{w}) = [Q, \Phi(w, \bar{w})]. \tag{59}$$

The operator, $T(z)$, expressible in terms of the fields, satisfies, for z close to w:

$$T(z)\Phi(w, \bar{w}) = \frac{h}{(z-w)^2}\Phi(w, \bar{w}) + \frac{1}{(z-w)}\partial_w\Phi(w, \bar{w}) + \cdots, \tag{60}$$

where the terms left out are nonsingular. Again, we have singular behavior as two fields are defined the same space-time point. A similar relation involving $\bar{T}$ defines $\bar{h}$. The quantities h and $\bar{h}$ are called conformal weights.

It turns out that, when one looks at the short-distance product of two T's, one obtains, from the requirement that the algebra should close, the relation

$$T(z)T(w) = \frac{c/2}{(z-w)^4}T(w) + \frac{2}{(z-w)^2}T(w) + \frac{1}{(z-w)}\partial_w T(w) + \cdots, \tag{61}$$

as $z \to w$. The quantity, c, as well as its analogue, $\bar{c}$, is completely arbitrary, not fixed by conformal invariance. If we define the quantum generators, L_n and $\bar{L}_n$, via

$$T(z) \equiv \sum_n z^{-n-2}L_n, \tag{62}$$

$$\bar{T}(z) \equiv \sum_n \bar{z}^{-n-2}\bar{L}_n, \tag{63}$$

they will obey an algebra which involves c and $\bar{c}$ as free parameters:

$$[L_n, L_m] = (m-n)L_{m+n} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}, \tag{64}$$

$$[\bar{L}_n, \bar{L}_m] = (m-n)\bar{L}_{m+n} + \frac{\bar{c}}{12}(n^3 - n)\delta_{n+m,0}, \tag{65}$$

$$[L_n, \bar{L}_m] = 0. \tag{66}$$

The quantities, c and $\bar{c}$, are called central charges and characterize different conformal theories. The conformal weights, h and $\bar{h}$, characterize the different representations of the quantum algebra, known as the Virasoro algebra [18]. Fields transform under scale changes with the exponent $\Delta \equiv h + \bar{h}$, whereas their spin (rotation) index is $s \equiv h - \bar{h}$.

The unitary representations of the Virasoro algebra with $c < 1$ belong to the, so-called, minimal series:

$$c = 1 - \frac{6}{m(m+1)}, \qquad m = 3, 4, \ldots. \tag{67}$$

The corresponding conformal weights satisfy

$$h_{p,q}(m) = \frac{[(m+1)p - mq]^2 - 1}{4m(m+1)}, \tag{68}$$

with $1 \leq p \leq m-1$ and $1 \leq q \leq p$. By identifying the properties of the fundamental fields with respect to scale transformations and spin, one can recognize that the dynamical variables for $m = 3, 4, 5, 6$ ($c = \frac{1}{2}, \frac{7}{10}, \frac{4}{5}, \frac{6}{7}$) are those of the Ising, tricritical Ising, 3-state Potts and 3-state tricritical Potts models, respectively. Furthermore, looking at the singular structure of correlations at short-distances, which is fixed by the conformal invariance, one is able to obtain the critical exponents as functions of c, $\bar{c}$, h and $\bar{h}$. Indeed

$$\langle \Phi(r)\Phi(0) \rangle \sim \frac{1}{r^{\eta}}, \tag{69}$$

$$\langle \Phi^2(r)\Phi^2(0) \rangle \sim \frac{1}{r^{4-2/\nu}}, \tag{70}$$

define η and ν; their values can, then, be extracted from the singular behavior predicted by conformal invariance. From these two exponents, we can obtain all others via scaling relations.

5. FIELD THEORY DESCRIPTION OF POLYMERS: AN EXAMPLE

To illustrate one of the current uses of field theory in condensed matter physics, we have chosen to discuss linearly conjugated polymers, with polyacetylene being the prototypical object of our interest.

Polyacetylene exists in nature in two forms: trans- and cis-polyacetylene. The former alternates single and double bonds between carbon atoms along a linear chain so that the single bonds adjacent to a double bond lie on opposite sides; the latter has the single bonds on the same side with respect to the double bond in between. For our purposes, it suffices to know that the trans-variety has a doubly degenerate ground-state, whereas the cis-variety has a unique ground-state which, however, has an energy very close to that of the first-excited state.

A simple model that captures the essential features of the physics of polyacetylene is the Su-Schrieffer-Heeger (SSH) model [19], whose Hamiltonian is

$$\hat{H}_{\mathrm{SSH}} = \sum_n \left[\frac{p_n^2}{2M} + K(y_n - y_{n+1})^2 + t_{n,n+1}(c_n c_{n+1}^{\dagger} + c_{n+1} c_n^{\dagger}) \right], \tag{71}$$

where M is the mass of the CH ions, K is a spring constant, y_n denotes the displacement of the ions from the nth site of an ideal linear lattice, c_n ($c_n^{\dagger}$) are annihilation (creation) operators of π-electrons and $t_{n,n+1}$ is a hopping amplitude for an electron between sites n and $n+1$. In a linearized approximation, the latter is given by

$$t_{n,n+1} = t_0 + (-1)^n(2\mu) + \gamma(y_n - y_{n+1}) + O([\Delta y]^2). \tag{72}$$

γ is related to the electron-phonon coupling; for $\mu = 0$, we have trans-polyacetylene, while $\mu \neq 0$ describes cis-polyacetylene.

We may introduce what is known as a dimerization parameter, $u_n \equiv (-1)^n y_n$, which measures deviations from an ideal lattice. If we are only interested in physical quantities measured at distances much larger than a lattice spacing, we may take a (naïve) continuum limit of the model, by letting the lattice parameter go to zero. Then, sums will become integrals and finite differences will turn into derivatives. The dimerization parameter $u_n(t)$ will give rise to a field, $\phi(x,t)$. If we linearize the dispersion relation for π-electrons, near the Fermi momentum, the fermionic degrees of freedom will become

$$\begin{bmatrix} c_n e^{-ik_F na} \\ c_n^\dagger e^{+ik_F na} \end{bmatrix} \sim \psi(x,t) = \begin{bmatrix} v_L(x,t) \\ v_R(x,t) \end{bmatrix}. \tag{73}$$

The SSH-model will, then, correspond to a Lagrangian density

$$\mathcal{L}_{\text{SSH}} = \tfrac{1}{2}\dot{\phi}^2 - \tfrac{1}{2}\phi^2 + \bar{\psi}(i\gamma^\nu \partial_\nu - \mu - g\phi)\psi, \tag{74}$$

where g is related to γ and we have used $\hbar = 1$, $v_F = 1$ (the Fermi velocity). We can investigate this continuum model by using field-theoretic methods. Indeed, integration over fermions yields an effective action, $S_{\text{eff}}[\phi]$, whose extrema can be found exactly, by means of inverse scattering methods, for both trans- [20] and cis-polyacetylene [21]. They correspond to dimerized, soliton and polaron configurations (defects).

These defects play an important role in the mechanism of conduction of electrical current at low doping, and can be observed indirectly through optical absorption experiments. The field theory analysis allows one to discover various types of defects, which may be stable, unstable or metastable, depending on the number of electrons which are bound to them [22]. It also allows one to calculate physical parameters of polyacetylene, such as the splitting between the first excited and ground-states of cis-polyacetylene, as well as a limiting velocity for the motion of defects [23].

6. CONCLUSION

In summary, the formulation of field theory has become, over the years, a common tool in the analysis of condensed matter systems. Recent examples are the studies of the quantum Hall effect, superfluids and superconductors.

Acting as a bridge between high and low-energy physics, the formalism has also led to the investigation of the statistical mechanics of high-energy systems. In fact, heavy-ion collisions and cosmological phase transitions have been extensively investigated using the techniques of condensed matter physics and statistical mechanics, made available through the formal closeness between these disciplines and field theory. Interesting progress is also being made in the analysis of systems out of equilibrium [24].

Field theory in condensed matter physics, as well as the study of the statistical mechanics of field theories, apart from pointing towards the unity of physics, will certainly play an important role in the discovery of fascinating new phenomena.

ACKNOWLEDGEMENTS

It is a pleasure to thank the organizers of the XXXIX meeting of the Mexican Physical Society for their kind invitation. In particular, the visit to the astronomical observatory at Monte Alban, a magnificent evidence of the advancement of the ancient Mexican cultures, is greatfully acknowledged. This work was funded by FUJB/UFRJ, CNPq and CLAF.

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