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Solution by the separation method of the motion of a rigid body with no forces

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ABSTRACT. The motion of a rotating rigid body with no torques, can be separated and solved in a spheroconal coordinate system, in classical mechanics and also in quantum mechanics. In this paper the solution to the Euler case of no torques is reviewed and afterwards the same case is solved in new coordinates, where the problem is separable. Nevertheless since it is not evident *a priori* the relation with the usual coordinates, the solution in the new coordinates is developed until various explicit relations with the traditional solution are made.

RESUMEN. El movimiento de un sólido rotante en ausencia de torcas puede ser separado y resuelto en un sistema coordenado esferoconal, tanto en mecánica clásica como cuántica. En este trabajo se revisa la solución del caso de Euler y a continuación se resulve en nuevas variables de separación hasta obtener relaciones explícitas con la solución tradicional.

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1. INTRODUCTION

The free motion of the rigid body without torques has been tied to the name of Leonard Euler, because he started the study of this motion in the XVIII century. The analysis of this motion reaches its highest development with Jacobi in the middle of the XIX century with the evaluation of Euler angles in terms of elliptic functions and its expression in terms of a Fourier series which converges extremely fast.

The known solution of the Euler's rigid body [1] was not sufficient for the quantization according to the methods of the old quantum mechanics, where the separability in the Liouville sense [2], is not directly applicable in the coordinates used originally to solve this problem. Reiche [3] finds that the problem of the Euler's no torque rigid body is separable whenever the component of the angular momentum vector in the inertial system, which is equal to the canonical moment corresponding to the gyration angle around the angular momentum, is zero. That results from a particular selection of the coordinate frame in the inertial system. In addition, proper elliptic coordinates are chosen.

Once the existence of a coordinate system where a mechanical problem is separable is recognized, the solution to it is reduced to quadratures and it is trivially integrable.

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However, by direct comparison of the Jacobi and Reiche coordinates, one discovers that both can be expressed in terms of Weber conical coordinates [4], but with an essential difference in the parameter m of the Jacobi elliptic functions that are used to define the two different conical coordinates.

We are therefore faced with the necessity of finding the explicit relation between two solutions to the same problem. This relation seems far from trivial because the existence of such essential difference in the parameters.

In another context, the question of solving the same mechanical problem in quantum mechanics by considering the Schrödinger equation of the rigid body model has been considered. Schrödinger's equation is simplified [5] when one takes into account that the angular momentum components in the inertial system commute with the Hamiltonian and with the square of the angular momentum vector in the body system. It is then sufficient to consider the simultaneous eigenfunctions of these two operators: Hamiltonian and square of the angular momentum, assuming that the eigenvalue of the operator associated to a given component of the angular momentum in the inertial system is zero. In terms of Euler angles this simplification leads us to a problem where only two of these angles appear. This implies no loss of generality because the known angular momentum techniques allow us to find the wave eigenfunctions corresponding to any non-zero eigenvalue of that component of the inertial angular momentum.

However in such a case the simplified Schrödinger equation is separable [5] in the same coordinates used by Reiche to separate the classical problem. It seems then quite interesting to make explicit the solution of the classical case in elliptic coordinates similar to the ones used by Reiche in order to easily study in the same coordinate system the quantum limit to the classical case.

This paper begins with the study in conical coordinates to construct the Jacobi solution. The form that we use has been presented previously by Piña [6, 7] by means of the hypothesis that a constant vector of the inertial system is transformed by the rotation matrix in a variable vector of the body system.

In the Sect. 2, the separation of variables according to an identity by Reiche [3] is achieved, with the introduction of conical coordinates also associated to the parametrization used by Piña [6], albeit with a different value of the parameter in t he Jacobi elliptic functions.

When we carry out explicitly the integration of the equations of motion in the new variables, there appear in a natural way, different new coordinates, which are dependent on Jacobi elliptic functions, with the same value of the parameter as the one used by Jacobi. We followed this hint until we were able to make transparent the relation between this solution and the classical one.

2. The motion of a rigid body without torques

Let us begin the study of the rotational motion of a rigid body in the absence of torques.

The sustitution of the torque equal to zero in the equation of motion immediately furnishes us with the conservation of the components of the angular momentum vector

$\mathbf{J}$ in the inertial frame:

$$\mathbf{J} = \text{ constant of motion.} \tag{1}$$

On the other hand, if the torques is set equal to zero in the Euler equations, we will get the equation of motion for the components of the angular momentum vector $\mathbf{L}$ in t he moving frame:

$$\dot{\mathbf{L}} + (\mathbf{I}^{-1}\mathbf{L}) \times \mathbf{L} = 0. \tag{2}$$

These equations are to be integrated in the system of principal axis of inertia, in which they acquire the form

$$\begin{aligned} \dot{L}_1 &= \left(\frac{1}{I_3} - \frac{1}{I_2}\right) L_2\, L_3, \\ \dot{L}_2 &= \left(\frac{1}{I_1} - \frac{1}{I_3}\right) L_3\, L_1, \\ \dot{L}_3 &= \left(\frac{1}{I_2} - \frac{1}{I_1}\right) L_1\, L_2. \end{aligned} \tag{3}$$

The scalar product of (3) with the vector $\mathbf{L}$ yields the constancy of the magnitude of this vector,

$$\mathbf{L}^T\,\mathbf{L} = \text{constant}, \tag{4}$$

that results also from the fact that the constant vector $\mathbf{J}$ is the rotation of the vector $\mathbf{L}$

$$\mathbf{J} = \mathbf{R}\,\mathbf{L}. \tag{5}$$

Let J to be the magnitude of this vector. Hence J^2 is the constant (4)

$$\mathbf{L}^T\,\mathbf{L} = \mathbf{J}^T\,\mathbf{J} = J^2. \tag{6}$$

The scalar product of the Euler equations with the angular velocity vector provides a second constant of motion, equal to the kinetic energy

$$\boldsymbol{\omega} = \mathbf{I}^{-1}\,\mathbf{L}, \tag{7}$$

yielding

$$T = \tfrac{1}{2}\mathbf{L}^T\,\mathbf{I}^{-1}\,\mathbf{L} = \text{constant}. \tag{8}$$

If we denote this constant by E, since in the absence of torques it represents the total energy,

$$\mathbf{L}^T\,\mathbf{I}^{-1}\,\mathbf{L} = 2E. \tag{9}$$

The sustitution of the two constants of motion, energy and angular momentum, in the equations of motion (3) will give us the time as an elliptic integral of one component of the angular momentum vector. The components of the vector $\mathbf{L}$ are therefore elliptic functions of time. In what follows its explicit dependence in time will be presented; for the time being, it will be enough to point out that it is possible to know the components of $\mathbf{L}$ as function of time.

We face the case 3 in which the components of two vectors $\mathbf{J}$ and $\mathbf{L}$ are known, with one vector being rotated with respect to the other by means of the rotation matrix. In Ref. 6 we see how to parametrize the rotation matrix in terms of these two vectors and of another parameter denoted by γ.

It is frequent in the literature [1] to associate $\mathbf{J}$ to the constant direction of the axis 3 and to $\mathbf{L}$ one assigns the Euler angles θ and ψ as spherical coordinates:

$$\mathbf{L} = J \begin{pmatrix} \sin\theta\sin\psi \\ \sin\theta\cos\psi \\ \cos\theta \end{pmatrix}. \tag{10}$$

From the integration of $\mathbf{L}$ the knowledge of those two Euler angles as a function of time results. However, the symmetry of the problem in no way implies the introduction of ordinary spherical coordinates for $\mathbf{L}$, except when the ellipsoid (8) happens to be a revolution ellipsoid. Only in such circumstance the use of Euler angles is recommended.

In general, Euler angles are not the best coordinates, and one should use the spheroconal coordinates instead. These include as coordinate curves on the sphere, its intersection with elliptic cones. These coordinates are expressed by means of the Jacobi elliptic functions furnishing the dependence in the time of the components of the angular momentum vector.

Denote by $\mathbf{l}$ the unitary vector in the variable direction $\mathbf{L}$

$$\mathbf{l}^{T}\mathbf{l} = 1. \tag{11}$$

The Euler Eqs. (3) in function of the vector $\mathbf{l}$ become

$$\begin{aligned} \dot{l}_1 &= l_2\, l_3 \left(\frac{J}{I_3} - \frac{J}{I_2}\right), \\ \dot{l}_2 &= l_3\, l_1 \left(\frac{J}{I_1} - \frac{J}{I_3}\right), \\ \dot{l}_3 &= l_2\, l_1 \left(\frac{J}{I_2} - \frac{J}{I_1}\right), \end{aligned} \tag{12}$$

where l_1, l_2, l_3, are the components of the vector $\mathbf{l}$.

The properties of the Jacobi elliptic functions that are used to integrate the equations of motion are the following five, two of them are not independent of the other three:

$$\mathrm{sn}^2(\xi, m) + \mathrm{cn}^2(\xi, m) = 1,$$
$$m\ \mathrm{sn}^2(\xi, m) + \mathrm{dn}^2(\xi, m) = 1,$$

$$\frac{d}{d\xi}\mathrm{sn}(\xi, m) = \mathrm{cn}(\xi, m)\,\mathrm{dn}(\xi, m),$$
$$\frac{d}{d\xi}\mathrm{cn}(\xi, m) = -\mathrm{sn}(\xi, m)\,\mathrm{dn}(\xi, m),$$
$$\frac{d}{d\xi}\mathrm{dn}(\xi, m) = -m\mathrm{sn}(\xi, m)\,\mathrm{cn}(\xi, m),$$

where m is the parameter of these Jacobi functions, previously denoted in the past literature by $k = m^{1/2}$.

Introduce the spheroconal coordinates defined in terms of the Jacobi's elliptic functions, sn, cn and dn, with parameters m and m'.

$$\begin{aligned} l_1 &= \mathrm{sn}(\alpha_2, m')\,\mathrm{dn}(\alpha_1, m), \\ l_2 &= \mathrm{dn}(\alpha_2, m')\,\mathrm{sn}(\alpha_1, m), \\ l_3 &= \mathrm{cn}(\alpha_2, m')\,\mathrm{cn}(\alpha_1, m). \end{aligned} \tag{13}$$

The spheroconal coordinates will be orthogonal on the sphere of unit radius when the equation

$$m + m' = 1 \tag{14}$$

for the parameters m and m' holds.

In addition, one asks that Eq. (9) should be satisfied identically when expression (13) of the spheroconal coordinates, for a constant value of the coordinate α_2 and for any value of the coordinate α_1 is substituted in it. This is satisfied when and only when (up to the sign of the square root)

$$\mathrm{sn}(\alpha_2, m') = \sqrt{\frac{\frac{2E}{J} - \frac{J}{I_3}}{\frac{J}{I_1} - \frac{J}{I_3}}}, \tag{15}$$

$$m = \frac{\left(\frac{J}{I_2} - \frac{J}{I_3}\right)\left(\frac{J}{I_1} - \frac{2E}{J}\right)}{\left(\frac{J}{I_1} - \frac{J}{I_2}\right)\left(\frac{2E}{J} - \frac{J}{I_3}\right)}, \tag{16}$$

and taking into account (14)

$$m' = \frac{\left(\frac{J}{I_1} - \frac{J}{I_3}\right)\left(\frac{2E}{J} - \frac{J}{I_2}\right)}{\left(\frac{J}{I_1} - \frac{J}{I_2}\right)\left(\frac{2E}{J} - \frac{J}{I_3}\right)}, \tag{17}$$

and the vector (13) expressed only as function of the variable α_1 satisfies identically (9) and (11):

$$l_1 = \sqrt{\frac{\frac{2E}{J} - \frac{J}{I_3}}{\frac{J}{I_1} - \frac{J}{I_3}}}\,\mathrm{dn}(\alpha_1, m),$$

$$l_2 = \sqrt{\frac{\frac{J}{I_1} - \frac{2E}{J}}{\frac{J}{I_1} - \frac{J}{I_2}}}\,\mathrm{sn}(\alpha_1, m),$$

$$l_3 = \sqrt{\frac{\frac{J}{I_1} - \frac{2E}{J}}{\frac{J}{I_1} - \frac{J}{I_3}}}\,\mathrm{cn}(\alpha_1, m). \tag{18}$$

These are the parametric equations of the curve where the sphere and the cone $\alpha_2 =$ constant meet.

But the principal reason for introducing the spheroconal coordinates comes after these expressions are replaced in any of the Euler Eqs. (12). This becomes

$$\frac{d\alpha_1}{dt} = \sqrt{\left(\frac{2E}{J} - \frac{J}{I_3}\right)\left(\frac{J}{I_1} - \frac{J}{I_2}\right)}, \tag{19}$$

showing that the spheroconal coordinate α_1 is a linear function of time. One discovers, as did Jacobi, that these coordinates are natural for the problem of the free asymmetric rigid body.

The three Jacobi functions, used to describe the motion of the vector $\mathbf{l}$, are periodic functions with the same period. The three can be seen also as inverse functions of elliptic integrals of first kind. The period of these three functions with respect to the variable α_1 is $4K$ where K is the complete elliptic integral of first kind

$$K(m) = \int_0^{\pi/2} \frac{dx}{\sqrt{1 - m\sin^2 x}}. \tag{20}$$

The period with respect to the time is then

$$4K(m)\sqrt{\frac{I_1 I_2 I_3}{(I_2 - I_1)(2EI_3 - J^2)}}. \tag{21}$$

We can now get the rotation matrix up to the parameter γ from Ref. 6, since the rotation matrix depends on the vectors $\mathbf{l}$ and $\mathbf{k}$ which are related by the equation

$$\mathbf{k} = \mathbf{R}\,\mathbf{l},$$

where $\mathbf{k}$ is the constant vector in the direction of the angular momentum of the inertial system $\mathbf{J}$ and $\mathbf{l}$ is the vector (18), function of t.

On the other hand, one knows [6] the angular velocity in terms of the parameters $\mathbf{l}$ and γ

$$\boldsymbol{\omega} = \frac{1}{1 + \mathbf{l}^T\mathbf{k}}\,[\dot{\mathbf{l}} \times (\mathbf{l} + \mathbf{k})] + \dot{\gamma}\,\mathbf{l}, \tag{22}$$

and the same is known in function of time after one knows the angular momentum vector $\mathbf{L} = J\mathbf{l}$, because of the relation (7), which give us

$$\boldsymbol{\omega} = \mathbf{I}^{-1}\,\mathbf{L} = J\,\mathbf{I}^{1}\,\mathbf{l}. \tag{23}$$

Solving $\dot{\gamma}$ from (21) by taking the scalar product on both sides of this equation with the vector $\mathbf{k} + \mathbf{l}$, becomes

$$\dot{\gamma} = \frac{(\mathbf{l} + \mathbf{k})^T \boldsymbol{\omega}}{1 + \mathbf{k}^T\,\mathbf{l}}, \tag{24}$$

and replacing (23) in (24) we have

$$\dot{\gamma} = J\frac{(\mathbf{l} + \mathbf{k})^T \mathbf{I}^{-1}\,\mathbf{l}}{1 + \mathbf{k}^T\,\mathbf{l}}. \tag{25}$$

The obtention of γ is reduced to the integration with respect to time of the right hand side of this equation which is a periodic function of time. The integral in (25) has not the standard form, used in the texts, because in them this problem is solved using Euler angles.

By means of Eq. (10), we know two Euler angles θ and ψ as spherical coordinates of the vector $\mathbf{l}$, and by means of the same Eq. (10), where $\mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, we can change the variables to Euler's angle ϕ

$$\dot{\phi} = \dot{\gamma} - \dot{\psi}, \tag{26}$$

where

$$\psi = \text{arc tan}\,\frac{l_1}{l_2} \tag{27}$$

and

$$\dot{\psi} = \frac{l_2\,\dot{l}_1 - l_1\,\dot{l}_2}{l_1^2 + l_2^2} = \frac{\mathbf{k}^T[(\mathbf{l} \times \boldsymbol{\omega}) \times \mathbf{l}]}{1 - (\mathbf{k}^T\,\mathbf{l})^2} = J\frac{\mathbf{k}^T\,\mathbf{I}^{-1}\,\mathbf{l} - \mathbf{k}^T\,\mathbf{l}\,\mathbf{l}^T\,\mathbf{I}^{-1}\,\mathbf{l}}{1 - (\mathbf{k}^T\,\mathbf{l})^2}. \tag{28}$$

Therefore

$$\dot{\phi} = \frac{\frac{2E}{J} - \frac{J}{I_3}(\mathbf{k}^T\mathbf{l})^2}{1 - (\mathbf{k}^T\mathbf{l})^2} = \frac{J}{I_3} + \frac{\frac{2E}{J} - \frac{J}{I_3}}{1 - \frac{J/I_1 - 2E/J}{J/I_1 - J/I_3}\,\text{cn}^2(\alpha^1, \text{m})}, \tag{29}$$

where the variable α_1 is linear in time according to (19):

$$\alpha_1 = t\sqrt{\left(\frac{2E}{J} - \frac{J}{I_3}\right)\left(\frac{J}{I_1} - \frac{J}{I_2}\right)} = a\,t.$$

Integration of (29), written as function of the constant parameter α_2, gives

$$\phi = \frac{Jt}{I_3} + \left(\frac{2E}{J} - \frac{J}{I_3}\right) \int_0^{\alpha_1} du \frac{1}{1 - \mathrm{cn}^2(\alpha_2, m')\,\mathrm{cn}^2(u, m)} \tag{30}$$

which by means of the elliptic integral of the third kind

$$\Pi(u, a) = \int_0^u du \frac{m\,\mathrm{sn}(a, m)\,\mathrm{cn}(a, m)\,\mathrm{dn}(a, m)\,\mathrm{sn}^2(u, m)}{1 - m\,\mathrm{sn}^2(a, m)\,\mathrm{sn}^2(u, m)} \tag{31}$$

is written in the form

$$\begin{aligned}\phi &= \frac{Jt}{I_3} + \frac{\mathrm{sn}(\alpha_2, m')\,\mathrm{dn}(\alpha_2, m')}{\mathrm{cn}(\alpha_2, m')} \int_0^{at} du \frac{1}{1 - \mathrm{cn}^2(\alpha_2, m')\,\mathrm{cn}^2(u, m)} \\ &= \frac{Jt}{I_3} + at \frac{\mathrm{dn}(\alpha_2, m')}{\mathrm{sn}(\alpha_2, m')\,\mathrm{cn}(\alpha_2, m')} + \frac{1}{i}\Pi(at, i\alpha_2 - iK') \\ &= \frac{Jt}{I_1} + \frac{1}{i}\Pi(at, i\alpha_2 - iK'). \end{aligned} \tag{32}$$

This could also have been expressed with the Jacobi's functions theta (Θ and H).

3. Solution by the method of separation of variables

In this section we present another form to solve the problem of the motion of a rigid body without torques.

We shall use the Lagrangian and Hamiltonian formalisms. Both Lagrange function and Hamilton function are equal to the kinetic energy because there are not torques.

Starting from the kinetic energy in the form

$$T = \tfrac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2), \tag{33}$$

which is written as a function of coordinates, similar to the previously ones used in Sect. 2, *i.e.*, assuming again that the rotation matrix transforms the variable vector $\mathbf{u}$ into the constant vector $\mathbf{v} = \begin{pmatrix}1\\0\\0\end{pmatrix}$ in the form

$$\mathbf{v} = \mathbf{R}\,\mathbf{u}. \tag{34}$$

The rotation matrix was parametrized in terms of these two vectors and of an angle η. The angular velocity in these coordinates is given in similar form to (22) by

$$\boldsymbol{\omega} = \frac{1}{1 + \mathbf{u}^T\mathbf{v}}[\dot{\mathbf{u}} \times (\mathbf{u} + \mathbf{v})] + \dot{\eta}\,\mathbf{u}. \tag{35}$$

Sustituting this form of $\boldsymbol{\omega}$ in the kinetic energy one finds that the coordinate η is a cyclic variable, hence one has the conservation of its conjugated canonical momentum

$$p_\eta = \frac{\partial T}{\partial \dot{\eta}} = \boldsymbol{\omega}^T\mathbf{I}\,\mathbf{u} = \mathbf{L}^T\,\mathbf{u} = \mathbf{J}^T\,\mathbf{v}. \tag{36}$$

In the following we shall use the identity published by Reiche [3] in 1918:

$$(\mathbf{u}\times\boldsymbol{\omega})^T\mathbf{I}^{-1}(\mathbf{u}\times\boldsymbol{\omega})+\frac{1}{I_1I_2I_3}[\boldsymbol{\omega}^T\mathbf{I}\,\mathbf{u}]^2=[I_1\omega_1^2+I_2\omega_2^2+I_3\omega_3^2]\left[\frac{u_1^2}{I_2I_3}+\frac{u_2^2}{I_3I_1}+\frac{u_3^2}{I_1I_2}\right], \quad (37)$$

and from it one deduces a suitable form of the kinetic energy

$$T=\frac{1}{2}\frac{(1/I_1)\dot u_1^2+(1/I_2)\dot u_2^2+(1/I_3)\dot u_3^2+(1/I_1I_2I_3)p_\eta^2}{(u_1^2/I_2I_3)+(u_2^2/I_3I_1)+(u_3^2/I_1I_2)} \quad (38)$$

where the equation

$$\dot{\mathbf{u}}=\mathbf{u}\times\boldsymbol{\omega}, \quad (39)$$

has been used.

But Reiche also proves that this form of the classical kinetic energy is separable in the Liouville sense when the conjugated canonical moment to the η angle is zero

$$p_\eta=0. \quad (40)$$

The quantum case is also separable provided a similar condition for the corresponding eigenvalue of one angular momentum component in the inertial frame holds [5].

The coordinates for $\mathbf{u}$ that separate the problem are also spheroconal but with different values of the parameters m than those in the solution of the previous section:

$$\mathbf{u}=\begin{pmatrix}\mathrm{dn}(c\phi_1,m_1) & \mathrm{sn}(c\phi_2,m_2)\\ \mathrm{cn}(c\phi_1,m_1) & \mathrm{cn}(c\phi_2,m_2)\\ \mathrm{sn}(c\phi_1,m_1) & \mathrm{dn}(c\phi_2,m_2)\end{pmatrix}, \quad (41)$$

which are orthogonal on the unit sphere, provided the following condition holds

$$m_1+m_2=1. \quad (42)$$

The parameters m_1 and m_2 are determined by the moments of inertia

$$m_1=\frac{\frac{1}{I_2}-\frac{1}{I_3}}{\frac{1}{I_1}-\frac{1}{I_3}}, \qquad m_2=\frac{\frac{1}{I_1}-\frac{1}{I_2}}{\frac{1}{I_1}-\frac{1}{I_3}}, \quad (43)$$

where one assumes $I_3>I_2>I_1$ and where c is the constant

$$c=\sqrt{\frac{1}{I_1}-\frac{1}{I_3}}. \quad (44)$$

When writting the energy or the Hamiltonian, two new functions appear

$$\mathcal{P}(\phi_1)=\frac{1}{I_3}+\left(\frac{1}{I_2}-\frac{1}{I_3}\right)\mathrm{sn}^2(c\phi_1,m_1), \quad (45)$$

$$\mathcal{P}(\phi_2) = \frac{1}{I_1} - \left(\frac{1}{I_1} - \frac{1}{I_2}\right) \mathrm{sn}^2(c\phi_2, m_2), \tag{46}$$

$$E = \frac{1}{2}\left[\dot{\phi}_1^2 \mathcal{P}(\phi_1) + \dot{\phi}_2^2 \mathcal{P}(\phi_2)\right]\left(\frac{1}{\mathcal{P}(\phi_1)} - \frac{1}{\mathcal{P}(\phi_2)}\right), \tag{47}$$

which is observed adopts the familiar structure [2] to be separable in Liouville sense.

From the energy, the conjugate canonical momenta to ϕ_1 and to ϕ_2,

$$\begin{aligned} p_1 &= \frac{\partial E}{\partial \dot{\phi}_1} = \dot{\phi}_1 \mathcal{P}(\phi_1)\left(\frac{1}{\mathcal{P}(\phi_1)} - \frac{1}{\mathcal{P}(\phi_2)}\right) \\ p_2 &= \frac{\partial E}{\partial \dot{\phi}_2} = \dot{\phi}_2 \mathcal{P}(\phi_2)\left(\frac{1}{\mathcal{P}(\phi_1)} - \frac{1}{\mathcal{P}(\phi_2)}\right) \end{aligned} \tag{48}$$

follow, and the Hamiltonian is

$$H = \frac{1}{2}\,\frac{p_1^2\,\mathcal{P}(\phi_2) + p_2^2\,\mathcal{P}(\phi_1)}{\mathcal{P}(\phi_2) - \mathcal{P}(\phi_1)}. \tag{49}$$

The Hamilton-Jacobi equation in these coordinates is separable.

The angular momentum vector $\mathbf{L}$ in the body system is written as a function of the two vectors orthogonal to $\mathbf{u}$

$$\mathbf{e}_1 = \frac{\partial \mathbf{u}}{\partial \phi_1}, \qquad \mathbf{e}_2 = \frac{\partial \mathbf{u}}{\partial \phi_2}. \tag{50}$$

These two vectors are orthogonal to each other and have the same magnitude, whose square is denoted by F:

$$F = \mathbf{e}_1 \cdot \mathbf{e}_1 = \mathbf{e}_2 \cdot \mathbf{e}_2 = \mathcal{P}(\phi_2) - \mathcal{P}(\phi_1). \tag{51}$$

We have

$$\mathbf{L} = \frac{1}{F}(-p_1 \mathbf{e}_2 + p_2 \mathbf{e}_1). \tag{52}$$

Hamilton equations of motion for these coordinates are then

$$\dot{\phi}_1 = \frac{\partial H}{\partial p_1} = \frac{p_1 \mathcal{P}(\phi_2)}{\mathcal{P}(\phi_2) - \mathcal{P}(\phi_1)}, \qquad \dot{\phi}_2 = \frac{\partial H}{\partial p_2} = \frac{p_2 \mathcal{P}(\phi_1)}{\mathcal{P}(\phi_2) - \mathcal{P}(\phi_1)}\ ; \tag{53}$$

subsequent sustitution of the momenta as a function of the coordinates, which are solved from the equation of energy conservation

$$2\ E = \frac{p_1^2\,\mathcal{P}(\phi_2) + p_2^2\,\mathcal{P}(\phi_1)}{\mathcal{P}(\phi_2) - \mathcal{P}(\phi_1)}, \tag{54}$$

and from the angular momentum square,

$$J^2 = \frac{p_1^2 + p_2^2}{\mathcal{P}(\phi_2) - \mathcal{P}(\phi_1)} \tag{55}$$

gives us

$$\dot{\phi}_1 = \frac{J\mathcal{P}(\phi_2)}{\mathcal{P}(\phi_2) - \mathcal{P}(\phi_1)}\sqrt{\frac{2E}{J^2} - \mathcal{P}(\phi_1)},$$

$$\dot{\phi}_2 = \frac{J\mathcal{P}(\phi_1)}{\mathcal{P}(\phi_2) - \mathcal{P}(\phi_1)}\sqrt{\mathcal{P}(\phi_2) - \frac{2E}{J^2}}. \tag{56}$$

These equations are explicitly separated by using an auxiliary variable σ, defined by:

$$\frac{d\phi_1}{d\sigma} = \frac{\sqrt{2E/J^2 - \mathcal{P}(\phi_1)}}{\mathcal{P}(\phi_1)}, \qquad \frac{d\phi_2}{d\sigma} = \frac{\sqrt{\mathcal{P}(\phi_2) - 2E/J^2}}{\mathcal{P}(\phi_2)}, \tag{57}$$

where the dependence on time of σ can be obtained from the equation

$$\frac{dt}{d\sigma} = \frac{1}{J}\left(\frac{1}{\mathcal{P}(\phi_1)} - \frac{1}{\mathcal{P}(\phi_2)}\right). \tag{58}$$

The problem has thus been reduced to quadratures.

The integrals in Eqs. (57) are elliptic if the coordinates $\mathcal{P}(\phi_1)$ and $\mathcal{P}(\phi_2)$, are used; Gradshteyn and Ryzhik [8] prescribe a change of variables that leads us to the coordinates transformation

$$\mathrm{sn}(h_1, m) = \frac{\sqrt{m_2}\,\mathrm{sn}(c\phi_1, m_1)}{\mathrm{dn}(c\phi_1, m_1)}, \tag{59}$$

by means of which the first equation of motion (57) is transformed into

$$\frac{d\sigma}{dh_1} = \frac{J}{a}\mathcal{P}(\phi_1) = \frac{J}{a}\left(\frac{1}{I_1} - \frac{(1/I_1) - (2E/J^2)}{1 - \mathrm{sn}^2(\alpha_2, m')\,\mathrm{dn}^2(h_1, m)}\right). \tag{60}$$

In a similar way, the transformation of the second equation in (57) by means of the change of variable [8]

$$\sqrt{m}\,\mathrm{sn}(h_2, m) = -\frac{\sqrt{m_1}\,\mathrm{sn}(c\phi_2, m_2)}{\mathrm{dn}(c\phi_2, m_2)}, \tag{61}$$

gives us the following equation of motion:

$$\frac{d\sigma}{dh_2} = \frac{J}{a}\mathcal{P}(\phi_2) = \frac{J}{a}\left(\frac{1}{I_3} + \frac{(2E/J^2) - (1/I_3)}{1 - \mathrm{cn}^2(\alpha_2, m')\,\mathrm{cn}^2(h_2, m)}\right). \tag{62}$$

The changes of coordinates (59) and (61) allow us to recover, in a natural way, not only the parameter m, but also the parameter α_2 that appears in (60) and (62). The Eqs. (60) and (62) are similar to those which are found in the traditional solution of the Euler case: compare for example Eqs. (30) and (62). Although the physical meaning is very different, the mathematics is identical, even for the numerical value of the parameters.

A quite simple property follows now by sustitution of the equations of motion into the equation relating the time with σ:

$$a\frac{dt}{d\sigma} = \frac{dh_1}{d\sigma} - \frac{dh_2}{d\sigma}, \tag{63}$$

and it follows

$$a\,t = h_1 - h_2. \tag{64}$$

One can arrive to this equation from an independent argument as follows. After rewriting the vector $\mathbf{u}$ as a function of the elliptic coordinates h_1 and h_2, we obtain the expression

$$\mathbf{u} = \frac{1}{\sqrt{1-\mathrm{sn}^2(\alpha_2,m')\,\mathrm{dn}^2(h_1,m)}}\,\frac{1}{\sqrt{1-\mathrm{cn}^2(\alpha_2,m')\,\mathrm{cn}^2(h_2,m)}} \times \begin{pmatrix} -\mathrm{cn}(\alpha_2,m')\,\mathrm{dn}(\alpha_2,m')\,\mathrm{sn}(h_2,m) \\ -\mathrm{sn}(\alpha_2,m')\,\mathrm{cn}(\alpha_2,m')\,\mathrm{cn}(h_1,m)\mathrm{dn}(h_2,m) \\ \mathrm{sn}(\alpha_2,m')\,\mathrm{dn}(\alpha_2,m')\,\mathrm{sn}(h_1,m) \end{pmatrix}. \tag{65}$$

The orthogonality condition between this vector and the vector $\mathbf{l}$ that appears in Eq. (13) follows immediately.

One thus finds Eq. (64) as the condition that should be satisfied in order for these vectors to be orthogonal, in the form

$$\mathrm{dn}(at,m)\,\mathrm{sn}(h_2,m) - \mathrm{sn}(at,m)\,\mathrm{cn}(h_1,m)\mathrm{dn}(h_2,m) + \mathrm{cn}(at,m)\,\mathrm{sn}(h_1,m) = 0. \tag{66}$$

Equations (61) and (62) are integrated in terms of the elliptic function $\Pi(u,a)$ and one finds

$$\sigma = \frac{J\,h_1}{aI_3} + \frac{1}{i}\Pi(h_1, i\alpha_2), \tag{67}$$

and

$$\sigma = \frac{J\,h_2}{aI_1} + \frac{1}{i}\Pi(h_2, i\alpha_2 - iK'). \tag{68}$$

Upon cancelling σ between these equations and using the next two following properties of the functions $\Pi(u,a)$ [9],

$$\frac{Ju}{aI_3} + \frac{1}{i}\Pi(u,i\alpha_2) - \frac{Ju}{aI_1} - \frac{1}{i}\Pi(u,i\alpha_2 - iK') = \frac{1}{2i}\ln\frac{\mathrm{sn}(u - i\alpha_2 + iK', m)}{\mathrm{sn}(u + i\alpha_2 - iK', m)} - \frac{\pi}{2} \tag{69}$$

and [10]

$$\begin{aligned}&\Pi(h_2 + at, i\alpha_2 - iK') - \Pi(h_2, i\alpha_2 - iK') + \Pi(at, i\alpha_2 - iK') \\ &\quad = -\frac{1}{2i}\ln\frac{1 - m\,\mathrm{sn}(i\alpha_2 - iK')\,\mathrm{sn}(h_2)\,\mathrm{sn}(at)\,\mathrm{sn}(h_2 + at - i\alpha_2 + iK')}{1 + m\,\mathrm{sn}(i\alpha_2 - iK')\,\mathrm{sn}(h_2)\,\mathrm{sn}(at)\,\mathrm{sn}(h_2 + at - i\alpha_2 - iK')},\end{aligned} \tag{70}$$

the equation

$$\begin{aligned}\phi(t) &= \frac{Jt}{I_1} + \frac{1}{i}\pi(at, i\mu - iK') \\ &= \frac{\pi}{2} + \frac{1}{2i}\ln\frac{\mathrm{sn}(h_2 + at - i\alpha_2, m) - \sqrt{m}\,\mathrm{sn}(i\alpha_2 - iK', m)\,\mathrm{sn}(h_2, m)\,\mathrm{sn}(at, m)}{\mathrm{sn}(h_2 + at + i\alpha_2, m) + \sqrt{m}\,\mathrm{sn}(i\alpha_2 - iK', m)\,\mathrm{sn}(h_2, m)\,\mathrm{sn}(at, m)},\end{aligned} \tag{71}$$

follows, where we have used also the property [10]

$$\sqrt{m}\,\mathrm{sn}(u + iK', m)\,\mathrm{sn}(u, m) = 1.$$

The left-hand side of Eq. (71) is the mathematical expression for the third Euler angle ϕ. This equation allows us to obtain the relation between the new coordinate h_2 and the time through this angle.

As an independent verification, one can obtain the same expression by a purely geometric argument. The vector $\mathbf{u}$ can be assumed equal to the first column of the rotation matrix, whereas the vector $\mathbf{l}$ is the third column. Then, starting from the expression in Euler angles, we deduce the identity

$$\mathbf{u} = \frac{1}{\sqrt{1 - (\mathbf{k}^T\mathbf{l})^2}}\left[\cos\phi\,\mathbf{l}\times\mathbf{k} + \sin\phi\,(\mathbf{k} - \mathbf{k}^T\,\mathbf{l}\mathbf{l})\right] \tag{72}$$

from which one can find the same expression (71) if we multiply the scalar product of the vector (65) with the complex vector $\mathbf{l}\times\mathbf{k} + i(\mathbf{k} - \mathbf{k}^T\mathbf{l}\mathbf{l})$. One finds, after droping out the irrelevant real factors of the square roots in (65),

$$\begin{aligned}&-\mathrm{cn}(\alpha_2, m')\,\mathrm{dn}^2(\alpha_2, m')\,\mathrm{sn}(h_2, m)\,\mathrm{sn}(at, m) \\ &\quad + \mathrm{sn}^2(\alpha_2, m')\,\mathrm{cn}(\alpha_2, m')\,\mathrm{dn}(at, m)\,\mathrm{cn}(h_2 + at, m)\,\mathrm{dn}(h_2, m) \\ &\quad + i\{\mathrm{cn}^2(\alpha_2, m')\,\mathrm{sn}(\alpha_2, m')\,\mathrm{dn}(\alpha_2, m')\,\mathrm{sn}(h_2, m)\,\mathrm{cn}(at, m)\,\mathrm{dn}(at, m) \\ &\quad + \mathrm{sn}(\alpha_2, m')\,\mathrm{cn}^2(\alpha_2, m')\,\mathrm{dn}(\alpha_2, m')\,\mathrm{cn}(h_2 + at, m)\,\mathrm{dn}(h_2, m)\,\mathrm{cn}(at, m)\,\mathrm{sn}(at, m) \\ &\quad + [1 - \mathrm{cn}^2(\alpha_2, m')\,\mathrm{cn}^2(at, m)]\,\mathrm{sn}(\alpha_2, m')\,\mathrm{dn}(\alpha_2, m')\,\mathrm{sn}(h_2 + at, m)\}.\end{aligned} \tag{73}$$

The imaginary part of this expression is simplified by means of the identity (66) to give us

$$\mathrm{sn}(\alpha_2, m')\,\mathrm{dn}(\alpha_2, m')\,\mathrm{sn}(h_2 + at, m), \tag{74}$$

whereas in the real part of (73) one makes the sustitution of the identity

$$\mathrm{dn}(at, m)\,\mathrm{dn}(h_2, m) = m\mathrm{cn}(h_2 + at)\,\mathrm{sn}(at, m)\,\mathrm{sn}(h_2 m) + \mathrm{dn}(h_2 + at, m),$$

in order to transform it into

$$\begin{aligned}&\mathrm{sn}^2(\alpha_2, m')\,\mathrm{cn}(\alpha_2, m')\,\mathrm{cn}(h_2 + at, m)\,\mathrm{dn}(h_2 + at, m) \\ &\qquad - \mathrm{cn}(\alpha_2, m')\,\mathrm{sn}(h_2, m)\,\mathrm{sn}(at, m)[1 - \mathrm{sn}^2(\alpha_2, m')\,\mathrm{dn}^2(h_2 + at, m)].\end{aligned} \tag{75}$$

Hence, we can write (73) in the form of the numerator in (71), multiplied by the imaginary factor

$$i[1 - \mathrm{sn}^2(\alpha_2, m') \,\mathrm{dn}^2(h_2 + at, m)] \,\mathrm{sn}(\alpha_2, m'), \tag{76}$$

which proves the equivalence of the two ways of relating the angle ϕ with the coordinate h_2.

Since we know the vector $\mathbf{u}$ as a function of time from the precedent section, we can as a consequence, solve the Jacobi function $\mathrm{sn}(h_2, m)$ by using expression (65). We therefore know the time dependence of the coordinates h_2.

References

1. L. Landau and E. Lifshitz, *Mechanics* (Pergamon Press, Oxford, 1976), third edition.
2. E.T. Whittaker, *Analytical Dynamics of Particles and Rigid Bodies* (Cambridge University Press, Cambridge, 1965) p. 47.
3. F. Reiche, *Phys. Zs.* **19** (1918) 394.
4. Ph.M. Morse and H. Feshback, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953) p. 659.
5. E. Piña in *Hamiltonian Systems and Celestial Mechanics*, edited by E.A. Lacomba and J. Llibre (World Scientific, 1993) p. 149.
6. E. Piña, *Am. J. Phys.* **51** (1983) 375.
7. E. Piña, *Rev. Mex. Fís.* **39** (1993) 10.
8. I.S. Gradshteyn and I.M. Ryzhik, *Tables of Integrals, Series and Products* (Academic Press, New York, 1965).
9. E. Piña, *Dinámica de rotaciones* (UAM-Iztapalapa, México 1996) p. 151.
10. E.T. Whittaker and G.N. Watson, *A course of modern analysis* (Cambridge University Press, Cambridge, 1965).