Recurrence-shift relations for the polynomial functions of Aldaya, Bisquert, and Navarro-Salas

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ABSTRACT. Using a simple factorization scheme we obtain the recurrence-shift relations of the polynomial functions of Aldaya, Bisquert and Navarro-Salas (ABNS), $F_n^N(\frac{\omega}{c}\sqrt{N}x)$, *i.e.*, one-step first-order differential relations referring to N, as follows. Firstly, we apply the scheme to the polynomial degree confirming the recurrence relations of Aldaya, Bisquert and Navarro-Salas, but also obtaining another slightly modified pair. Secondly, the factorization scheme is applied to the Gegenbauer polynomials to get the recurrence relations with respect to their parameter. Next, we make use of Nagel's result, showing the connection between Gegenbauer polynomials and the ABNS functions, to write down the recurrence-shift relations for the latter ones. Such relations may be used in the study of the spatial structure of pair-creation processes in an Anti-de Sitter gravitational background.

RESUMEN. Se obtienen relaciones de recurrencia de los polinomios usados por Aldaya, Bisquert y Navarro-Salas para describir osciladores armónicos relativistas, $F_n^N(\frac{\omega}{c}\sqrt{N}x)$, usando un esquema de factorización simple, incluyendo un desplazamiento discreto en el argumento, es decir, relaciones diferenciales de primer orden en un paso con respecto a N, como sigue. Primeramente aplicamos el esquema al grado del polinomio confirmando las relaciones de recurrencia de Aldaya *et al.*, pero también obtenemos otro par ligeramente modificado; además para los polinomios de Gegenbauer, obtenemos una relación de recurrencia con respecto a su parámetro. Enseguida, haciendo uso de la relación de Nagel entre los polinomios de Gegenbauer y las funciones ABNS obtenemos un par de relaciones de recurrencia-desplazamiento para esas funciones. Tales relaciones podrían ser usadas en el estudio de la estructura espacial de procesos de creación de pares en el fondo gravitacional de un universo de anti de Sitter.

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1. Introduction

Some time ago, Aldaya, Bisquert and Navarro-Salas (ABNS) [1] discussed a relativistic generalization of the quantum harmonic oscillator for which the spacing of the energy levels is kept constant in the relativistic domain. By group-theoretical methods they obtained the wavefunctions for such a "relativistic" oscillator, which up to phase factors and a "weight function", contain a polynomial part dubbed by ABNS the "relativistic Hermite polynomials", $F_n^N(\xi)$, of degree n, parameter $N = mc^2/\hbar\omega$ (considered as discrete) and variable $\xi_N = (\omega/c)\sqrt{N}x$. In this paper, we shall call them the ABNS functions, because Nagel [2] showed that they actually are a product of a factorial, a simple polynomial, and the Gegenbauer polynomial of the same degree and parameter but of a different argument.

In the ABNS paper, recurrence relations with respect to the polynomial degree n are written down. One might think of similar relations for N, which, however, appears both as a polynomial parameter and as a discrete factor in the argument of the ABNS functions. We have here a case of both shift relations with respect to a discrete variable from the point of view of the ABNS functions and recurrence relations from the point of view of the polynomial parameter. We recall that shifts with respect to a continuous variable are a common procedure for relativistic oscillators of finite difference type [3]. In the following, we shall show that the recurrence-shift relations for the ABNS functions can be obtained almost trivially from the corresponding recurrence relations of the Gegenbauer polynomials. Moreover, we shall give some hints on the physical relevance of these recurrence-shift relations, which from their definition appear to be related to pairs creation processes. A very short presentation of our results has been published in the *Proceedings of Wigner lV Symposium* [5], and here one may find a more detailed study. We mention that Zarzo *et al.* [4] studied the asymptotic distribution of the zeros of ABNS functions, and other algebraic and spectral properties. The ABNS functions satisfy the following second-order linear differential equation

$$(1 + \xi_N^2/N)y'' - (2/N)(N + n - 1)\xi_N y' + (n/N)(2N + n - 1)y = 0, \qquad (1)$$

where the derivatives are with respect to the ξ_N variable. The limit $N \to \infty$ is the "nonrelativistic one", $c \to \infty$, in which the ABNS functions go into the usual Hermite polynomials.

The base for getting our results is a simple yet sufficient general factorization method that can be inferred from the factorization techniques in the book of Miller [7]. A particular case has been used by Piña [6] to work out many examples.

2. The factorization scheme

Given the following family of second order linear differential equations

$$P(\xi)\frac{d^2 y_s(\xi)}{d\xi^2} + Q_s(\xi)\frac{dy_s(\xi)}{d\xi} + R_s(\xi)y_s(\xi) = 0, \qquad (2)$$

let us suppose that the solutions y_s to Eq. (2) satisfy the following recurrence (ladder) relations:

$$A_s^\pm \, y_s = r_s^\pm \, y_{s\pm 1}, \tag{3}$$

where $A_s^\pm$ are first order differential operators of the type

$$A_s^+ = f_s^+ \frac{d}{d\xi} + g_s^+ , \tag{4}$$

$$A_s^- = f_s^- \frac{d}{d\xi} + g_s^- . \tag{5}$$

In Eqs. (4) and (5), $f_s^\pm$ and $g_s^\pm$ might be functions of ξ, whereas in Eqs. (3) $r_s^\pm$ depend only on the parameter s. By means of the ladder operators we can write down two types of second-order differential equations

$$A_{s+1}^- \, A_s^+ \, y_s = r_s^+ \, r_{s+1}^- \, y_s \tag{6}$$

and

$$A_s^+ \, A_{s+1}^- \, y_{s+1} = r_s^+ \, r_{s+1}^- \, y_{s+1}, \tag{7}$$

involving the same constant $k_s = r_s^+ \, r_{s+1}^-$.

By substituting Eqs. (4) and (5) in Eqs. (6) and (7), respectively, one should obtain the second-order linear differential Eq. (2), for the indices s and $s+1$, respectively. This allows to get the following five equations as sufficient for Eqs. (2) and (6)–(7) to have the same solutions

$$P = f_s^+ \, f_{s+1}^- , \tag{8}$$

$$Q_{s+1} = f_s^+ \frac{df_{s+1}^-}{d\xi} + f_s^+ \, g_{s+1}^- + f_{s+1}^- \, g_s^+ , \tag{9}$$

$$Q_s = f_{s+1}^- \frac{df_s^+}{d\xi} + f_{s+1}^- \, g_s^+ + f_s^+ \, g_{s+1}^- , \tag{10}$$

$$R_{s+1} + k_s = f_s^+ \frac{dg_{s+1}^-}{d\xi} + g_s^+ \, g_{s+1}^- , \tag{11}$$

$$R_s + k_s = f_{s+1}^- \frac{dg_s^+}{d\xi} + g_s^+ \, g_{s+1}^- . \tag{12}$$

Taking the derivative of P with respect to ξ and subtracting Eq. (10) from (9) one will obtain

$$\frac{1}{2}\left(\frac{dP}{d\xi} + Q_{s+1} - Q_s\right) = f_s^+ \frac{df_{s+1}^-}{d\xi} = P \frac{d}{d\xi} \ln f_{s+1}^- . \tag{13}$$

From Eq. (13) one gets easily

$$f_{s+1}^- = \sqrt{P} \exp\left(\frac{1}{2} \int \frac{Q_{s+1} - Q_s}{P} \, d\xi\right) = \sqrt{P} E_s, \tag{14}$$

and using Eq. (8)

$$f_s^+ = \sqrt{P} \exp\left(-\frac{1}{2}\int \frac{Q_{s+1} - Q_s}{P}\,d\xi\right) = \sqrt{P}E_s^{-1}. \tag{15}$$

To get g_s^+ and g_s^- one should proceed as follows. Firstly, one can obtain easily

$$\frac{R_{s+1} - R_s}{\sqrt{P}} = E_s^{-1}\frac{dg_{s+1}^-}{d\xi} - E_s\frac{dg_s^+}{d\xi} \tag{16}$$

and

$$\frac{1}{2}\left(Q_{s+1} + Q_s - \frac{dP}{d\xi}\right) \equiv \sqrt{P}W_s = f_s^+\,g_{s+1}^- + f_{s+1}^-\,g_s^+. \tag{17}$$

It follows

$$W_s = E_s^{-1}\,g_{s+1}^- + E_s\,g_s^+. \tag{18}$$

After some algebra one gets

$$\frac{R_{s+1} - R_s}{\sqrt{P}} + \frac{dW_s}{d\xi} = \frac{d}{d\xi}\left(2E_s^{-1}g_{s+1}^-\right) + W_s\frac{d\ln E_s}{d\xi}. \tag{19}$$

Thus

$$g_{s+1}^- = \frac{E_s}{2}\left[W_s + \int\left(\frac{R_{s+1} - R_s}{\sqrt{P}} - W_s\frac{Q_{s+1} - Q_s}{2P}\right)d\xi\right]. \tag{20}$$

The other coefficient g_s^+ is obtained from $g_s^+ = W_s/E_s - g_{s+1}^-/E_s^2$ giving

$$g_s^+ = \frac{1}{2E_s}\left[W_s - \int\left(\frac{R_{s+1} - R_s}{\sqrt{P}} - W_s\frac{Q_{s+1} - Q_s}{2P}\right)d\xi\right]. \tag{21}$$

Finally, the k-constant can be found from the most convenient of Eqs. (11) and (12). In order to find the r-coefficients one needs either supplementary information on the polynomial solutions of second order linear differential equations or some tricks as one can see in the applications to follow.

3. Application to ABNS polynomial functions

3.1. Factorization with respect to the polynomial degree

In this case we have $s = n$, $P(\xi) = 1 + \xi_N^2/N$, $Q_n = -2(N + n - 1)\xi_N/N$, $R_n = n(2N+n-1)/N$. The calculations are straightforward. However we present some details. The convenient variable to work with is $u = \xi_N/\sqrt{N}$. Other variables will change the constants of integration by logarithmic terms, which anyway we shall not include here as far as we are interested in the simplest solutions. The first thing to do is to calculate the integral in the exponential for the f coefficients, which reads

$$\int \frac{Q_{n+1} - Q_n}{P}\,d\xi = \int \frac{-2\xi\,d\xi}{N + \xi^2} = \int \frac{-2u\,du}{1 + u^2} = -\ln\left(1 + u^2\right), \tag{22}$$

and by substituting in Eqs. (14), (15) one gets $f^- = 1$ and $f^+ = 1 + u^2$, respectively, both not depending on n. W_n can be obtained from the identity in Eq. (17) leading to $W_n = -\frac{2(N+n)}{\sqrt{N}} \frac{u}{\sqrt{1+u^2}}$. The first term in the integrals for $g^\pm$ reads

$$\int \frac{R_{n+1} - R_n}{\sqrt{P}} \, d\xi = \frac{2(N+n)}{\sqrt{N}} \ln\left(u + \sqrt{1+u^2}\right) = \frac{2(N+n)}{\sqrt{N}} \operatorname{arcsinh}(u). \qquad (23)$$

The second part of the integral in Eqs. (20) and (21) is amenable to the following simple one:

$$\frac{2(N+n)}{\sqrt{N}} \int \frac{u^2 \, du}{(1+u^2)^{3/2}} = \frac{2(N+n)}{\sqrt{N}} \left[\ln\left(u + \sqrt{1+u^2}\right) - \frac{u}{\sqrt{1+u^2}}\right]$$

$$= \frac{2(N+n)}{\sqrt{N}} \left[\operatorname{arcsinh}(u) - \frac{u}{\sqrt{1+u^2}}\right]. \qquad (24)$$

The final result for the g-terms is

$$g^\mp = \frac{(N+n)}{\sqrt{N}} \left(\sqrt{1+u^2}\right)^{\mp 1} \left[-\frac{u}{\sqrt{1+u^2}} \pm \operatorname{arcsinh}(u) \mp \operatorname{arcsinh}(u) \pm \frac{u}{\sqrt{1+u^2}}\right]. \qquad (25)$$

So, in this case, $g_{n+1}^- = 0$ and $g_n^+ = -2(1 + n/N)\xi_N$. Since g^- is naught, from Eq. (11) one will find out that $k_n = -R_{n+1}$.

The recurrence relations for ABNS functions can be written as follows:

$$A_n^+ y_n \equiv \left[\left(1 + \frac{\xi_N^2}{N}\right)\frac{\partial}{\partial \xi_N} - 2\left(1 + \frac{n}{N}\right)\xi_N\right] y_n = r_n^+ y_{n+1} \qquad (26)$$

and

$$A_n^- y_n \equiv \frac{\partial}{\partial \xi_N} y_n = r_n^- y_{n-1}. \qquad (27)$$

In order to proceed further we have to be aware of the fact that the factorization scheme above does not allow to find out the r-coefficients, but only the k-constant, *i.e.*, their product. As a matter of fact, the ambiguities of factorization procedures have been known since Infeld and Hull [8]. For the present method, the k-constant comes out quite often in factorized form and then one can make some selection of the r-coefficients identified with the k-factors (though this is not a rule) on the base of further criteria. This situation is clearly illustrated by the ABNS functions. Indeed, from Eq. (11) one gets $k_n = -R_{n+1} = -[(n+1)(2N+n)]/N$. Because of the three factors contained in R, to which one should add the minus sign, there are sixteen r-pairs leading to the same k-constant. However, if one asks for the first two ABNS functions to be identical to the first two Hermite polynomials, *i.e.*, $F_0^N = 1$ and $F_1^N = 2\xi_N$, respectively, one can show easily that $r_0^+ = -1$ and $r_1^- = 2$. In this way most of the r-pairs are discarded, and one ends up with just two cases, *i.e.*, (*i*) $r_n^+ = -1$ and $r_{n+1}^- = [(n+1)(2N+n)]/N$, (*ii*) $r_n^+ = -(n+1)$ and $r_{n+1}^- = (2N+n)/N$. However the latter pair is merely a rescaling of the pure numerical coefficients entering the ABNS functions. On the other hand, the first pair corresponds exactly to the recurrence relations obtained by ABNS (see Eqs. (13) and (14) in their paper, which are our Eqs. (26) and (27), respectively, when the first r-pair is used).

3.2. FACTORIZATION WITH RESPECT TO THE PARAMETER

To write down recurrence relations with respect to the parameter N seems impossible from the point of view of the factorization scheme because it occurs also in the variable of the ABNS functions and consequently one deals in fact with recurrence-shift relations involving ABNS functions of different variables. Therefore a direct application of our method is not possible. We have found a way to obtain such relations by making use of the aforementioned result of Nagel [2], who proved that ABNS functions are a product of a factorial, a simple polynomial and a Gegenbauer polynomial. Thus, the idea is first to obtain recurrence relations for the Gegenbauer polynomials with respect to their parameter, which fits in our factorization scheme, since the Gegenbauer variable is not a N-depending quantity. More exactly we derive recurrence relations for the equation

$$x^2(1-x^2)y'' - (2N+1)x^3y' + n(2N+n)x^2y = 0. \tag{28}$$

Then $P = x^2(1-x^2)$, $Q_N = -(2N+1)x^3$, $R_N = n(2N+n)x^2$. The application of the factorization scheme is straightforward leading to the following factorizing coefficients $f^+ = x$, $f^- = x(1-x^2)$, $g_N^+ = 2N+n$, $g_{N+1}^- = -(2N+n+1-nx^2)$. The constant $k_N = -(2N+n+1)(2N+n)$ is in factorized form but we shall not make a direct identification of the r-coefficients with the k-factors. Instead we take $r_N^+ = 2N$ and thus $r_{N+1}^- = -(2N+n+1)(2N+n)/2N$. The recurrence relations read

$$\left[x\frac{\partial}{\partial x} + (2N+n) \right] C_n^N = 2N C_n^{N+1} \tag{29}$$

and

$$\left[x\left(1-x^2\right)\frac{\partial}{\partial x} - \left(2N+n-1-nx^2\right) \right] C_n^N = -\frac{(2N+n-1)(2N+n-2)}{2N-2} C_n^{N-1}. \tag{30}$$

The first one can also be obtained by combining Eqs. (8.933.3) and (8.935.2) in Gradshteyn and Ryzhik [9], while the latter one can be reached from the set (8.933.2–4), (8.935.2) in the same reference.

Nagel [2] proved the following relationship:

$$F_n^N(u\sqrt{N}) = n! \left(\frac{1+u^2}{N}\right)^{n/2} C_n^N(u/\sqrt{1+u^2}). \tag{31}$$

Plugging Nagel's result into equations (29) and (30) one gets the following recurrence-shift relations for the ABNS functions:

$$\left[\xi_N \left(1+\xi_N^2/N\right)\frac{\partial}{\partial\xi_N} - \frac{n}{N}\xi_N^2 + 2N+n \right] F_n^N(\xi_N)$$

$$= 2N \left(\sqrt{1+\frac{1}{N}}\right)^n F_n^{N+1}\left(\sqrt{1+\frac{1}{N}}\,\xi_N\right) \tag{32}$$

and

$$\left[\xi_N \frac{\partial}{\partial \xi_N} - (2N + n - 1)\right] F_n^N(\xi_N) = -\frac{(2N + n - 1)(2N + n - 2)}{2N - 2}$$
$$\times \left(\sqrt{1 - \frac{1}{N}}\right)^n F_n^{N-1}\left(\sqrt{1 - \frac{1}{N}}\xi_N\right). \quad (33)$$

Have these relationships any application in physics? Our answer is as follows. The wave-functions containing the ABNS polynomial functions satisfy the Klein-Gordon equation associated with the Anti-de Sitter (AdS) metric (of negative curvature $R = -\omega^2/c^2$) as stated by ABNS:

$$\left(\Box + \frac{m^2 c^2}{\hbar^2} + NR\right)\psi = 0. \quad (34)$$

The true limiting process (or contraction in group theory terminology) should be taken as $c \to \infty$ and simultaneously $R = -\omega^2/c^2 \to 0$, keeping $c\sqrt{|R|} = \omega$, leading from the AdS relativity to the Newton one [10]. The AdS symmetry group is a deformation of both the relativistic free particle and the harmonic oscillator. At the quantum level, the double covering $SU(1,1)$ of the AdS group is represented by the discrete series of representations, which are indexed by the dimensionless number $\eta = N + \frac{1}{2} + O(\sqrt{|R|})$ [10]. Our claim is that the recurrence-shift relations of ABNS oscillator functions can be important in the spatial analysis of the pair-production effects in a wave approach in the AdS gravitational background, since their action is of connecting ABNS functions with consecutive parameters related to discrete changes in the gravitational curvature (in the Klein-Gordon equation, Eq. (34), N is like a discrete coupling constant for the curvature) and simultaneously discrete changes in the polynomial variable. Indeed, one can see that $N = 1/k\lambda_C$, where k is the usual wavevector and λ_C is the Compton wavelength. Thus, one can think of N as the ratio of the spatial resolution of a common wave $\propto 1/k$ and the spatial resolution given by the Compton wavelength. We recall the similar suggestion of Noll [11] in Optics on the usefulness of ladder operations of Zernike polynomials, whenever the gradient of a wave front is required. Moreover, to be recalled is the idea of hadrons as AdS microuniverses of huge curvature [12]. Pair creation processes in such a context might be examined in the aforementioned perspective as well.

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