Revista Mexicana de Física **43**, *No. 3 (1997) 388–394*

Influence of high order aberrations induced by atmospheric turbulence on the image centroid

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Recibido el 20 de agosto de 1996; aceptado el 22 de enero de 1997

ABSTRACT. The study of a star image produced by an earth based telescope in the presence of atmospheric turbulence is presented. The phase of the wavefront at the telescope pupil is represented as a linear superposition of Zernike polynomials. The statistical relative error of the tilt angle in the x-direction compared with its estimator defined by the image centroid position and the optical axes is calculated for the Kolmogorov model of turbulence and the value 0.125 is obtained. This result implies the existence of an additional contribution for the exact determination of the centroid position from high order aberrations. The relative error calculations also are shown, which consider the influence of the outer scale of the turbulence using the Von Karman model. Results obtained in the two cases are consistent.

RESUMEN. Se presenta un estudio en donde se considera la imagen de una estrella, producida por un telescopio instalado en tierra firme en presencia de la turbulencia atmosférica. La fase de la onda de luz en la pupila del telescopio se expresa como una combinación lineal de polinomios de Zernike. Se calcula el error relativo estadístico del ángulo de *"tilt"* en la dirección x con respecto a su estimador definido por la posición del centroide con el eje óptico, obteniéndose un valor de 0.125 para el modelo de Kolmogorov de la turbulencia. Este valor residual significa que existe una contribución de las aberraciones de alto orden en la determinación exacta del centroide. Se muestran también los cálculos del error relativo considerando la influencia de la escala exterior de la turbulencia, utilizando el modelo de Von Karman. Los resultados obtenidos en los dos casos son consistentes.

PACS: 42.68.Bz

1. INTRODUCTION

It is well known that when one observes a star with a ground based telescope, one of the most important effects of atmospheric turbulence is an agitation of the image in the focal plane [1] as a consecuence of local variations of the angle of the tangent plane of the wavefront at the telescope pupil. Agitation of the image corresponds to the wandering of the image centroid position. In order to compensate such an undesirable effect it is necessary to find all possible sources of this error. Importance of this subject in adaptive optics is fundamental since the simplest adaptive optical system attempts to correct the image centroid offset by employing a tip-tilt rotating mirror, which is not very efficient if one compensates only tip and tilt as the name suggests.

The aims of this paper are twofold. Firstly, to show that high order aberrations are important in calculating the mean deviation of the image centroid with respect to its ideal position, when the Kolmogorov model of turbulence is applied for the phase fluctuation correlation function. Secondly, to investigate the influence on the previous results of the outer scale of turbulence using the Von Karman model.

2. TILT ANGLE AND IMAGE CENTROID ANGLE

Let the wavefront $w(\rho,\varphi)$ be represented as a linear superposition of Zernike polynomials [2], also known as modal representation of the wavefront, by the expression

$$w(\rho,\varphi)=\sum_{j=1}^{\infty} a_j Z_j(\rho,\varphi), \tag{1}$$

where ρ,φ are polar coordinates and the Zernike polynomials or wavefront modes $Z_j(\rho,\varphi)$ are expressed as

$$Z_{j(\mathrm{even})}(\rho,\varphi)=\sqrt{2(n+1)}\,R_n^m(\rho)\cos(m\varphi), \qquad m\neq 0\,, \tag{2a}$$

$$Z_{j(\mathrm{odd})}(\rho,\varphi)=\sqrt{2(n+1)}\,R_n^m(\rho)\sin(m\varphi), \qquad m=0\,, \tag{2b}$$

$$Z_j(\rho,\varphi)=\sqrt{n+1}\,R_n^0(\rho)\,, \qquad m=0\,, \tag{2c}$$

and R_n^m are radial Zernike polynomials [3]. The normalized coefficients a_j are given by

$$a_j=\frac{1}{\pi R^2}\int_{G_R} d^2\rho\, w(\rho/R,\varphi)\, Z_j(\rho/R,\varphi), \tag{3}$$

where the integral is evaluated over a circular region G_R of radius R within the wavefront is defined. The coefficients $a_1, a_2, a_3, a_4, \ldots$ correspond to the piston mode, tilt about x mode, tilt about y mode, focus mode, etcetera. In addition, observe that the wave phase is expressed by $S(\rho/R,\varphi)=kw(\rho/R,\varphi)$, where $k=2\pi/\lambda$ is the wave number, so the coefficients in the Eq. (3) can be written in the form

$$a_j=\frac{1}{k\pi R^2}\int_{G_R} d^2\rho\, S(\rho/R,\varphi)\, Z_j(\rho/R,\varphi). \tag{4}$$

Let us consider, on the other hand, a tilted wavefront $w=\alpha x$ at the telescope pupil and the offset of the image centroid caused by the atmospheric turbulence as shown in Fig. 1. With this, we exclude other aberrations to be present in the wavefront, so we can write $w(\rho/R,\varphi)=a_2 Z_2(\rho/R,\varphi)$. It is not difficult to show from these two last equations that the angle α is proportional to the tilt coefficient a_2 and this coefficient can be obtained from Eq. (4), so that α can be written according to the formula

$$\alpha=\frac{2}{k\pi R^3}\int_{G_R} d^2\rho\, S(\rho/R,\varphi)\, Z_2(\rho/R,\varphi). \tag{5}$$

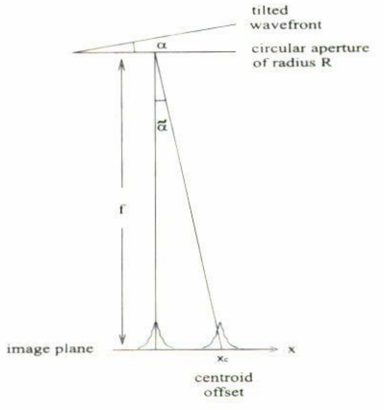

FIGURE 1. Displacement of the image centroid due to a tilted wavefront.

Now, in order to find an analytical expression for $\tilde{\alpha}$, the angle corresponding to the image centroid, let us supose that the optical field at the aperture is given by

$$U(\vec{\rho}) = A\, e^{iS(\vec{\rho})}\,, \tag{6}$$

where A is a constant and $S(\vec{\rho})$ is the wave phase at the position $\vec{\rho}$ in the telescope aperture. The field at the image plane is proportional to the Fourier transform of $U(\vec{\rho})$ and the intensity is easily calculated by its square modulus. The result is

$$I(\vec{r}) = K \int_{G_R} d^2\rho_1 \int_{G_R} d^2\rho_2\, U(\vec{\rho}_1)\, U^*(\vec{\rho}_2)\, \exp\left[\frac{-ik\vec{r}}{f} \cdot (\vec{\rho}_1 - \vec{\rho}_2)\right], \tag{7}$$

where K is a constant, f is the telescope focal distance, $\vec{r}$ is a vector in the image plane and * denotes the complex conjugate. The image centroid x coordinate is determined [4] by

$$x_c = \frac{\int_{-\infty}^{\infty} d^2r\, x I(\vec{r})}{\int_{-\infty}^{\infty} d^2r\, I(\vec{r})}. \tag{8}$$

After substitution of Eq. (7) into (8), this can be written in the form [4]

$$x_c = \frac{f}{k\pi R^2} \int_{G_R} d^2\rho\, \frac{\partial S(\vec{\rho})}{\partial \rho_x}. \tag{9}$$

So, from geometry in Fig. 1, the tilt angle estimator $\tilde{\alpha}$ is found to be represented by the formula

$$\tilde{\alpha} = \frac{1}{k\pi R^2} \int_{G_R} d^2\rho\, \frac{\partial S(\vec{\rho})}{\partial \rho_x}. \tag{10}$$

3. RELATIVE ERROR BETWEEN α AND $\tilde{\alpha}$ FOR KOLMOGOROV MODEL

An adequate way to compare the tilt angle α and its estimator $\tilde{\alpha}$ is to calculate the statistical relative error in the form

$$\varepsilon^2 = \frac{\langle(\alpha - \tilde{\alpha})^2\rangle}{\langle\alpha^2\rangle} = \frac{\langle\alpha^2\rangle - 2\langle\alpha\tilde{\alpha}\rangle + \langle\tilde{\alpha}^2\rangle}{\langle\alpha^2\rangle}, \tag{11}$$

where $\langle\cdots\rangle$ denotes an ensemble average. Developing the averages using Eqs. (5) and (10) we have

$$\langle\alpha^2\rangle = \left(\frac{2}{k\pi R^3}\right)^2 \int_{G_R} d^2\rho_1\, Z_2\left(\vec{\rho}_1/R\right) \int_{G_R} d^2\rho_2\, Z_2\left(\vec{\rho}_2/R\right) \langle S(\vec{\rho}_1)S(\vec{\rho}_2)\rangle, \tag{12a}$$

$$\langle\alpha\tilde{\alpha}\rangle = \frac{2}{k\pi R^3}\,\frac{1}{k\pi R^2} \int_{G_R} d^2\rho_1\, Z_2\left(\vec{\rho}_1/R\right) \int_{G_R} d^2\rho_2\, \frac{\partial}{\partial x_2}\langle S(\vec{\rho}_1)S(\vec{\rho}_2)\rangle, \tag{12b}$$

$$\langle\tilde{\alpha}^2\rangle = \left(\frac{1}{k\pi R^2}\right)^2 \int_{G_R} d^2\rho_1 \int_{G_R} d^2\rho_2\, \frac{\partial^2}{\partial x_1 \partial x_2}\langle S(\vec{\rho}_1)S(\vec{\rho}_2)\rangle, \tag{12c}$$

where $\langle S(\vec{\rho}_1)S(\vec{\rho}_2)\rangle = B_S(\vec{\rho}_1, \vec{\rho}_2)$ is the correlation function of phase fluctuations that can be written in the form [4]

$$B_S(\vec{\rho}_1, \vec{\rho}_2) = 0.49\, r_0^{-5/3} \int_0^\infty d^2\kappa\, \kappa^{-11/3} e^{i\vec{\kappa}\cdot(\vec{\rho}_1 - \vec{\rho}_2)}, \tag{13}$$

where r_0 is the atmospheric coherence diameter or Fried's parameter, $\kappa = |\vec{\kappa}| = 2\pi/L$ is a spatial frecuency vector and L is the size of eddies of the turbulence model. Substituting Eq. (13) into Eqs. (12a)–(12c) and evaluating the integrals we find

$$\langle\alpha^2\rangle = 5.7151\, \frac{1}{k^2 R^2} \left(\frac{R}{r_0}\right)^{5/3}, \tag{14a}$$

$$\langle\alpha\tilde{\alpha}\rangle = 5.4841\, \frac{1}{k^2 R^2} \left(\frac{R}{r_0}\right)^{5/3}, \tag{14b}$$

$$\langle\tilde{\alpha}^2\rangle = 5.3432\, \frac{1}{k^2 R^2} \left(\frac{R}{r_0}\right)^{5/3}. \tag{14c}$$

Substituting the expressions (14a)–(14c) into (11) we get the final result

$$\varepsilon = 0.1255. \tag{15}$$

This result means that the tilt angle and its estimator differ by 12.5%. At this point, observe that in the formula (10) for $\tilde{\alpha}$ all wavefront aberrations are present while α only represents the tilt term.

4. Influence of the outer scale of turbulence

For the Von Karman model of turbulence the correlation function of phase fluctuations takes the form [5]

$$B_S(\vec{\rho}_1,\vec{\rho}_2) = 0.49\, r_0^{-5/3} \int_0^\infty d^2\kappa \left(\kappa^2+\kappa_0^2\right)^{-11/6} e^{i\vec{\kappa}\cdot(\vec{\rho}_1-\vec{\rho}_2)}, \tag{16}$$

where $\kappa_0 = 1.071/L_0$. Here L_0 is a maximum critical size of eddies which describe the turbulent atmosphere and is called the outer scale. Substituting Eq. (16) into Eqs. (12a)–(12c) results in the folowing:

$$\langle\alpha^2\rangle = \frac{A_0}{k^2}\left(\frac{R}{r_0}\right)^{5/3} R^{-2}\int_0^\infty t^{-1}\left(t^2+\kappa_0^2R^2\right)^{-11/6} J_2^2(t)\,dt, \tag{17a}$$

$$\langle\alpha\tilde{\alpha}\rangle = \frac{A_1}{k^2}\left(\frac{R}{r_0}\right)^{5/3} R^{-2}\int_0^\infty \left(t^2+\kappa_0^2R^2\right)^{-11/6} J_1(t)J_2(t)\,dt, \tag{17b}$$

$$\langle\tilde{\alpha}^2\rangle = \frac{A_2}{k^2}\left(\frac{R}{r_0}\right)^{5/3} R^{-2}\int_0^\infty t\left(t^2+\kappa_0^2R^2\right)^{-11/6} J_1^2(t)\,dt, \tag{17c}$$

where the constants $A_0 = 98.52$, $A_1 = 24.63$ and $A_2 = 6.15$, $t = \kappa R$ and J_1, J_2 are Bessel functions of the first kind. Evaluation of these integrals yields

$$\int_0^\infty t^{-1}\left(t^2+\kappa_0^2R^2\right)^{-11/6} J_2^2(t)\,dt = 0.0577\; {}_2F_3\left(\tfrac{14}{6},\tfrac{11}{6};\tfrac{5}{6},\tfrac{29}{6},\tfrac{17}{6};\kappa_0^2R^2\right) - 0.0561\,(\kappa_0R)^{1/3}\; {}_2F_3\left(\tfrac{5}{2},2;\tfrac{7}{6},3,5;\kappa_0^2R^2\right), \tag{18a}$$

$$\int_0^\infty \left(t^2+\kappa_0^2R^2\right)^{-11/6} J_1(t)J_2(t)\,dt = 0.2217\; {}_2F_3\left(\tfrac{14}{6},\tfrac{11}{6};\tfrac{5}{6},\tfrac{23}{6},\tfrac{17}{6};\kappa_0^2R^2\right) - 0.2250\,(\kappa_0R)^{1/3}\; {}_2F_3\left(\tfrac{5}{2},2;\tfrac{7}{6},3,4;\kappa_0^2R^2\right), \tag{18b}$$

$$\int_0^\infty t\left(t^2+\kappa_0^2R^2\right)^{-11/6} J_1^2(t)\,dt = 0.8639\; {}_1F_2\left(\tfrac{4}{3};\tfrac{5}{6},\tfrac{17}{6};\kappa_0^2R^2\right) - 0.9\,(\kappa_0R)^{1/3}\; {}_1F_2\left(\tfrac{3}{2};\tfrac{7}{6},3;\kappa_0^2R^2\right), \tag{18c}$$

where ${}_pF_q(\alpha_1,\alpha_2,\ldots,\alpha_p;\,\beta_1,\beta_2,\ldots,\beta_q;\,z)$ is the generalized hypergeometric function defined by [6]

$${}_pF_q(\alpha_1,\alpha_2,\ldots,\alpha_p;\,\beta_1,\beta_2,\ldots,\beta_q;\,z) = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n(\alpha_2)_n\ldots(\alpha_p)_n\,z^n}{(\beta_1)_n(\beta_2)_n\ldots(\beta_q)_n\,n!}. \tag{19}$$

After calculations the final expression for the relative error given by Eq. (11) depends on the outer scale L_0 of turbulence and the pupil diameter $D = 2R$. Figure 2 shows the relative error behavior with the outer scale. As one can see the magnitude of the error depends on the value of L_0 and is close to that obtained from the Kolmogorov Model.

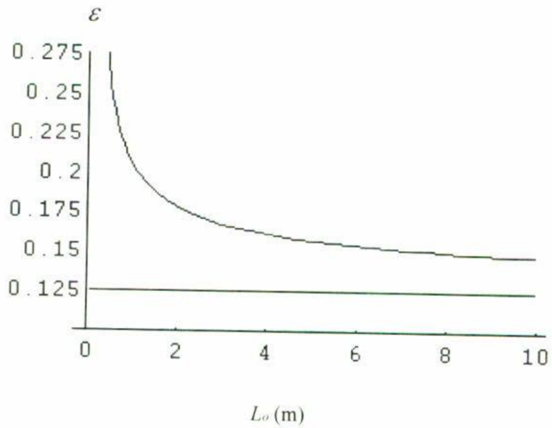

FIGURE 2. Relative error versus L_0. Aperture size: $D = 0.5$ m. The horizontal line corresponds to the value obtained for Kolmogorov model and should be taken as a reference. Observe that for $L_0 = D$ the error increases to 0.25 approximately.

5. CONCLUSIONS

From the result given in Eq. (15) the first thing we conclude is that the tilt angle and its estimator differ by an amount that depends on other aberrations different than tilt which contribute also to the agitation of the image; that is to say, the image centroid is influenced by high order aberrations.

One may also conclude that in designing adaptive optical sistems the so-called "tip-tilt" mirror is not sufficient to compensate for the image agitation if only tip and tilt terms are considered in the wavefront measurement process. This means that in order to correct properly the image centroid offset all linear contributions included within the high order aberrations should be considered. From this, one could rename the mentioned mirror as "image centroid mirror".

A third conclusion is the agreement between the Von Karman and Kolmogorov models of turbulence when outer scale approximates to infinity ($L_0 \to \infty$), as shown in Fig. 2, which corresponds to an error of 12.5% in determining the image centroid. In addition, the outer scale does not affect significantly the image centroid unless it is of the order of the telescope aperture ($L_0 = D = 0.5$ m, in the considered case) whereupon the error increases to 25% approximately. This fact is very important to consider in designing adaptive optical telescopes and experimental work on atmospheric optics.

ACKNOWLEDGMENTS

The author would like to thank Dr. V.V. Voitsekhovich for helpful advise and Consejo Nacional de Ciencia y Tecnología (Mexico) for the doctorate studies scholarship.

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