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A variational principle of the scalar wave diffraction problem and its applications

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ABSTRACT. A new variational formulation of the problem of the scalar wave diffraction on an isolated inhomogeneity is proposed. The corresponding variational principle is based on the forward amplitude of the scattering field. An approximate solution of the diffraction problem for a spherical anisotropic inclusion is obtained with the help of this variational formulation. It is assumed that the wave field inside the inclusion is a plane wave with unknown amplitude and wave number. The latter are found from the stationarity of the variational functional of the problem. The comparison of the exact and approximate solutions in the case of an isotropic spherical inclusion is presented.

RESUMEN. En este artículo se propone una nueva formulación variacional del problema de difracción de ondas escalares sobre una inclusión aislada. El principio variacional correspondiente se basa en la amplitud de avance del campo de dispersión. Mediante la utilización de esta formulación variacional, se obtiene una solución aproximada del problema de difracción para una inclusión anisotrópica esférica. Se asume que el campo de la onda dentro de la inclusión es una onda plana con amplitud y número de onda no conocidos. Estos últimos se obtuvieron a partir de la condición estacionaria de la funcional variacional del problema. Se presenta la comparación entre las soluciones exacta y aproximada para el caso de una inclusión esférica isótropa.

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1. INTRODUCTION

Diffraction of waves on isolated inhomogeneities (inclusions) in a homogeneous medium is an important problem of the wave theory. Exact solutions of this problem are known only for rather simple cases: for instance, isotropic inclusions of canonical forms (ellipsoids and limit forms of ellipsoids) in an isotropic homogeneous medium [1, 2]. Even in these cases the exact solutions have the forms of series of spherical (or cylindrical) Bessel's function and Legendre's polynomials with a poor convergence. It is necessary to keep about hundred of terms in these series to obtain a reliable result in the region of middle and short waves [1]. For anisotropic inclusions or inclusions of non canonical forms, only approximate solutions of the diffraction problem are available. A natural way to build approximate solutions is a variational formulation of the diffraction problem. Some of such formulations were considered in Refs. 2–4.

In this work a new variational formulation of the diffraction problem is proposed. This formulation is based on the forward amplitude of the scattering field. The corresponding variational functional differs from the ones considered in Refs. 2–4 and depends on the wave field and its gradient inside the volume occupied by the inclusion. It is shown that this functional has a clear physical meaning: it is proportional to the forward amplitude of the scattering field and its imaginary part coincides with the full scattering cross section of monochromatic waves on the inclusion. This fact may be used for the correction of approximate solutions, particularly in the short wave region where exact asymptotics of the full scattering cross sections are known [1, 2].

The proposed variational formulation allows us to construct an approximate solution of the diffraction problem. It is assumed in this work that the wave field and its gradient inside the inclusion are plane waves with unknown amplitudes and wave vectors. The latter are found from the condition of stationarity of the mentioned variational functional of the problem. Such an approximation is built here for an anisotropic spherical inclusion. The comparison of the exact and approximate solutions for the isotropic spherical inhomogeneity is presented. In the Conclusion the area of the possible application of the plane wave approximation is discussed.

2. Integral equations of the diffraction problem

Let us consider an infinite homogeneous medium with an isolated inclusion ideally conjugated with the medium along the interface. We study here the diffraction of scalar waves of fixed frequency ω (monochromatic waves) on such an inhomogeneity. If the dependence on time t is defined by the multiplier $e^{i\omega t}$, the amplitude $u(x)$ of the wave field in the medium with the inclusion satisfies the following equation of motion:

$$\nabla_i C_{ij}(x)\nabla_j u(x) + \rho(x)\omega^2 u(x) = -q(x), \qquad \nabla_i = \partial/\partial x_i. \tag{1}$$

Here $x(x_1, x_2, x_3)$ is a point of 3D space, two rank tensor $C_{ij}(x)$ and $\rho(x)$ are parameters of elasticity and density of the medium. They are equal to C_{0ij}, ρ_0 in the medium and C_{ij}, ρ inside the inclusion; $q(x)$ is the amplitude of the sources of the field. Here and further low latin indexes are tensorial, summation in respect to the repeating indexes is implied.

The functions $C_{ij}(x)$ and $\rho(x)$ may be represented in the forms

$$\begin{aligned} C_{ij}(x) &= C_{0ij} + C_{1ij}(x), & \rho(x) &= \rho_0 + \rho_1(x), \\ C_{1ij}(x) &= C_{1ij}V(x), & \rho_1(x) &= \rho_1 V(x), \\ C_{1ij} &= C_{ij} - C_{0ij}, & \rho_1 &= \rho - \rho_0, \end{aligned} \tag{2}$$

where $V(x)$ is the characteristic function of the area V occupied by the inclusion ($V(x) = 1$ if $x \in V$ and $V(x) = 0$ if $x \notin V$).

After rewriting Eq. (1) in the form

$$\nabla_i C_{0ij}\nabla_j u(x) + \rho_0\omega^2 u(x) = -q(x) - \nabla_i C_{1ij}(x)\nabla_j u(x) - \rho_1(x)\omega^2 u(x) \tag{3}$$

one can go to the integral equation for the wave field $u(x)$ in the medium with the inclusion

$$u(x) = u_0(x) + \int_V \nabla_i G_0(x - x') C_{1ij} \nabla_j u(x')\, dx' + \omega^2 \int_V G_0(x - x')\rho_1 u(x')\, dx', \qquad u_0(x) = \int G_0(x - x') q(x') dx'. \quad (4)$$

Here $G_0(x)$ is Green's function of the homogeneous medium with the elastic constant tensor C_{0ij} and density ρ_0. This function satisfies the following equation:

$$\left[\nabla_i C_{0ij} \nabla_j + \rho\omega^2\right] G_0(x) = -\delta(x), \quad (5)$$

where $\delta(x)$ is Dirac's delta-function, and the "radiation" condition at infinity [5]. The explicit form of $G_0(x)$ for an isotropic medium ($C_{0ij} = C_0\delta_{ij}$, δ_{ij} is Kronecker's symbol) is

$$G_0(x) = \frac{e^{-ik_0 r}}{4\pi C_0 r}, \qquad r = |x|, \qquad k_0 = \omega\sqrt{\frac{\rho_0}{C_0}}, \qquad i = \sqrt{-1}. \quad (6)$$

Note that $u_0(x)$ in Eq. (4) is an "exciting" field which would have existed in the homogeneous medium (C_0, ρ_0) and the same sources $q(x)$ of the field and the conditions at infinity.

To obtain Eq. (4) it is necessary to apply the integral operator with the kernel $G_0(x)$ to the both parts of Eq. (3) and then to take into account the definition (5) of Green's function and Gauss's theorem. The integral equation (4) is totally equivalent to the original equation (1) and has been used by many authors for the solution of diffraction problems [2, 6, 7].

It is evident from Eq. (4) that the amplitude of the wave field gradient vector $\varepsilon(x) = \nabla u(x)$ in the medium satisfies the equation

$$\varepsilon_i(x) = \varepsilon_{0i}(x) - \int_V K_{0ij}(x - x') C_{1jk}\varepsilon_k(x')\, dx' + \omega^2 \int_V \nabla_i G_0(x - x')\rho_1 u(x')\, dx',$$
$$K_{0ij}(x) = -\nabla_i\nabla_j G_0(x), \qquad \varepsilon_{0i}(x) = \nabla_i u_0(x). \quad (7)$$

Symbolically the equations (4),(7) may be written in the forms

$$u = u_0 + \nabla G \cdot C_1 \cdot \varepsilon + \omega^2 G\rho_1 u, \qquad \varepsilon = \varepsilon_0 - K \cdot C_1 \cdot \varepsilon + \omega^2 \nabla G \rho_1 u. \quad (8)$$

Note that Eq. (8) are in essence the equations for the fields $u(x)$ and $\varepsilon(x)$ inside the inclusion. The wave fields in the medium can be reconstructed from Eqs. (4), (7) if the wave fields inside V are known.

In the case of isotropic medium and inclusion ($C_{0ij} = C_0\delta_{ij}$, $C_{ij} = C\delta_{ij}$ and C_0, C are scalars) the wave field $u(x)$ inside the inclusion of unit radius $a = 1$ may be represented in the form of the series of spherical Bessel's functions $j_m(kr)$ and Legendre's polynomials $P_m(\cos\theta)$ of order m [2, 5]

$$u(x) = u(r,\theta) = \sum_{m=0}^{\infty} \alpha_m j_m(kr) P_m(\cos\theta), \quad k = \omega\sqrt{\frac{\rho}{C}},$$

$$\alpha_m = \frac{(2m+1)(-i)^{m+1} C_0}{k_0[C_0 k_0 j_m(k) h'_m(k_0) - C k j'_m(k) h_m(k_0)]}, \quad h_m(z) = j_m(z) - i y_m(z). \quad (9)$$

Here $y_m(z)$ are spherical Bessel's functions of the second rank, $f'(x) = df/dz$; (r, θ, φ) are the spherical coordinates in 3D-space with the origin in the center of the inclusion and the polar axis directed along the wave vector $\mathbf{k}_0$ of the exciting field $u_0(x)$. The latter is a plane wave with the unit amplitude ($u_o(x) = e^{-i\mathbf{k}_0 \cdot x}$).

3. A VARIATIONAL PRINCIPLE OF THE DIFFRACTION PROBLEM

Let us rewrite the integral equations (4),(7) of the diffraction problem in the form

$$u(x) = u_0(x) + u^s(x), \qquad \varepsilon(x) = \varepsilon_0(x) + \varepsilon^s(x),$$
$$u^s(x) = \int_V \nabla G_0(x - x') \cdot C_1 \cdot \varepsilon(x')\, dx' + \omega^2 \int_V G_0(x - x')\rho_1 u(x')\, dx', \tag{10}$$
$$\varepsilon^s(x) = -\int_V K_0(x - x') \cdot C_1 \cdot \varepsilon(x')\, dx' + \omega^2 \int_V \nabla G_0(x - x')\rho_1 u(x')\, dx'.$$

Further the exciting fields $u_0(x)$ and $\varepsilon_0(x)$ are assumed to be plane waves with the wave vector $\mathbf{k}_0$ of the medium:

$$u_0(x) = e^{-i\mathbf{k}_0 \cdot x}, \quad \varepsilon_0 = -i\mathbf{k}_0 e^{i\mathbf{k}_0 \cdot x}, \quad \mathbf{k}_0 = k_0 \mathbf{m}, \quad \mathbf{k}_0 \cdot x = k_0 m_i x_i. \tag{11}$$

The fields $u^s(x)$ and $\varepsilon^s(x)$ in the right hand sides of Eq. (10) may be intepreted as the wave fields scattered on the inclusion. In the case of an isotropic medium the main terms of these fields in the far zone from the inclusion($r = |x| \gg d$, d is a characteristic linear size of the inclusion) have the forms

$$u^s(x) \approx \frac{e^{-ik_0 r}}{r} F_u(\mathbf{n}), \quad \varepsilon^s(x) \approx \frac{e^{-ik_0 r}}{r} F_\varepsilon(\mathbf{n}), \quad F_\varepsilon(\mathbf{n}) = -ik_0 \mathbf{n} F_u(\mathbf{n}), \quad \mathbf{n} = \frac{x}{r},$$
$$F_u(\mathbf{n}) = \frac{1}{4\pi C_0}\left[-ik_0 \mathbf{n} \cdot \int_V C_1 \cdot \varepsilon(y) e^{ik_0(\mathbf{n}\cdot y)}\, dy + \omega^2 \int_V \rho_1 u(y) e^{ik_0(\mathbf{n}\cdot y)}\, dy\right]. \tag{12}$$

To obtain these expressions we use the following asymptotic representation of Green's function and its derivative in the far zone ($|x - y| \approx |x| - \mathbf{n} \cdot y$ in this zone):

$$G_0(x - y) \approx \frac{e^{-ik_0|x|}}{4\pi C_0} e^{ik_0(\mathbf{n}\cdot y)}, \qquad \nabla G_0(x - y) \approx -\frac{e^{-ik_0|x|}}{4\pi C_0}(ik_0 \mathbf{n}) e^{ik_0(\mathbf{n}\cdot y)}.$$

Let us consider the amplitud $F_u(\mathbf{n})$ of the scattering field in the direction $\mathbf{n} = \mathbf{m}$ (the direction of the exciting field propagation). $F_u(\mathbf{n})$ is called the forward scattering amplitude. Taking into account Eq. (11) one can represent $F_u(\mathbf{m})$ in the form

$$F_u(\mathbf{m}) = \frac{V}{4\pi C_0}\left[-(C_1 \cdot \varepsilon, \overline{\varepsilon}_0) + \omega^2(\rho_1 u, \overline{u}_0)\right], \quad (f, \phi) = \frac{1}{V}\int_V f(x)\phi(x)\, dx. \tag{13}$$

Here and further the line over functions means the complex conjugation; the last integral includes the convolution if f and ϕ are tensor functions of the same rank.

It is known (see, *e.g.*, Refs. 1 and 2) that the imaginary part of the forward scattering amplitude is connected with the normalized full scattering cross-section Q of the inclusion by the relation

$$Q = -\frac{4\pi}{k_0 S_0}\,\mathrm{Im}\,[\mathrm{F_u}(\mathbf{m})]. \tag{14}$$

Here S_0 is the maximal area of the intersection of the inclusion V by the plane orthogonal to the direction $\mathbf{m}$. Note that the short wave asymptotics (ω, $k_0 \to \infty$) of Q is equal to 2 (the exctinction paradox [1, 2]).

In the case of a spherical isotropic inhomogeneity of a unit radius the field $u(x)$ inside the inclusion has the form (9) and form Eqs. (13), (14) we get the following expressions for $F_u(\mathbf{n})$ and Q:

$$F_u(\mathbf{n}) = -\frac{k_0^2}{3}\sum_{m=0}^{\infty} i^m \alpha_m \left[\frac{C_1}{C_0} g_{1m} - \frac{\rho_1}{\rho_0} g_m\right] P_m(\cos\theta),$$
$$Q = \frac{4}{3} k_0\,\mathrm{Im}\left[\frac{C_1}{C_0} H_c(k, k_0) - \frac{\rho_1}{\rho_0} H_\rho(k, k_0)\right], \tag{15}$$

where θ is the angle between the vectors $\mathbf{n}$ and $\mathbf{m}$, the functions H_ρ and H_c are defined by the series

$$H_\rho = \frac{1}{V}\int_V u(x) e^{i\mathbf{k}_o\cdot x}\,dx = \sum_{m=0}^{\infty} i^m \alpha_m g_m,$$
$$H_c = \frac{i}{k_0^2 V}\mathbf{k}_0 \cdot \int_V \nabla u(x) e^{i\mathbf{k}_0\cdot x}\,dx = \sum_{m=0}^{\infty} i^m \alpha_m g_{1m}, \tag{16}$$

$$g_m = \frac{3}{k^2 - k_0^2}[k j_{m+1}(k) j_m(k_0) - k_0 j_{m+1}(k_0) j_m(k)], \qquad g_{1m} = \frac{3}{k_0} j_m(k) j_m'(k_0) + g_m,$$

and α_m has the form (9).

Let us express u_0 and ε_0 through u and ε using the original equations (8) and substitute the result in Eq. (13). For the function $\overline{F_u(\mathbf{m})}$ we get the following expresion:

$$\overline{F_u(\mathbf{m})} = \frac{V}{4\pi C_0}\Big[-\left(\overline{C_1\cdot\varepsilon}, \varepsilon\right) - \left(\overline{C_1\cdot\varepsilon}, K\cdot C_1\cdot\varepsilon\right) + \omega^2\left(\overline{C_1\cdot\varepsilon}, \nabla G_0\rho_1 u\right)$$
$$+\omega^2(\overline{\rho_1 u}, u) - \omega^2(\overline{\rho_1 u}, \nabla G_0\cdot C_1\cdot\varepsilon) - \omega^4(\overline{\rho_1 u}, G_0\rho_1 u)\Big]. \tag{17}$$

Let us introduce the functional $JQ(u,\varepsilon)$:

$$JQ(u,\varepsilon) = \frac{V}{k_0 C_0 S_0}\Big[-\left(\overline{C_1\cdot\varepsilon}, \varepsilon\right) - \left(\overline{C_1\cdot\varepsilon}, K\cdot C_1\cdot\varepsilon\right) + \omega^2\left(\overline{C_1\cdot\varepsilon}, \nabla G_0\rho_1 u\right)$$
$$+\,\omega^2\left(\overline{\rho_1 u}, u\right) - \omega^2\left(\overline{\rho_1 u}, \nabla G_0\cdot C_1\cdot\varepsilon\right) - \omega^4\left(\overline{\rho_1 u}, G_0\rho_1 u\right)$$
$$+\left(\overline{C_1\cdot\varepsilon}, \varepsilon_0\right) - \omega^2\left(\overline{\rho_1 u}, u_0\right) + (C_1\cdot\varepsilon, \overline{\varepsilon}_0) - \omega^2\left(\rho_1 u, \overline{u}_0\right)\Big]. \tag{18}$$

Because of Eqs. (13) and (17) the value of this functional on the exact solution of the system (8) is proportional to the forward scattering amplitude $F(\mathbf{m})$

$$JQ(u,\varepsilon) = \frac{4\pi}{k_0 S_0}\left[\overline{F_u(\mathbf{m})} - \overline{F_u(\mathbf{m})} - F_u(\mathbf{m})\right] = -\frac{4\pi}{k_0 S_0} F_u(\mathbf{m}). \tag{19}$$

It is evident from Eq. (14) that the imaginary part of JQ coincides with the full scattering cross section Q of the given inclusion $[\mathrm{Im}\,(JQ) = Q]$.

On the other hand the variational equations

$$\frac{\delta(JQ)}{\delta u} = 0, \qquad \frac{\delta(JQ)}{\delta \varepsilon} = 0 \tag{20}$$

are equivalent to the original system of the integral equations (8). Really, the first equation (20) may be written in the form

$$\begin{aligned}\left(\frac{\delta(JQ)}{\delta u}, \delta u\right) &= -\omega^2 \rho_1 \Big[\Big((\overline{u} - \nabla G_0 C_1 \cdot \overline{\varepsilon} - \omega^2 G_0 \rho_1 \overline{u} - \overline{u}_0), \delta u\Big) \\ &\quad + \Big((u - \nabla G_0 \cdot C_1 \cdot \varepsilon - \omega^2 G_0 \rho_1 u - u_0), \overline{\delta u}\Big)\Big] = 0.\end{aligned} \tag{21}$$

Here we take into accout that C_1 is a two rank symmetric tensor and the kernel $G_0(x)$ of the operator G_0 has the property $G_0(-x) = G_0(x)$ that follows the expression (6) for $G_0(x)$.

The real $\mathrm{Re}\,(\delta u)$ and imaginary $\mathrm{Im}\,(\delta u)$ parts of the variation δu of the wave field inside inclusion can be considered as independent. Thus the last equation may be rewritten in the form

$$\begin{aligned}&\Big[\Big(\mathrm{Re}\,(\mathrm{u}) - \nabla \mathrm{G}_0 \cdot \mathrm{C}_1 \cdot \mathrm{Re}\,(\varepsilon) - \omega^2 \mathrm{G}_0 \rho_1 \,\mathrm{Re}\,(\mathrm{u}) - \mathrm{Re}\,(\mathrm{u}_0)\Big),\, \mathrm{Re}\,(\delta \mathrm{u})\Big] \\ &\quad + \Big[\Big(\mathrm{Im}\,(\mathrm{u}) - \nabla \mathrm{G}_0 \cdot \mathrm{C}_1 \cdot \mathrm{Im}\,(\varepsilon) - \omega^2 \mathrm{G}_0 \rho_1 \,\mathrm{Im}\,(\mathrm{u}) - \mathrm{Im}\,(\mathrm{u}_0)\Big),\, \mathrm{Im}\,(\delta \mathrm{u})\Big] = 0.\end{aligned} \tag{22}$$

After putting the multipliers in front of $\mathrm{Re}\,(\delta u)$ and $\mathrm{Im}\,(\delta u)$ equal to zero and joining these equations into the one we get the first equation of the system (8). At the same way it may be demonstrated that the equation $\delta(JQ)/\delta\varepsilon = 0$ gives the second equation of the system (8). Thus the solution of Eqs. (8) is a stationary point of the functional (18).

Variational principles based on the forward amplitude of the scattering field in application to the diffraction problems were considered in Ref. 2 (See Chapters 9.4 and 12.3) and Ref. 4. If one changes the complex conjugated function in Eq. (18) for the original ones the resulting functional will olso have a stationer value on the exact solution of Eqs. (8). (For the problem of the elastic wave diffraction the similar funtional was proposed in Ref. 3.) But by such a definition the functional JQ loses its mentioned physical meaning (to be proportional to the forward amplitude of the scattering field and to have the imaginary part equal to the full scattering cross section Q of the given inclusion).

4. An approximate solution of the diffraction problem

In order to build an approximate solution of the diffraction problem let us assume that the wave field $u(x)$ and $\varepsilon(x)$ inside the inclusion are plane waves with unknown amplitudes b and $\mathbf{B}$ and a wave vector $\mathbf{l}$:

$$u(x) = be^{-i\mathbf{l}\cdot x}, \qquad \varepsilon(x) = \mathbf{B}e^{-i\mathbf{l}\cdot x}, \qquad x \in V. \tag{23}$$

For simplicity the medium is assumed to be isotropic with the dynamic properties equal to one ($C_0 = 1$, $\rho_0 = 1$) and the inclusion is a sphere of a unit radius $a = 1$. (The dimensions are of no importance for the present analysis)

In orden to obtain the amplitudes b and $\mathbf{B}$ in the representation (23) let us substitute $u(x)$ and $\varepsilon(x)$ from Eq. (23) into the functional JQ (18). As a result JQ will be the function of three variables

$$JQ = JQ(b, \mathbf{B}, \mathbf{l}), \tag{24}$$

where b is a scalar, $\mathbf{l}$ and $\mathbf{B}$ are vectors. According to Ritz's scheme the equations for the amplitudes b and $\mathbf{B}$ follow from the condition of stationarity of JQ $[\partial(JQ)/\partial b = 0$, $\partial(JQ)/\partial \mathbf{B} = 0]$ and take the forms

$$\begin{aligned} b - \omega^2\rho_1 g(k_0, \mathbf{l})b - \mathbf{G}(k_0, \mathbf{l})\cdot C_1 \cdot \mathbf{B} &= F(|\mathbf{k}_0 - \mathbf{l}|), \\ \mathbf{B} + K(k_0, \mathbf{l})\cdot C_1\cdot \mathbf{B} - \omega^2\rho_1 \mathbf{G}(k_0, \mathbf{l})b &= -i\mathbf{k}_0 F(|\mathbf{k}_0 - \mathbf{l}|). \end{aligned} \tag{25}$$

Here $g(k_0, \mathbf{l})$, $\mathbf{G}(k_0, \mathbf{l})$, $K(k_0, \mathbf{l})$ are the following integrals:

$$\begin{aligned} g(k_0, \mathbf{l}) &= \int G_0(x)e^{i\mathbf{l}\cdot x} f(x)\, dx, \\ \mathbf{G}(k_0, \mathbf{l}) &= \int [\nabla G_0(x)]e^{i\mathbf{l}\cdot x} f(x)\, dx, \\ K(k_0, \mathbf{l}) &= -\int [\nabla \otimes \nabla G_0(x)]e^{i\mathbf{l}\cdot x} f(x)\, dx, \end{aligned} \tag{26}$$

where the function $f(x)$ has the form

$$f(x) = f(|x|) = \frac{1}{V}\int V(y)V(x+y)\, dy = \begin{cases} 1 - \frac{3}{4}|x| + \frac{1}{16}|x|^3, & |x| \le 2 \\ 0, & |x| > 2 \end{cases} \tag{27}$$

Here $V(y)$ is the characteristic function of the spherical are V of radius $a = 1$ with the center at point $y = 0$.

The function $F(k)$ on the right hand sides of Eq. (25) has the form

$$F(k) = \frac{1}{V}\int_V e^{i\mathbf{k}\cdot x}\, dx = \frac{j_1(k)}{3k}, \qquad |\mathbf{k}| = k, \tag{28}$$

where $j_1(k)$ is the spherical Bessel's function of the first order.

The Eqs. (25) may be obtained also according to Galerkin's scheme if we substitute the representation (23) into Eqs. (8), multiply both parts on $e^{i\mathbf{l}\cdot x}$ and then average the result over the volume of the inclusion.

Note that for $\mathbf{l} = 0$ (the long wave limit) the equations similar to Eqs. (25) were proposed in Ref. 7 for the solution of the problem of the elastic wave diffraction on a spherical inhomogeneity.

After using Gauss's theorem and integrating over the unit sphere the integrals (26) take the forms

$$g(k_0, \mathbf{l}) = \frac{g_0(k_0, l)}{C_0}, \quad g_0(k_0, l) = \int_0^2 e^{-ik_0r} f(r) j_0(lr) r\, dr, \quad |\mathbf{l}| = l, \quad \mathbf{l} = l\mathbf{e},$$

$$\mathbf{G}(k_0, \mathbf{l}) = \int G_0(x) \nabla[e^{i\mathbf{l}\cdot x} f(x)]\, dx = -\frac{i}{C_0} \mathbf{l} G(k_0, l),$$

$$G(k_0, l) = \int_0^2 e^{-ik_0r} [f'(r) j_1(lr) + r f(r) j_0(lr)]\, dr, \tag{29}$$

$$K(k_0, \mathbf{l}) = -\int G_0(x) \nabla \otimes \nabla[e^{i\mathbf{l}\cdot x} f(x)]\, dx = \frac{1}{C_0}[K^{(1)}(k_0, l)\mathbf{I} + K^{(2)}(k_0, l)\mathbf{e} \otimes \mathbf{e}],$$

$$K^{(1,2)}(k, l) = \int_0^2 e^{-ikr} \Phi^{(1,2)}(r)\, dr, \qquad \Phi^{(1)}(r) = f'(r) j_0(r) + [r f''(r) - f'(r)] \frac{j_1(lr)}{lr},$$

$$\Phi^{(2)}(r) = -[r f''(r) - f'(r)] j_2(lr) - 2lr f'(r) j_1(r) - r f(r) l^2 j_0(lr).$$

Here $\mathbf{I}$ is the two rank unit tensor. Note that all these integrals can be calculated in the explicit forms (see Appendix).

The solution of the system (25) has the form

$$b = \frac{1 + i\mathbf{R}_1 \cdot \mathbf{D}_0^{-1} \cdot \mathbf{k}_0}{d_0 - \mathbf{R}_1 \cdot \mathbf{D}_0^{-1} \cdot \mathbf{R}_2} F(|\mathbf{k}_0 - \mathbf{l}|), \qquad \mathbf{B} = -i\mathbf{D}_0^{-1} \cdot \mathbf{k}_0 F(|\mathbf{k}_0 - \mathbf{l}|) - \mathbf{R}^2 \cdot \mathbf{D}_0^{-1} b,$$

$$d_0 = 1 - k_0^2 \frac{\rho_1}{\rho_0} g_0, \qquad \mathbf{D}_0 = \mathbf{I} + K \cdot C_1, \qquad \mathbf{R}_1 = iG\mathbf{l} \cdot C_1, \qquad \mathbf{R}_2 = ik_0^2 G\mathbf{l}. \tag{30}$$

After substituting (23), (30) into the functional JQ (18) the latter take the form

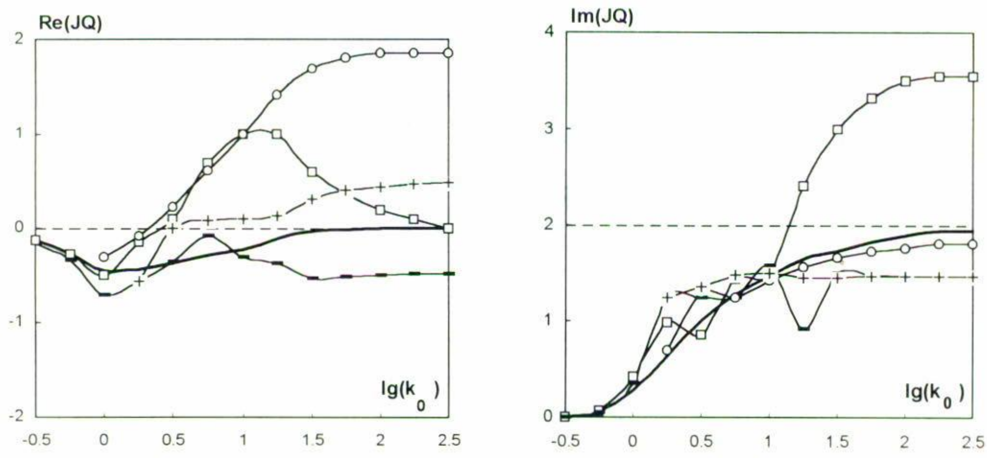
$$JQ = \frac{4}{3} k_0 \left[-\frac{i}{C_0 k_0^2} \mathbf{k}_0 \cdot C_1 \cdot \mathbf{B} + \frac{\rho_1}{\rho_0} b \right] F(|\mathbf{k}_0 - \mathbf{l}|). \tag{31}$$

In order to check the quality of this approximation let us compare the values of the functional JQ (18) on the exact solution of the Eqs. (8) and on the approximate solution [(13), (30)] in the case of an isotropic inclusion. It is natural to assume that the directions of the wave vectors $\mathbf{k}_0$ and $\mathbf{l}$ are the same for the isotropic inclusion:

$$\mathbf{k}_0 = k_0 \mathbf{m}, \qquad \mathbf{l} = l\mathbf{m}.$$

In this case b and $\mathbf{B}$ are represented in the forms

$$b = \tilde{H}\rho(k_0, l), \qquad \mathbf{B} = -i\mathbf{k}_0 \tilde{H} c(k_0, l), \tag{32}$$



(a) (b)

FIGURE 1. The dependences of the real (a) and imaginary (b) parts of the functional JQ on the wave number k_0 of the medium for a soft and light inclusion ($\rho = 0.1$, $C = 0.001$; $\rho_0 = 1$, $C_0 = 1$). Solid lines correspond to the exact solution of the diffraction problem; lines of pluses (+) and minuses (–) correspond to the assumption that the wave number l of the wave field inside inclusion provides stationary to the functional JQ; lines of squares (□) correspond to the assumption $l = k_0$; lines of circles (○) are obtained from the condition that l is real and provides a maximum to the real part of the functional JQ.

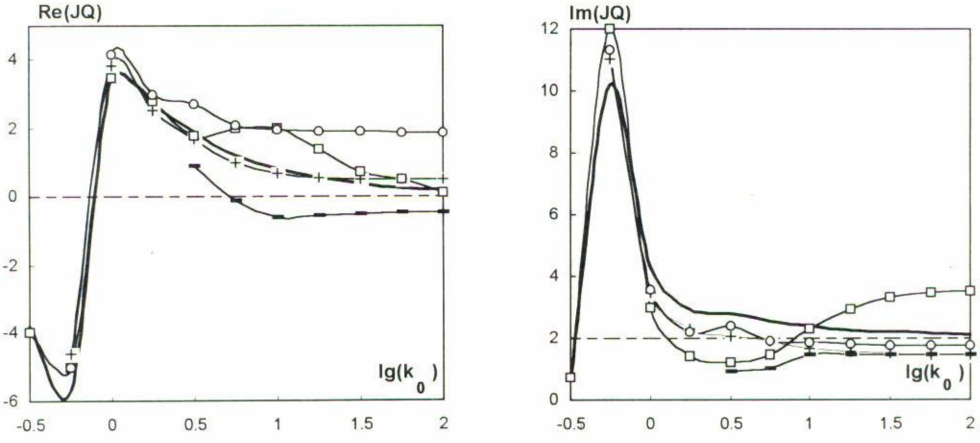
where the scalar functions $\tilde{H}\rho$ and $\tilde{H}c$ are defined by the following relations:

$$
\begin{aligned}
\tilde{H}\rho(k_0,l) &= \frac{1}{\Delta(k_0,l)}\left[De(k_0,l) - \frac{C_1}{C_0} l k_0 G(k_0,l)\right] F(|k_0 - l|),\\
\tilde{H}c(k_0,l) &= \frac{1}{\Delta(k_0,l)}\left[de(k_0,l) - \frac{\rho_1}{\rho_0} k_0 l G(k_0,l)\right] F(|k_0, l|),\\
\Delta(k_0,l) &= De(k_0,l) de(k_0,l) + \frac{\rho_1 C_1}{\rho_0 C_0} k_0^2 l^2 G^2(k_0,l),\\
De(k_0,l) &= 1 - \frac{C_1}{C_0}[K^{(1)}(k_0,l) + K^{(2)}(k_0,l)], \quad de(k_0,l) = l - \frac{\rho_1}{\rho_0} k_0^2 g(k_0,l).
\end{aligned}
\tag{33}
$$

The functional JQ for this case takes the form

$$
JQ(k_0,l) = \frac{4}{3} k_0 \left[\frac{C_1}{C_0}\tilde{H}c(k_0,l) - \frac{\rho_1}{\rho_0}\tilde{H}\rho(k_0,l)\right] F(|k_0,l|). \tag{34}
$$

The solid lines on Figs. 1a and 1b are the exact dependences of the real and imaginary parts of functional JQ on the wave number k_0 for the soft and light inclusion ($C = 0.1$, $\rho = 0.01$). The same dependences for the hard and heavy inclusion ($C = 1000$, $\rho = 10$) are presented in Figs. 2a and 2b. In order to build these dependences the equations (19), (15), (16) were used. (Note that Im(JQ) coincides with the full scattering cross section Q of the inclusion).



(a) (b)

FIGURE 2. The dependences of the real (a) and imaginary (b) parts of the functional JQ on the wave number k_0 of the medium for a hard and heavy inclusion ($\rho = 10$, $C = 1000$; $\rho = 1, C_0$). The denotations of the lines are the same as in Fig. 1.

In order to build the approximate dependences of JQ on k_0 it is necessary to find the wave number l of the wave field inside the inclusion. Let us assume at first that the wave number l coincides with the wave number k_0 of the medium. The dependences of Re(JQ) and Im(JQ) on k_0 for this case are presented in Figs. 1 and 2 (the lines with squares). It is clear that this approximation serves only in the long wave region. In the short wave limit when $k_0, \omega \to \infty$ it gives for $Q =$ Im(JQ) the asymptotic value $32/9 \approx 3.55\ldots$ instead of 2 (the exact limit of Im(JQ)).

In order to correct the approximate solution in the short wave region let us consider the functional JQ (34) as a function of the wave number l. According to the logic of the calculus of variations one should chose l from the condition of stationary of $JQ(l)$. It turns out that in in the limit $k_0 \to \infty$ the functional JQ does not depend on the dynamic properties of the inclusion and is only a function of the difference $z = l - k_0$. The limit expression $JQ_\infty(z)$ of the functional JQ has the form (see Appendix)

$$\lim_{k_0 \to \infty} JQ(k_0, l) = JQ_\infty(z) = \frac{192 z^2 j_1^2(z)}{8z^3 + 3i(e^{2iz} - 1) + 6z(e^{2iz} - iz)}. \tag{35}$$

The dependences of the real (solid line) and imagynary (dashed line) parts of the functional $JQ_\infty(z)$ on z (z are real) are presented in Fig. 3a. The line with triangles is the dependence of the modulus of the derivative $|JQ'_\infty(z)|$ on z. It is evident from Fig. 3a that for real z in the region where Im(JQ_∞) is closed to 2 there are no stationary points of $JQ_\infty(z)$. (The derivative $JQ'_\infty(z)$ is not equal to zero in this region.)

Let us consider $JQ_\infty(z)$ as a function of the complex variable $z = \varsigma + i\eta$. In turns out that there are two stationary points z_+ and z_- of $JQ_\infty(z)$ in the region where Im(JQ_∞) is close to 2 ($JQ'_\infty(z) = 0$ at these points):

$$z_\pm = \pm 2.471 - i2.129, \qquad JQ(z_\pm) = \pm 0.485 + i1.468\,. \tag{36}$$

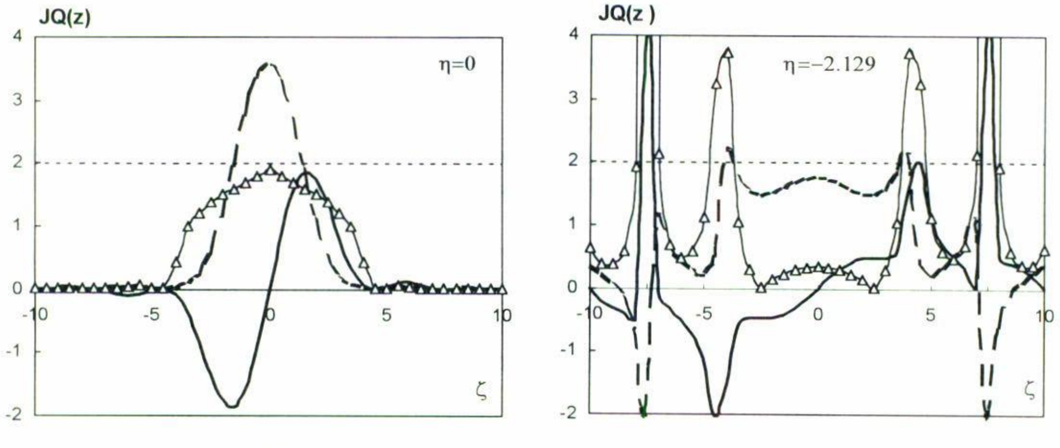


FIGURE 3. The dependences of the real (solid lines) and imaginary (dashed lines) parts of the functional JQ_∞ on $z = l - k_0$. The lines with triangles ($\triangle$) are the modulus of the derivative $|JQ'_\infty(z)|$. (3a) is these dependences for real $z, (z = \zeta + i\eta,\ \eta = 0)$, (3b) is the same dependences for $z = \zeta - i2.129$.

The dependences of the real (solid line) and imaginary (dashed line) parts of functional $JQ_\infty(z)$ on ς for $\eta = -2.129$ are presented Fig. 3b. The line with triangles is the dependence of the modulus of the derivative $|JQ'_\infty(z)|$ on ς for this η.

Let us consider the roots of the equation

$$\frac{\partial[JQ(k_0, l)]}{\partial l} = 0 \tag{37}$$

in the region of short and middle waves if l is a complex variable. Here JQ is the complete functional (34). There are two branches of these roots that correspond to two roots (36) of the Eq. (37) in the short wave region. The first branch where $\mathrm{Re}\,(l) < k_0$ if $k_0 \to \infty$ we denote as (-) and the second branch ($\mathrm{Re}\,(l) > k_0$ if $k_0 \to \infty$) as (+). The lines of pluses and minuses on Figs. 1 and 2 correspond to these branches of the solution of (37). In turns out that (+)-branch for soft inclusion and (−)-branch for hard ones break off in the long wave region. Note that the short wave limit of $Q = \mathrm{Im}\,(JQ)$ is equal to 1.468 but not 2 for this approximation.

One can accept that

$$Q = \mathrm{Im}\,[JQ_\infty(z)] = 2 \tag{38}$$

and consider this equation as an additional conditional for the variational problem in the region of very short waves. This equation defines a line on the complex plane (ς, η). In the physically acceptable region ($\eta \leq 0$) the functional $JQ_\infty(z)$ does not have any stationary point on this line but maximun and minimum of the real part of this functional are achieved on the real axis ($\varsigma = \pm 1.561,\ \eta = 0$). As it can be seen from Fig. 3a that these points are close to the stationary points $z = \pm 1.579$ of the real part of the functional

$JQ_\infty(z)$ on the real axis ($\eta = 0$). Note that

$$JQ_\infty(\pm 1.579) = \pm 1.856 + i1.759$$

and Im (JQ_∞) is close to 2 at these points.

One can use this fact in order to obtain the dependence $l = l(k_0)$ in the region of short and middle waves from the condition of stationary (maximum) of Re $[JQ(k_0, l)]$ for real l. The corresponding values of the functional JQ are the lines of circles in Figs. 1 and 2 (The choosing l from the condition of the minimum of Re $[JQ(k_0, l)]$ does not change practically the imaginary part of JQ for all k_0 but the real part of JQ will have another sign in the short wave region.) As it can be seen this approximation allows us to describe better the dependence of the full scattering cross-section $Q =$ Im (JQ) on k_0 then the same dependence of Re (JQ).

As it the follows from the Eqs. (12) and (23), the amplitude $F_u(\mathbf{n})$ of scattering field in the plane wave approximation takes the form

$$F_u(\mathbf{n}) = -\frac{k_0^2}{3}\left[\frac{i}{C_0 k_0}\mathbf{n}\cdot C_1\cdot\mathbf{B} - \frac{\rho_1}{\rho_0}b\right]F(|k_0\mathbf{n} - \mathbf{l}|), \tag{39}$$

where b and $\mathbf{B}$ are defined in Eq. (30).

The differential cross-section $DQ(\mathbf{n})$ of the inclusion is defined by the relation

$$DQ(\mathbf{n}) = |F_u(\mathbf{n})|^2 \tag{40}$$

and characterizes the friction of the energy scattered in $\mathbf{n}$-direction. The comparison of the exact [(15), (40)] and approximate [(32), (39), (40)] differential cross-sections for the hard isotropic inclusion is presented in Fig. 4. The wave numbers l of the wave field inside the inclusion for various k_0 were choosen from the maximum of the real part of the functional JQ [Eq. (34)]. The discrepancy between the exact and approximate solutions for soft inclusions has the same character.

5. DIFFRACTION ON AN ANISOTROPIC INCLUSION

Let us consider the diffraction of a plane monochromatic wave on an anisotropic spherical inclusion when C is a two rank symmetric positive tensor. We assume again that the wave field inside the inclusion is a plane wave with unknown amplitude and wave vector $\mathbf{l}$. For the anisotropic inclusion the direction of the vector $\mathbf{l}$ does not coincide with the wave vector $\mathbf{k}_0$ of the exciting field in general case.

Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are the principal vectors of the tensor $C_1 = C - C_0$ and c_1, c_2, c_3 are the corresponding principal values. We assume for simplicity that the wave vector $\mathbf{k}_0$ is located on the plane of the vectors $\mathbf{e}_1, \mathbf{e}_2$:

$$\mathbf{k}_0 = k_0(m_1\mathbf{e}_1 + m_2\mathbf{e}_2).$$

It follows from the symmetry of the problem that the vector $\mathbf{l}$ is located on the same plane:

$$\mathbf{l} = l_1\mathbf{e}_1 + l_2\mathbf{e}_2.$$

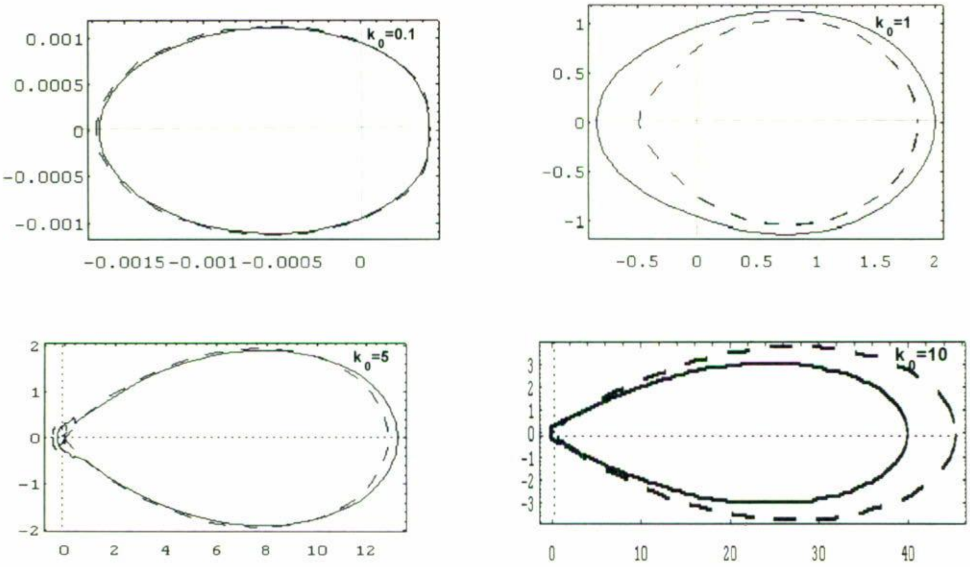


FIGURE 4. The differential cross-sections of the isotropic spherical inclusion ($C_0 = 1$, $\rho = 1$; $C = 1000$, $\rho = 10$; $a = 1$) for different wave numbers k_0 of the exciting field. Solid lines are exact solutions, dashing lines are the results of the plane wave approximation.

The components l_1, l_2 of this vector should be found from the condition of stationary of the functional JQ (31). Here as in the case of the isotropic inclusion we find the real numbers l_1, l_2 from the condition of the maximum of the real part of JQ. The graphs of the differential cross section of the hard anisotropic inclusion ($c_1 = 100$, $c_2 = c_3 = 10$, $\rho = 10$) in the isotropic medium ($C_0 = 1$, $\rho = 1$) are presented in Figs. 5. Polar angles on these graphs are the angles between the wave vector $\mathbf{k}_0 = k_0\mathbf{m}$ of the exciting field and the direction vector $\mathbf{n}$ of the point $(\mathbf{e}_1, \mathbf{e}_2)$. The direction of the exciting field wave vector $\mathbf{m}$ was chosen in the form

$$\mathbf{m} = \frac{1}{\sqrt{2}}(\mathbf{e}_1 + \mathbf{e}_2).$$

For the given values of the wave numbert k_0 of the medium the corresponding values of the components l_1, l_2 of the vector $\mathbf{l}$ are:

$$k_0 = 0.1, \qquad l_1 = 0.071, \qquad l_2 = 0.071,$$

$$k_0 = 1, \qquad l_1 = -0.638, \qquad l_2 = 1.307,$$

$$k_0 = 5, \qquad l_1 = 2.437, \qquad l_2 = 4.111,$$

$$k_0 = 10, \qquad l_1 = 6.451, \qquad l_2 = 7.942.$$

Note that in the long wave region ($k_0 < 0.5$) maximum $\mathrm{Re}\,(JQ)$ disappears and we chose $\mathbf{k_0} = \mathbf{l}$ for $k_0 = 0.1$. In the short wave region the direction of the wave vector $\mathbf{l}$ turns to the direction of $\mathbf{k_0}$ when $k_0 \to \infty$.

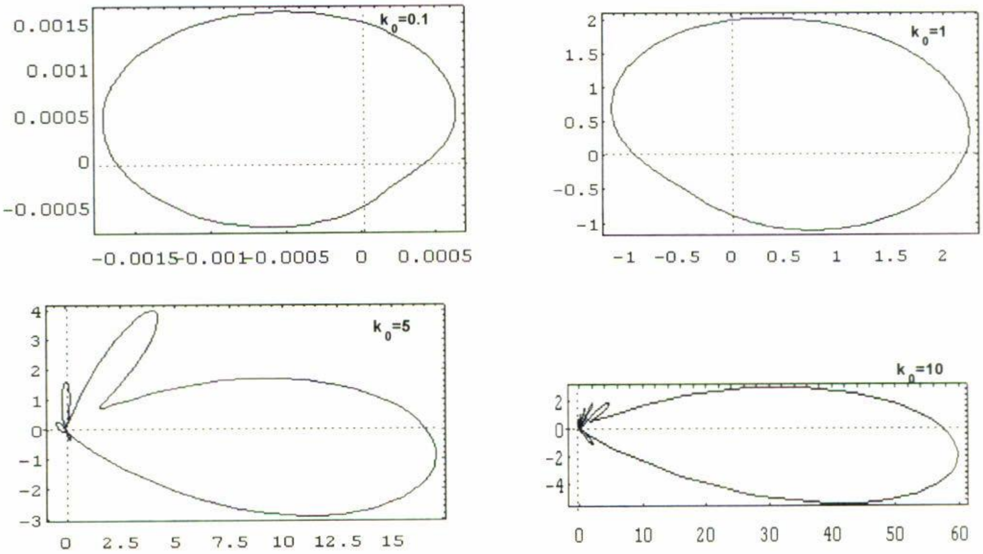


FIGURE 5. The differential cross sections of the anisotropic spherical inclusion ($C_0 = 1$, $\rho_0 = 1$; $c_1 = 100$, $c_2 = 10$, $\rho = 10$; $a = 1$) for different wave numbers k_0 of the exciting field.

6. CONCLUSION

The approximate solution of the diffraction problem which has been built in this work may be called the plane wave approximation. This approximation gives us the correct asymptotics for the full scattering cross-section of a spherical inclusion in the long wave region and it is close to the exact result in the short wave region. The formulae for the calculation of the differential and full scattering cross-section of the spherical anisotropic inclusion in the framework of this approximation consist of some combinations of exponential functions and polynoms of finite powers (see Appendix). The numerical calculations according to these formulae are more simple that the summation of the series of the exact solutions similar to Eq. (9), particulary in the short wave region. Of course with the help of only one plane wave it is difficult to describe a rather complicate wave field inside the inclusion in the region of the middle and short waves. In the middle wave region this approximation describes only a general trend of the dependence of the full scattering cross-section on the frequency of the exciting field. But in many cases such a description is very close to the exact solution. Note that in the long wave region ($k_0 < 1$) an agreement with exact solutions can be obtained in the assumption that $\mathbf{k}_0 = \mathbf{l}$ (the wave vector inside the inclusion coincides with the wave vector of exciting field). In the region of the middle and short waves one has to find the wave vector $\mathbf{l}$ from the condition of the maximum of the real part of the functional JQ (31).

It is worth to emphasize that the plane wave approximation may be applied to the solution of the diffraction problem for inclusions of noncanonical forms. This approximation on the base of the corresponding varactional principle may be also used in the problems of the elastic and electromagnetic waves diffraction on isolated inhomogeneities. A possible area of applications of this approximation is the problem of wave propagation through the medium with a random set of inclusions. For this problem an exact account of details of the scattering field from every inclusion is not necessary because of further averaging the result over the sizes of the inclusions and their spatial positions. That is because using some simple approximations of the diffraction problem is more preferable then to operate with the huge series of the exact solutions no to say of the cases when such solutions are not available. For instance the plane wave approximation is simple enough to be used inside self-consistent schemes for the calculation of effective dynamic properties of composite materials [7, 8].

APPENDIX

For the calculation of the integrals (29) it is necessary to take into account that the spherical Bessel's functions j_0, j_1, j_2 may be represented as combinations of trigonometrical and power functions

$$j_0(x) = \frac{\sin(z)}{z}, \qquad j_1(z) = \frac{\sin(z)}{z} - \frac{\cos(z)}{z}, \qquad j_2(z) = \frac{3j_1(z)}{z} - j_0(z). \tag{41}$$

The function $f(r)$ in Eq. (29) and its derivatives are also power functions:

$$f(r) = 1 - \tfrac{3}{4}r + \tfrac{1}{16}r, \qquad f'(r) = -\tfrac{3}{4} + \tfrac{3}{16}r^2, \qquad f''(r) = \tfrac{3}{8}r. \tag{42}$$

That is because the integrals (29) may be represented as linear combinations of the integrals $I_n(c)$ of the following type:

$$I_n(c) = \int_0^2 r^n e^{icr}\, dr, \qquad n = -3, -2, -1, 0, 1, 2, 3 \tag{43}$$

For negative n these integrals diverge. But it is necessary to take into account that the integrand functions in the integrals (29) are bounded when $r = 0$. That is because these integrals can be understood as the following.

If $n < 0$ let us substitute instead of 0 in the low limit of the integrals (43) a small numbert δ and take into account only the terms that have the order of 1 in comparison with..., δ^{-1}, $\ln(\delta)$, δ, etc. The other terms should disappear when $\delta \to 0$ in the full expression of the integrals (29). As a result one should understand the integrals (43) for negative n, $(n = -m,\ m > 0)$ as following

$$I_{-m}(c) = -\frac{1}{m-1}\left[\frac{e^{2ic}}{2^{m-1}} - \frac{(ic)^{m-1}}{(m-1)!} - icI_{-(m-1)}(c)\right], \qquad m = \dots, 3, 2. \tag{44}$$

If $m = 1$ this integrals is

$$I_{-1}(c) = -E_1(-2ic) - \ln(c) - \gamma + i\frac{\pi}{2}, \qquad E_1(z) = \int_1^\infty \frac{e^{-zt}}{t}\, dt, \tag{45}$$

when γ is Euler's constant.

For positive n the integrals (43) converge and have the forms

$$I_0(c) = \frac{1}{ic}(e^{2ic} - 1); \qquad I_n(c) = \frac{1}{ic}\left[2^n e^{2ic} - nI_{n-1}(c)\right], \qquad n > 0. \tag{46}$$

Let us turn to the integrals (29). After substituting Eqs. (41), (42) in Eq. (29) we get

$$\begin{aligned} g_0(k,l) &= j0(k,l,1) - \tfrac{3}{4}j0(k,l,2) + \tfrac{1}{16}j0(k,l,4), \\ G(k,l) &= j0(k,l,1) - \tfrac{3}{4}j0(k,l,2) + \tfrac{1}{16}j0(k,l,4) - \frac{3}{4l}j1(k,l,1) + \frac{3}{16l}j1(l,k,3), \\ K(k,l) &= K^{(1)}(k,l) + K^{(2)}(k,l) = -\tfrac{3}{4}j0(k,l,0) - l^2 j0(k,l,1) + \tfrac{3}{16}j0(k,l,2) \\ &\quad + \tfrac{3}{4}l^2 j0(k,l,2) - \tfrac{1}{16}l^2 j0(k,l,4) + \frac{3}{4l}j1(k,l,-1) + \frac{3}{16l}j1(k,l,1) \\ &\quad + \tfrac{3}{2}lj1(k,l,1) - \tfrac{3}{8}lj1(k,l,3) - \tfrac{3}{4}j2(k,l,0) - \tfrac{3}{16}j2(k,l,2). \end{aligned} \tag{47}$$

Here $jm(b,a,n)$ are the following integrals

$$jm(k,l,n) = \int_0^2 r^n e^{-ikz} j_m(lr)\,dr, \qquad m = 0,1,2; \qquad n = -1,0,1,2,3$$

and $j_m(z)$ are spherical Bessel's functions (41).

The integrals $jm(k,l,n)$ may be represented in the forms which follow from Eq. (41):

$$\begin{aligned} j0(k,l,n) &= \frac{1}{l}I^s_{n-1}(k,l), \qquad j1(k,l,n) = \frac{1}{l^2}I^s_{n-1}(k,l) - \frac{1}{l}I^c_{n-1}(k,l), \\ j^2(k,l,n) &= \frac{3}{l^3}I^s_{n-3}(k,l) - \frac{3}{l^2}I^c_{n-2}(k,l) - \frac{1}{b}I^s_{n-1}(b,a), \end{aligned} \tag{48}$$

where

$$\begin{aligned} I^s_n(k,l) &= \int_0^2 r^n e^{-ikr}\sin(lr)\,dr = \frac{1}{2i}[I_n(l-k) - I_n(-k-l)], \\ I_n c &= \int_0^2 r^n e^{-ikr}\cos(lr)\,dr = \frac{1}{2}[I_n(l-k) + I_n(-k-l)], \end{aligned} \tag{49}$$

and integrals $I_n(c)$ have forms (43)–(46).

The explicit expressions of the integrals (29) are rather huge. We wrote here only the asymptotics of these integrals for long and short waves.

1. The long wave asymptotics $[(k,l) \sim \omega;\ \omega \to 0]$

$$\begin{aligned} g_0(k,l) &= \tfrac{2}{5} - \tfrac{2}{35}l^2 - \tfrac{6}{35}k^2 + i\tfrac{1}{15}k^3 + i(-\tfrac{1}{3} + \tfrac{1}{15}l^2)k + O(\omega^4), \\ G(k,l) &= \tfrac{2}{15} - \tfrac{2}{175}l^2 + k^2(\tfrac{2}{35} - \tfrac{8}{945}l^2) - \tfrac{2i}{45} + O(\omega^4), \\ K(k,l) &= -\tfrac{1}{3} - \tfrac{4}{75}l^2 + i(\tfrac{1}{9} - \tfrac{1}{45}l^2)k^3 - \tfrac{2}{15}k^2 + O(\omega^4). \end{aligned} \tag{50}$$

2. The short wave asymptotics ($z = k - l$; $k \sim \omega \to \infty$)

$$
\begin{aligned}
g_0(k,z) &= \frac{1}{16kz^4}[3i(e^{2iz}-1)+6(e^{2iz}-iz)z+8z^3]+O(k^{-2}),\\
G(k,z) &= \frac{1}{k}\left[\left(\frac{3i}{16z^4}+\frac{3}{8z^3}\right)(e^{2iz}-1)+\frac{1}{2z}\right]+O(k^{-2}),\\
K(k,z) &= \frac{k}{16z^4}\left[3i(1-e^{2iz})-6z(e^{2iz}-iz)-8z^3\right]-\frac{3}{4}+\\
&\quad \frac{3i}{16z^3}(e^{2iz}-1)+\frac{3}{8z^2}(e^{2iz}-iz)+\frac{33i-4z}{32k}+O(k^{-2}).
\end{aligned}
\tag{51}
$$

The Eq. (35) for $JQ_\infty(z)$ is the consequence of these expressions.

REFERENCES

1. C.F. Bohren and D.R. Huffman, *Absorption and Scattering of Light by Small Particles*, (John Wiley and Sons, New York, 1983).
2. Ph.M. Morse and H. Feshbach, *Methods of Theoretical Physics*, (McGraw-Hill, New York, Toronto, London, 1953), Vol. 2.
3. J.R. Willis, *Wave Motion* **3** (1980) 1.
4. K.T. Ladas, T.A. Maniatis, and N.K. Uzunoglu, *Journal of electromagnetic Waves and Applications* **10** (1996) 3.
5. A.N. Tichonov and A.A. Samarsky, *Equations of Mathematical Physics*, (Nauka, Moscow, 1953).
6. W.C. Chew, *Wave and Fields in Inhomogeneous Media*, (Van Nostrand Reinhold, New York, 1990).
7. F.J. Sabina and J.R. Willis, *Wave Motion* **10** (1988) 127.
8. S.K. Kanaun and V.M. Levin, *Advances in Mathematical Modeling of Composite Materials*, edited by Markov K.Z., Series on Advances in Mathematics for Applied Sciences, **15** (1994) 1.