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The free relativistic particle of arbitrary spin*

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ABSTRACT. The energy eigenvalues of a relativistic particle of arbitrary spin are derived explicitly in this paper, and an outline of the corresponding states, with ordinary and sign spin, is discussed by a procedure indicated in a previous publication.

RESUMEN. Los eigenvalores de una partícula relativista de espín arbitrario se derivan explícitamente en este trabajo. Se indica esquemáticamente la derivación de los correspondientes eigenestados usando los conceptos de espín ordinario y de signo definidos en una publicación anterior.

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As is well known the Dirac equation for a system of n non-inter acting particles can be written as

$$\sum_{u=1}^{n}(\boldsymbol{\alpha}_u \cdot \mathbf{p}_u + \beta_u)\psi = E\psi, \tag{1}$$

in units in which $\hbar = m = c = 1$ and where

$$\beta_u = I \otimes I \otimes \cdots \otimes I \otimes \beta \otimes I \otimes \cdots \otimes I, \tag{2}$$

is a direct product in 4×4 matrices where $n-1$ of them are unity and in the u position we have a β. A similar definition holds for the $\boldsymbol{\alpha}_u$. The $\boldsymbol{\alpha}$, β have the standard 4×4 matrix form

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{3}$$

where $\boldsymbol{\sigma}$ is the vector of the Pauli spin matrices. The validity of (1) is justified [1] by the fact that by squaring, rearranging, squaring again, etc. and using the anticommuting properties of the $\boldsymbol{\alpha}_u$, β_u we can obtain a 2^n degree algebraic equation involving only the E and the $\mathbf{p}_u$ and its 2^n roots turn out to be

$$E = \pm\sqrt{p_1^2+1} \pm \sqrt{p_1^2+1} \pm \cdots \pm \sqrt{p_n^2+1}\,, \tag{4}$$

as the Einstein relation leads us to expect.

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Nothing prevents us considering the case when all the momenta are equal, *i.e.* $\mathbf{p}_u = \mathbf{p}$, $u = 1, 2, \ldots, n$ and in that case we have an equation for a single particle, but as each $\boldsymbol{\alpha}_u$, has spin $\frac{1}{2}$, the presence of n of them indicate that our particle would, in general, have a mixture of spins with values,

$$\frac{n}{2}, \frac{n}{2} - 1, \ldots, \frac{1}{2} \text{ or } 0. \tag{5}$$

The equation we have to deal with has then the form

$$\sum_{u=1}^{n} (\boldsymbol{\alpha}_u \cdot \mathbf{p} + \beta_u)\psi = nE\psi. \tag{6}$$

Note that in (6) the energy E of Eq. (1) is replaced by nE as each of the terms in the left hand side of (6) makes a contribution and we would like to denote the energy as the average of these contributions [1].

Note that the momentum $\mathbf{p}$ is an integral of motion, as it obviously commutes with the operator on the left hand side of (6). We can then replace the operator $\mathbf{p}$ by its constant vector eigenvalue $\mathbf{k}$ and, if we choose our axis x_3 of coordinates in direction of this vector, the equation (6) takes the form

$$\sum_{u=1}^{n} (\alpha_{3u} k + \beta_u)\psi = nE\psi. \tag{7}$$

Now our objective is to find E as a function of k and for this purpose we note that

$$[\alpha_{3u} k + \beta_u, \, \alpha_{3v} + \beta_v] = 0, \quad \text{when} \quad u \neq v \tag{8}$$

while

$$[\alpha_{3u} k + \beta_u]^2 = k^2 + 1, \tag{9}$$

as $(\alpha_{3u})^2$, β_u^2 are unit 4×4 matrices while $\alpha_{3u}\beta_u + \beta_u\alpha_{3u} = 0$.

Thus when squaring Eq. (6), rearranging, squaring again, etc. we are carrying exactly the same procedure that took us from Eq. (1) to the relation (4) for the energy, but where E is replaced by nE and all p_u, $u = 1, 2, \ldots, n$ by k. Thus we have

$$nE = \pm\sqrt{k^2 + 1} \pm \sqrt{k^2 + 1} \pm \cdots \pm \sqrt{k^2 + 1}\,, \tag{10}$$

and all possible values of the energy can be expressed as

$$E_m = [(n - 2m)/n]\sqrt{k^2 + 1}\,, \quad m = 0, 1, \ldots, n. \tag{11}$$

Clearly all the different energies E_m correspond to highly degenerate states that may have the values indicated in Eq. (5) for both the ordinary spin and the new one we introduced in Ref. 1 as sign spin.

To obtain the ordinary and sign spins associated with the values of the energies E_m, we would have to express $\sum_{u=1}^{n} \alpha_{3u}$, $\sum_{u=1}^{n} \beta_u$ in terms of the generators of a U(4) group as is done in Ref. 1, and determine then the matrix of the operator on the left hand of

Eq. (7) with respect to the eigenstates that are basis of irreducible representations of the chain of groups $\mathrm{U}(4) \supset \widehat{\mathrm{U}}(2) \otimes \check{\mathrm{U}}(2)$. By diagonalizing this finite matrix, as the irreps of U(4) are integrals of motion, we shall get, associated with the eigenvalues E_m of (11) the eigenstates of definite ordinary and sign spin that correspond to this energy.

We shall not present this last analysis here as it is a simpler aspect of the one given in Ref. 1, and thus we conclude by stating that the relativistic free particle with arbitrary spin can be fully discussed in the formalism presented here.

REFERENCES

*. As a participant in the 1996 National Meeting of the Sociedad Mexicana de Física, on October 16, I presented the paper entitled "Supermultiplets and relativistic problems: The free particle with arbitrary spin in a magnetic field" written in collaboration with Yu. F. Smirnov. As this paper had then been submitted for publication and, in fact, is already published in Ref. 1, I considered it more appropriate to present another contribution, with the above title, that admits a very simple presentation.

1. M. Moshinsky and Yu. F. Smirnov, *J. Phys. A: Math. Gen.* **29** (1996) 6027.
2. M. Moshinsky and Yu. F. Smirnov, *The Harmonic Oscillator in Modern Physics*, (Hardwood Academic Publishers, New York, 1996), p. 9.