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Aharonov-Bohm potential via spin weight

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Abstract. It is shown that the shifting of the spin weight of the wave function of a particle, expressed in terms of the circular, parabolic or elliptic cylindrical coordinates, is equivalent to introducing the interaction with the field of a thin infinitely long solenoid along the z axis. Using this equivalence, the Schrödinger equation with an Aharonov-Bohm potential is solved in circular cylindrical coordinates.

Resumen. Se muestra que el desplazar el peso de espín de la función de onda de una partícula, expresada en términos de coordenadas cilíndricas circulares, parabólicas o elípticas, es equivalente a introducir la interacción con el campo de un solenoide delgado infinitamente largo colocado sobre el eje z. Usando esta equivalencia, se resuelve la ecuación de Schrödinger con un potencial de Aharonov-Bohm en coordenadas cilíndricas circulares.

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1. Introduction

In a recent paper [1] it has been shown that, up to a gauge transformation, the interaction of a charged particle with the field of a magnetic monopole can be accounted for by simply shifting by q units the spin weight of the spherical components of the particle's wave function, where $q \equiv eg/\hbar c$, e is the electric charge of the particle and g is the magnetic charge of the monopole (see also Ref. 2).

A closely related procedure to reproduce the interaction with the field of a magnetic monopole has been employed in Ref. 3 (see also the references cited therein), where the wave function is expressed in terms of a two-component spinor. The three cartesian coordinates x_i can be represented in the form $x_i = \xi^\dagger \sigma_i \xi$, in terms of the two complex components of a spinor ξ and the Pauli matrices σ_i; in this manner, the wave function and the equations governing it can be written in terms of ξ and, by assuming a dependence on the phase of ξ through an exponential factor, the interaction with a magnetic monopole is reproduced. [The spin weight of a quantity is determined by its behavior under the transformation $\xi \mapsto e^{i\alpha/2}\xi$ (see, *e.g.*, Ref. 4).]

Since the spin weight is also defined in connection with cylindrical coordinates (circular, parabolic and elliptic) [5,6], one can look for the interaction produced by shifting the spin weight of the wave functions expressed in terms of these coordinates. As we shall show below, the interaction generated in this way is that corresponding to a thin infinitely long solenoid along the z axis (Aharanov-Bohm potential). By contrast with the spherical case, where the shift q must be integral or half-integral (which is equivalent to Dirac's quantization condition), in the case considered here, the shift of the spin weight (which is related to the magnetic flux inside the solenoid) can be any real number.

In Sect. 2 we briefly review the relationship between a magnetic monopole field and the spin weight with respect to the spherical basis. Then we consider the effect of shifting the spin weight with respect to a cylindrical basis and, in Sect. 3, the Schrödinger equation for a charged particle in an Aharonov-Bohm potential is solved.

2. Spin weight and electromagnetic interactions

Let $\{\hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_\varphi, \hat{\mathbf{e}}_r\}$ be the orthonormal basis induced by the spherical coordinates. A quantity η has spin weight s if under the rotation about $\hat{\mathbf{e}}_r$ given by $\hat{\mathbf{e}}_\theta + i\hat{\mathbf{e}}_\varphi \mapsto e^{i\alpha}(\hat{\mathbf{e}}_\theta + i\hat{\mathbf{e}}_\varphi)$, transforms as $\eta \mapsto e^{is\alpha}\eta$. The raising and lowering operators $\eth$ and $\bar{\eth}$ are defined by [7]

$$\eth\eta \equiv -(\partial_\theta + i\csc\theta\,\partial_\varphi - s\cot\theta)\eta, \qquad \bar{\eth}\eta \equiv -(\partial_\theta - i\csc\theta\,\partial_\varphi + s\cot\theta)\eta, \tag{1}$$

if η has spin weight s. Then $\eth\eta$ and $\bar{\eth}\eta$ have spin weight $s+1$ and $s-1$, respectively.

Since the interaction of a particle of electric charge e with an electromagnetic field given by the vector potential $\mathbf{A}$ is obtained by replacing the operator $-i\hbar\nabla$ by $-i\hbar\nabla - e\mathbf{A}/c$, the replacement of the spin weight s by $s+q$ in Eqs. (1) amounts to introduce an electromagnetic field given by

$$A_\varphi = -\frac{q\hbar c}{e}\frac{\cot\theta}{r}, \qquad A_\theta = 0 = A_r, \tag{2}$$

in some specific gauge. It is easy to see that the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ generated by the vector potential (2) is

$$\mathbf{B} = \frac{q\hbar c}{e}\frac{\hat{\mathbf{e}}_r}{r^2}, \tag{3}$$

which is the magnetic field produced by a magnetic monopole at the origin of charge $g = q\hbar c/e$; thus, the spin weight shift q is related to the charges through

$$q = \frac{eg}{\hbar c} \tag{4}$$

and, according to Dirac's quantization condition, q can take integral and half-integral values only. As shown in Refs. 1 and 2, given the solution of the Dirac or of the Schrödinger equation, without a magnetic monopole field, in terms of the spin-weighted spherical harmonics ${}_sY_{jm}$ (which reduce to the usual spherical harmonics when $s=0$), the corresponding solution when a magnetic monopole field is added is obtained, essentially, by replacing s by $s+q$.

Now let $\{\hat{\mathbf{e}}_\rho, \hat{\mathbf{e}}_\phi, \hat{\mathbf{e}}_z\}$ be the orthonormal basis induced by the circular cylindrical coordinates ρ, ϕ, z. A quantity η has spin weight s if under the rotation about $\hat{\mathbf{e}}_z$ given by $\hat{\mathbf{e}}_\rho + i\hat{\mathbf{e}}_\phi \mapsto e^{i\alpha}(\hat{\mathbf{e}}_\rho + i\hat{\mathbf{e}}_\phi)$, transforms as $\eta \mapsto e^{is\alpha}\eta$. In this case, the raising and lowering operators $\eth$ and $\bar{\eth}$ are defined by [5]

$$\eth\eta \equiv -\left(\partial_\rho + \frac{i}{\rho}\partial_\phi - \frac{s}{\rho}\right)\eta, \qquad \bar{\eth}\eta \equiv -\left(\partial_\rho - \frac{i}{\rho}\partial_\phi + \frac{s}{\rho}\right)\eta, \tag{5}$$

if η has spin weight s and therefore

$$\bar{\eth}\,\eth\,\eta = \eth\,\bar{\eth}\,\eta = \partial_{\rho\eta}^2 + \frac{1}{\rho}\partial_\rho \eta + \frac{1}{\rho^2}\partial_{\phi\eta}^2 + \frac{2is}{\rho^2}\partial_\phi \eta - \frac{s^2}{\rho^2}\eta. \tag{6}$$

The cylindrical harmonics of spin weight s, ${}_sZ_{\alpha m}(\rho, \phi)$, satisfy the eigenvalue equations [5]

$$\bar{\eth}\,\eth({}_sZ_{\alpha m}) = -\alpha^2\,{}_sZ_{\alpha m}, \qquad -i\partial_\phi({}_sZ_{\alpha m}) = m\,{}_sZ_{\alpha m}, \tag{7}$$

and (for $\alpha \neq 0$) their normalization is given by

$$sZ_{\alpha m}(\rho, \phi) = Z_{m+s}(\alpha\rho)e^{im\phi}, \tag{8}$$

where Z_ν is a Bessel function $[e.g., {}_sJ_{\alpha m}(\rho, \phi) = J_{m+s}(\alpha\rho)e^{im\phi}]$.

The change of s by $s+q$ in Eqs. (5) is equivalent to introduce the interaction with the magnetic field corresponding to the vector potential

$$A_\phi = -\frac{q\hbar c}{e}\frac{1}{\rho}, \qquad A_\rho = 0 = A_z. \quad . \tag{9}$$

From Eq. (9) one finds that $\mathbf{B} = \mathbf{0}$ for $\rho \neq 0$ and that the magnetic flux through any surface crossed once by the z axis is given by

$$F \equiv \int \mathbf{B} \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l} = -\frac{q\hbar c}{e}\oint d\phi = -\frac{qhc}{e}. \tag{10}$$

Therefore, Eq. (9) is a vector potential for the magnetic field of a thin infinitely long solenoid along the z axis containing a flux F given by Eq. (10).

In the case of the parabolic cylindrical or elliptic cylindrical coordinates, u, v, z, a quantity η has spin weight s if under the rotation about $\hat{\mathbf{e}}_z$ given by $\hat{\mathbf{e}}_u + i\hat{\mathbf{e}}_v \mapsto e^{i\alpha}(\hat{\mathbf{e}}_u + i\hat{\mathbf{e}}_v)$, where $\{\hat{\mathbf{e}}_u, \hat{\mathbf{e}}_v, \hat{\mathbf{e}}_z\}$ is the orthonormal basis induced by the coordinates u, v, z, transforms according to $\eta \mapsto e^{is\alpha}\eta$. Now

$$\begin{aligned} \eth\eta &\equiv -\frac{1}{h}\left[\partial_u + i\partial_v - \frac{s}{h}(h_{,u} + ih_{,v})\right]\eta, \\ \bar{\eth}\eta &\equiv -\frac{1}{h}\left[\partial_u - i\partial_v + \frac{s}{h}(h_{,u} - ih_{,v})\right]\eta, \end{aligned} \tag{11}$$

where h is the (common) scale factor of the coordinates u and v [6]. The change of the spin weight s by $s+q$ amounts to introduce the interaction with the vector potential

$$A_u = \frac{q\hbar c}{e}\frac{h_{,v}}{h^2}, \qquad A_v = -\frac{q\hbar c}{e}\frac{h_{,u}}{h^2}, \qquad A_z = 0, \tag{12}$$

where e is the electric charge of the particle, which again corresponds to the field of a thin infinitely long solenoid along the z axis.

3. Solution of the Schrödinger equation with an Aharonov-Bohm potential

The (time-independent) Schrödinger equation for a free particle of mass M, written in circular cylindrical coordinates is

$$-\frac{\hbar^2}{2M}\left(\partial_\rho^{\,2}\psi+\frac{1}{\rho}\partial_\rho\psi+\frac{1}{\rho^2}\partial_\phi^{\,2}\psi+\partial_z^{\,2}\psi\right)=E\psi. \tag{13}$$

Comparing with Eq. (6), we see that Eq. (13) is equivalent to

$$\bar{\eth}\eth\psi+\partial_z^{\,2}\psi=-\frac{2ME}{\hbar^2}\psi, \tag{14}$$

taking into account that ψ is a function of spin weight 0. This equation admits separable solutions of the form

$$\psi={}_0J_{\alpha m}(\rho,\phi)e^{ik_3z}=J_m(\alpha\rho)e^{im\phi}e^{ik_3z} \tag{15}$$

[see Eq. (7)], where m is an integer, and α and k_3 are real numbers such that

$$\alpha^2+k_3^{\,2}=\frac{2ME}{\hbar^2}. \tag{16}$$

According to the discussion in the preceding section, the wave function

$$\psi={}_qJ_{\alpha m}(\rho,\phi)e^{ik_3z}=J_{m+q}(\alpha\rho)e^{im\phi}e^{ik_3z}, \tag{17}$$

is a solution of the Schrödinger equation for a particle of mass M and electric charge e with the Aharonov-Bohm potential

$$A_\phi=\frac{F}{2\pi\rho},\qquad A_\rho=0=A_z, \tag{18}$$

where

$$q=-\frac{eF}{hc}. \tag{19}$$

It must be noticed that in the spherical case things are not so simple because the operators $\eth$ and $\bar{\eth}$ do not commute and, by contrast with Eq. (7), the eigenvalue of $\bar{\eth}\eth$ involves the spin weight, which affects the value of the separation constants. In the present case, α is left unchanged when the spin weight of the cylindrical harmonic in Eq. (15) is shifted and therefore the separation constant k_3 needs not be changed.

The wave function (17) is an eigenfunction of the operators $-i\hbar\partial_\phi$ and $\mathcal{M}_z\equiv[\mathbf{r}\times(-i\hbar\nabla-e\mathbf{A}/c)]_z=-i\hbar\partial_\phi+\hbar q$ [see Eqs. (18) and (19)], with eigenvalues $m\hbar$ and $(m+q)\hbar$, respectively.

4. Final remarks

Since, as in the case of the electromagnetic field, the interaction with a non-Abelian gauge field or with the gravitational field is obtained by replacing the partial derivatives with respect to the space-time coordinates by the corresponding "covariant" derivatives, the results of Refs. 1 and 2 and of this paper suggest that one can easily obtain the interaction with certain field configurations by shifting the spin weight or another weight related with rotations in the "internal" space.

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