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Evolution and entanglement of Fock states

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ABSTRACT. The evolution and entanglement of quantum states are studied for the particular case of two harmonic oscillators coupled through a quadratic Hamiltonian. A general formalism is presented that permits to calculate the elements of the unitary transformation that leads from an initial product state to an entangled superposition of product states. Two examples are worked out in details.

RESUMEN. La evolución y el enredamiento de estados cuánticos se estudia para el caso particular de dos osciladores armónicos acoplados por medio de un hamiltoniano cuadrático. Se presenta un formalismo general que permite calcular los elementos de una transformación unitaria que conduce un estado inicial tipo producto a una superposición enrredada de productos de estados. Se estudian dos casos particulares a modo de ilustración.

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1. INTRODUCTION

Shortly after the publication of the famous Einstein, Podolsky and Rosen paper [1], Schrödinger [2] discussed the paradox presented by these authors and reached the conclusion that non-local effects in quantum mechanics are due to the peculiar phenomenon of entanglement. Entanglement implies that two o more systems that once were in interaction can no longer be described as individual entities after they separate, even if they are no longer causally connected. In Schrödinger's words, entanglement is not " ... *one* but rather *the* characteristic trait of quantum mechanics" [2].

The entanglement of states in quantum mechanics implies a type of correlation between separate systems that can not be explained in terms of classical physics. This fact can be quantitatively established with the use of the Bell inequalities [3].

Bell's argument in its standard version involves a pair of particles with spin emitted by a common source. However, entanglement is not necessarly related to the spin. Indeed, Yurke and Stoler [4] have shown that non classical correlations can also be obtained with a model involving four harmonic oscillators in interaction.

In order to describe the dynamical aspects of an entangled system, we present in this paper a general formalism that permits to describe the evolution of two harmonic oscillators interating through a time dependent Hamiltonian. If the oscillators are initially in a product state, a finite time interaction between them leads to an entangled state;

this final state is a superposition of product states and is related to the initial state by a unitary transformation. As shown in Sect. 2, the coefficients of this transformation can be evaluated quite generally for any quadratic Hamiltonian describing the system under consideration; this can be done using a formalism developed by Dodonov, Malkin and Man'ko [5] in combination with a technique of generating functions. As an application of our formalism, some particular cases are studied in Sect. 3, where we consider the entanglement of two harmonic oscillator states with two possible types of interactions; the first is the interaction proposed by Yurke and Stoler [4] in their study of the Bell's inequalities, and the second corresponds to the case of a particle in a Penning trap [6].

2. THEORY

Consider a system of two harmonic oscillators described by a Hamiltonian of the form:

$$H = \tfrac{1}{2}A_{ij}p_ip_j + B_{ij}p_ix_j + \tfrac{1}{2}C_{ij}x_ix_j\,, \tag{1}$$

where A_{ij}, B_{ij} and C_{ij} are two-by-two time dependent matrices. Following the formalism of Dodonov *et al.* [5], we notice that the system admits constants of motions P and X that are linear combinations of the position and momentum variables, namely:

$$P = \Lambda_1(t)p + \Lambda_2(t)x, \tag{2}$$

$$X = \Lambda_3(t)p + \Lambda_4(t)x, \tag{3}$$

where p and x are column vectors with elements p_i and x_i, and Λ_i are two-by-two matrices.

Defining the four-by-four matrices

$$\Lambda = \begin{pmatrix} \Lambda_1 & \Lambda_2 \\ \Lambda_3 & \Lambda_4 \end{pmatrix}, \tag{4}$$

$$\mathbf{B} = \begin{pmatrix} A_{ij} & B_{ij} \\ B_{ji} & C_{ij} \end{pmatrix}, \tag{5}$$

and the sympletic metric

$$\Upsilon = \begin{pmatrix} \mathbf{0} & I \\ -I & \mathbf{0} \end{pmatrix}, \tag{6}$$

it follows that the classical equations of motion can be written as:

$$\frac{d}{dt}\mathbf{\Lambda} = \mathbf{\Lambda \Upsilon B}, \tag{7}$$

which implies in particular that:

$$\mathbf{\Lambda \Upsilon \Lambda}^T = \mathbf{\Upsilon}. \tag{8}$$

Consider now the Schrödinger equation associated to the quadratic Hamiltonian (1). The Green function, defined quite generally by the relation

$$\Psi(x_1, x_2, t) = \int G(x_1, x_2, y_1, y_2, t)\, \Psi(y_1, y_2, 0)\, dy_1\, dy_2, \tag{9}$$

can be calculated with the formalism of Ref. 5, valid for any quadratic Hamiltonian. It turns out to be:

$$G(x_1, x_2, y_1, y_2, t) = \frac{i}{2\pi} \frac{1}{\sqrt{\det \Lambda_3}} \exp\left[-\frac{i}{2}\left(x^T \Lambda_3^{-1} \Lambda_4 x + y^T \Lambda_1 \Lambda_3^{-1} y - 2x^T \Lambda_3^{-1} y\right)\right], \tag{10}$$

where x and y are column vectors with elements x_i and y_i.

We have now all the mathematical tools necessary to determine the evolution of a quantum state. Consider two non interacting harmonic oscillators, with masses m_i and frequencies ω_i, that are initially in the product state

$$\psi_{ab}^{\text{in}}(x_1, x_2) = \psi_a(x_1)\, \psi_b(x_2), \tag{11}$$

at time $t = 0$. Here ψ_a are the usual Fock states in coordinate representation,

$$\psi_a(x_i) = (\sqrt{\pi} 2^a a!)^{-1/2}\, e^{-\frac{1}{2}\xi_i^2}\, H_a(\xi_i), \tag{12}$$

where $\xi_i = (m_i \omega_i / \hbar)^{1/2} x_i$. If an interaction is switched on at time $t = 0$, the system will evolve afterwards to the new state

$$\Psi_{ab}(x_1, x_2, t) = \sum_{cd} U_{cd,ab}(t)\, \psi_c(x_1)\, \psi_d(x_2), \tag{13}$$

where $U_{cd,ab}(t)$ are elements of a unitary transformation.

Quite generally, Eq. (13) describes an entangled state; our aim now is to calculate the coefficients $U_{ab,cd}$. This can be done by noticing first that, due to the normalization of the Fock states,

$$U_{ab,cd}(t) = \int \psi_a(x_1)\, \psi_b(x_2)\, G(x_1, x_2, y_1, y_2, t)\, \psi_c(y_1)\, \psi_d(y_2)\, dx_1\, dx_2\, dy_1\, dy_2. \tag{14}$$

More explicitely:

$$\begin{aligned} U_{ab,cd} &= \pi^{-1} 2^{-\frac{1}{2}(a+b+c+d)} (a!\, b!\, c!\, d!)^{-\frac{1}{2}} \\ &\times \int \exp\Big\{ -\frac{1}{2\hbar}(x^T \mu^2 x + y^T \mu^2 y) G(x_1, x_2, y_1, y_2, t)\Big\} \\ &\times H_a(\xi_1) H_b(\xi_2) H_c(\eta_1) H_d(\eta_2)\, dx_1\, dx_2\, dy_1\, dy_2, \end{aligned} \tag{15}$$

where x and y are column vectors and μ is the matrix

$$\mu = \begin{pmatrix} (m_1\omega_1)^{1/2} & 0 \\ 0 & (m_2\omega_2)^{1/2} \end{pmatrix}. \tag{16}$$

In order to proceed further, we use the technique of generating funtions. In particular, the Hermite polynomials follow from the standard formula:

$$\exp\left(-\alpha^2 + 2\alpha x\right) = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} H_n(x)\,, \tag{17}$$

and this suggests to define the generating function

$$I(\alpha_1, \alpha_2, \beta_1, \beta_2) = \sum_{a,b,c,d=0}^{\infty} \frac{\alpha_1^a\, \alpha_2^b\, \beta_1^c\, \beta_2^d}{\sqrt{a!\, b!\, c!\, d!}}\, U_{ab,cd}, \tag{18}$$

or more explicitely

$$I(\alpha_1, \alpha_2, \beta_1, \beta_2) = \frac{i}{2\pi^2\hbar\sqrt{\det \Lambda_3}}$$
$$\times \int \exp\Bigg[-\frac{1}{2}\alpha^T\alpha - \frac{1}{2}\beta^T\beta + \sqrt{\frac{2}{\hbar}}\alpha^T\mu x + \sqrt{\frac{2}{\hbar}}\beta^T\mu y - \frac{1}{2\hbar}\left(x^T\mu^2 x + y^T\mu^2 y\right)$$
$$- \frac{i}{2\hbar}\left(x^T\Lambda_3^{-1}\Lambda_4 x + y^T\Lambda_1\Lambda_3^{-1}y - 2x^T\Lambda_3^{-1}y\right)\Bigg] dx_1\, dx_2\, dy_1\, dy_2. \tag{19}$$

Using now the relation

$$\int \exp\left(-x^T M x + 2N^T x\right) dx^n = \frac{\pi^{n/2}}{\sqrt{\det M}} \exp\left(N^T M^{-1} N\right), \tag{20}$$

valid for any $n \times n$ matrix M and n-vector N, we find with some lenghty but straighforward algebra that

$$I\left(\alpha_1, \alpha_2, \beta_1, \beta_2\right) = \frac{2i\hbar}{(m_1 m_2 \omega_1 \omega_2)^{1/2}} \frac{\det \Lambda_3}{\sqrt{\det \mathbf{S}}} \exp\left(-\tfrac{1}{2}\Gamma^T \mathbf{S}^{-1}\mathbf{S}^*\Gamma\right), \tag{21}$$

where we have defined the four-vector $\Gamma^T = (\alpha_1, \alpha_2, \beta_1, \beta_2)$ and the 4×4 matrix

$$\mathbf{S} = \begin{pmatrix} \Lambda_4\mu^{-1} - i\Lambda_3\mu & -\mu^{-1} \\ -\mu^{-1} & \Lambda_1^T\mu^{-1} - i\Lambda_3^T\mu \end{pmatrix}, \tag{22}$$

given in terms of the matrices Λ_i. Equations (21) and (22) are our basic result.

3. Two examples

Equations (21) and (22) permit to calculate the evolution coefficients $U_{ab,cd}$ through a series expansion. Although a general formula can be obtained, the result is too cumbersome to be of any use, and it is more illustrating to consider only some particular examples. This will be done in the following.

3.1. YURKE-STOLER INTERACTION

Let us consider the Hamiltonian used by Yurke and Stoler [4] for an interacting pair of harmonic oscillators; it has the form:

$$H = \tfrac{1}{2}\left(p_1^2 + p_2^2\right) + \tfrac{1}{2}\left(x_1^2 + x_2^2\right) + \kappa\left(p_1 p_2 + x_1 x_2\right) \tag{23}$$

where it is assumed for simplicity that the two oscillators have the same masses and frequencies, both taken to be unity (also $\hbar = 1$ in the following).

The solution of Eq. (7) is

$$\Lambda_1 = \Lambda_4 = \cos\Omega t, \qquad \Lambda_2 = -\Lambda_3 = \sin\Omega t, \tag{24}$$

where

$$\Omega = \begin{pmatrix} 1 & \kappa \\ \kappa & 1 \end{pmatrix}, \tag{25}$$

as can be checked easily.

Thus, the matrix $\mathbf{S}$ defined by Eq. (22) turns out to be in this particular case:

$$\mathbf{S} = \begin{pmatrix} \exp(i\Omega t) & -I \\ -I & \exp(i\Omega t) \end{pmatrix}, \tag{26}$$

where

$$\exp(i\Omega t) = e^{it}\begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix}, \tag{27}$$

with $\theta = \kappa t$.

It follows from these last relations that the generating function takes the form

$$I = e^{-it}\exp\Big\{e^{-it}\left[(\alpha_1\beta_1 + \alpha_2\beta_2)\cos\theta - i(\alpha_1\beta_2 + \alpha_2\beta_1)\sin\theta\right]\Big\}. \tag{28}$$

A simple analysis of this formula reveals that the only non-vanishing coefficients $U_{ab,cd}$ are those with $a + b = c + d$. Furthermore, this same relation (28) can be simplified by noticing that it admits a straightforward Taylor series expansion in the coefficients β_i. The result is

$$U_{ab,cd} = \exp^{-i(E_c+E_d)t}\, V_{ab,cd}, \tag{29}$$

where $E_n = (\frac{1}{2}+n)$ is the energy of a single oscillator and $V_{ab,cd}$ follow from the generating function

$$\sum_{a,b=0}^{\infty} \frac{\alpha_1^a\,\alpha_2^b}{\sqrt{a!\,b!}} V_{ab,cd} = \frac{1}{\sqrt{c!\,d!}}(\alpha_1\cos\theta - i\alpha_2\sin\theta)^c\,(\alpha_2\cos\theta - i\alpha_1\sin\theta)^d. \tag{30}$$

Some of the first few coefficients are:

$$V_{00,00} = 1; \tag{31}$$

$$V_{01,01} = V_{10,10} = \cos\theta,$$

$$V_{10,01} = V_{01,10} = -i\sin\theta; \tag{32}$$

$$V_{02,02} = V_{20,20} = \cos^2\theta,$$

$$V_{11,02} = V_{11,20} = -\sqrt{2}i\sin\theta\cos\theta,$$

$$V_{20,02} = V_{02,20} = -\sin^2\theta; \tag{33}$$

and

$$V_{11,11} = \cos 2\theta,$$

$$V_{02,11} = V_{20,11} = -\sqrt{2}i\sin\theta\cos\theta. \tag{34}$$

3.2. PENNING TRAP

As a second example, we consider an Hamiltonian of the form:

$$H = \tfrac{1}{2}(p_1^2 + p_2^2) + \tfrac{1}{2}(x_1^2 + x_2^2) + \omega(x_1 p_2 - x_2 p_1). \tag{35}$$

This correspond to a particle (with unit mass) in a Penning trap, such that the cyclotron frequency is 2ω and the frequency of oscilation in the z direction [not considered in Eq. (35)] is $\omega_z = \sqrt{2(\omega^2 - 1)}$, (see, *e.g.*, Brown and Gabrielse [6]).

The equations of motion for this Hamiltonian admit the solutions

$$\Lambda_1 = \Lambda_4 = R(\omega t)\cos t\,, \qquad \Lambda_2 = -\Lambda_3 = R(\omega t)\sin t\,, \tag{36}$$

where R is the rotation matrix defined as

$$R(\phi) = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}. \tag{37}$$

The algebraic procedure to follow is the same as in the case of the Yurke-Stoler Hamiltonian. It follows that the generating function is:

$$I = e^{-it}\exp\Big\{e^{-it}\Big[(\alpha_1\beta_1 + \alpha_2\beta_2)\cos\theta + (-\alpha_1\beta_2 + \alpha_2\beta_1)\Big]\sin\theta\Big\}, \tag{38}$$

where now $\theta = \omega t$.

Following exactly the same steps as in the previous subsection, it can be seen that again $U_{ab,cd} = 0$ unless $a + b = c + d$; furthermore, Eq. (38) can also be simplified using a Taylor series expansion in the coefficients β_i. It follows that an equation of the form (29) remains valid, but with the coefficients $V_{ab,cd}$ given now by the generating function

$$\sum_{a,b=0}^{\infty} \frac{\alpha_1^a\,\alpha_2^b}{\sqrt{a!\,b!}} V_{ab,cd} = \frac{1}{\sqrt{c!\,d!}}(\alpha_1\cos\theta + \alpha_2\sin\theta)^c\,(\alpha_2\cos\theta - \alpha_1\sin\theta)^d. \tag{39}$$

Some of the first few coefficients are:

$$V_{00,00} = 1; \tag{40}$$

$$V_{01,01} = V_{10,10} = \cos\theta,$$

$$V_{10,01} = V_{01,10} = \sin\theta; \tag{41}$$

$$V_{02,02} = V_{20,20} = \cos^2\theta,$$

$$V_{11,02} = V_{11,20} = -\sqrt{2}\sin\theta\cos\theta,$$

$$V_{20,02} = V_{02,20} = \sin^2\theta, \tag{42}$$

and

$$V_{11,11} = \cos 2\theta,$$

$$V_{02,11} = V_{20,11} = \sqrt{2}\sin\theta\cos\theta. \tag{43}$$

4. DISCUSSION

The formalism developed in this paper permits to analize the evolution of two harmonic oscillators in entangled states. The elements of the unitary operator defined by Eq. (13) were calculated in the preceding section for two particular cases, and some of these elements given explicitely. In particular, it is easy to check that these elements do satisfy the unitarity condition:

$$\sum_{a,b=0}^{\infty} |U_{ab,cd}|^2 = 1, \tag{44}$$

for all integers a and b. Some of the coefficients given in Eqs. (31) to (33) were obtained by Yurke and Stoler with a different method and for a particular value of the interaction time; we stress that our result is entirely general.

It is also worth mentioning that the formalism of Ref. 5 is not restricted to the two dimensional case considered in the present paper. In fact, it is rather straightforward to generalize formulas such as Eqs. (21) and (22) to any number of dimensions; however, the result is too cumbersome and does not add anyhing fundamental to the discussion presented here.

Quadratic Hamiltonians with time dependent coefficients have attracted much attention in recent years in connection with several problems, such as the behavior of particles in ion traps (see, *e.g.*, Refs. 7 and 8). It is expected that the present formalism may be useful in studying the important effect of entanglement in these systems.

As shown in a separate publication [9], our formalism is useful in the study of model situations in which quantum correlations are involved. A new application of the Bell's inequalities for the entangled states of harmonic oscillators is analized in Ref. 9.

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