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Solution of the Schrödinger equation in the field of a magnetic monopole

L.C. Cortés Cuautli
Departamento de Física
Centro de Investigación y de Estudios Avanzados del IPN
Apartado postal 14-740, 07000 México, D.F., Mexico

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Abstract. The Schrödinger equation for a charged particle in the field of a magnetic monopole and a Coulomb potential is solved making use of the spin-weighted spherical harmonics. It is shown that the separable solutions obtained are eigenfunctions of the z component and of the square of the total angular momentum.

Resumen. Se resuelve la ecuación de Schrödinger para una partícula cargada en el campo de un monopolo magnético y un potencial de Coulomb usando los armónicos esféricos con peso de espín. Se muestra que las soluciones separables que se obtienen son eigenfunciones de la componente z y del cuadrado del momento angular total.

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1. Introduction

The spin-weighted spherical harmonics are very useful in the solution of systems of partial differential equations for vector, tensor, and spinor fields by separation of variables. The first systematic treatment of these functions was given by Newman and Penrose [1], though equivalent sets of functions were previously introduced in various problems of quantum mechanics and general relativity.

In this paper the Schrödinger equation for a particle in a Coulomb field and the field of a magnetic monopole is solved applying the method of separation of variables with the aid of the spin-weighted spherical harmonics. The Schrödinger equation for a particle in the field of a magnetic monopole alone has been solved making use of certain *ad hoc* angular functions, which are equivalent to the spin-weighted spherical harmonics (see, *e.g.*, Refs. 2 and 3).

In Sect. 2 the basic notions about the spin-weighted spherical harmonics are presented. In Sect. 3 the Schrödinger equation for a particle in the combined field of a point charge and a magnetic monopole is solved and it turns out that the interaction with the monopole is taken into account by assigning a nonzero spin weight to the wave function. In Sect. 4 it is shown that the separable solutions thus obtained are eigenfunctions of the z component and of the square of the angular momentum of the particle and the electromagnetic field.

2. SPIN-WEIGHTED SPHERICAL HARMONICS

The spin-weighted spherical harmonics, ${}_sY_{jm}(\theta,\phi)$, are defined, up to a normalization factor, by [1, 4]

$$\begin{aligned} \bar{\eth}\eth({}_sY_{jm}) &= \big[s(s+1) - j(j+1)\big]\ {}_sY_{jm}, \\ -i\frac{\partial}{\partial\phi}\ {}_sY_{jm} &= m\ {}_sY_{jm}. \end{aligned} \tag{1}$$

The operators $\eth$ and $\bar{\eth}$, acting on an arbitrary function η with spin weight s, are given by

$$\eth\eta \equiv -\left(\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\phi} - s\cot\theta\right)\eta, \qquad \bar{\eth}\eta \equiv -\left(\frac{\partial}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial}{\partial\phi} + s\cot\theta\right)\eta. \tag{2}$$

$\eth$ raises the spin weight in one unit, $\bar{\eth}$ lowers the spin weight in one unit, and ${}_sY_{jm}$ has spin weight s. From the definitions (2) it follows that

$$\bar{\eth}\,\eth\,\eta - \eth\,\bar{\eth}\,\eta = 2s\,\eta, \tag{3}$$

where s is the spin weight of η and that

$$\bar{\eth}\eth\eta = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\eta}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2\eta}{\partial\phi^2} + 2is\frac{\cos\theta}{\sin^2\theta}\frac{\partial\eta}{\partial\phi} - \frac{s^2}{\sin^2\theta}\eta + s(s+1)\eta. \tag{4}$$

The functions ${}_sY_{jm}$ are normalized according to

$$\int_0^{2\pi}\int_0^{\pi} {}_s\overline{Y}_{jm}(\theta,\phi)\ {}_sY_{jm}(\theta,\phi)\sin\theta\, d\theta\, d\phi = 1,$$

and the phase of these functions is chosen in such a way that

$$\begin{aligned} \eth({}_sY_{jm}) &= \sqrt{(j-s)(j+s+1)}\ {}_{s+1}Y_{jm}, \\ \bar{\eth}({}_sY_{jm}) &= -\sqrt{(j+s)(j-s+1)}\ {}_{s-1}Y_{jm}, \end{aligned} \tag{5}$$

${}_0Y_{jm}$ are the usual spherical harmonics Y_{jm}. The indices s, j and m of ${}_sY_{jm}$ can take the values

$$\begin{aligned} s &= 0, \pm\tfrac{1}{2}, \pm 1, \dots, \\ j &= |s|, |s|+1, |s|+2, \dots, \\ m &= -j, -j+1, \dots, j. \end{aligned} \tag{6}$$

When s is an integer, j and m are integers and when s is a half-integer, j and m are also half-integers. As a consequence of Eqs. (1), for a given value of s, the functions ${}_sY_{jm}$ are orthogonal, thus

$$\int_0^{2\pi}\int_0^{\pi} {}_s\overline{Y}_{jm}(\theta,\phi)\ {}_sY_{j'm'}(\theta,\phi)\sin\theta\, d\theta\, d\phi = \delta_{jj'}\,\delta_{mm'}. \tag{7}$$

Furthermore, the set of functions ${}_sY_{jm}$, with s fixed, is complete in the sense that any quantity with spin weight s can be expanded in a series of the ${}_sY_{jm}$ [1].

Using Eqs. (1) and (2) one can show that [5] (*cf.* also Ref. 3)

$$ {}_sY_{jm}(\theta,\phi) = C(1-\cos\theta)^{\alpha/2}(1+\cos\theta)^{\beta/2}P_n^{(\alpha,\beta)}(\cos\theta)e^{im\phi}, \tag{8} $$

where $P_n^{(\alpha,\beta)}$ are the Jacobi polynomials,

$$ \alpha = |m+s|, \qquad \beta = |m-s|, \qquad n = j - \tfrac{1}{2}(\alpha+\beta) = j - \max\{|m|,|s|\}, \tag{9} $$

and C is a normalization constant. Owing to the restrictions on the values of the indices j, m and s, n is always an integer.

3. SOLUTION OF THE SCHRÖDINGER EQUATION

The time-independent Schrödinger equation for a particle of mass M and electric charge e in the presence of an electromagnetic field produced by a point charge $-Ze$ and a magnetic monopole g placed at the origin is given by

$$ -\frac{\hbar^2}{2M}\left(\nabla - \frac{ie}{\hbar c}\mathbf{A}\right)^2\Psi + e\varphi\Psi = E\Psi, \tag{10} $$

where the electromagnetic potentials can be taken as

$$ \varphi = -\frac{Ze}{r}, \qquad \mathbf{A} = g\frac{(\mp 1 - \cos\theta)}{r\sin\theta}\hat{\mathbf{e}}_\phi. \tag{11} $$

With the positive sign, the vector potential $\mathbf{A}$ is singular on the negative z axis, while with the negative sign, $\mathbf{A}$ diverges on the positive z axis. Thus, we shall consider both signs in Eqs. (11) in order to find a well-behaved solution of the Schrödinger equation everywhere. As shown in Ref. 3, the solutions corresponding to the two choices of $\mathbf{A}$ can be joined to form a section on a line bundle provided that

$$ \frac{eg}{\hbar c} = \frac{n}{2}, \tag{12} $$

where n is an integer. Condition (12) is precisely the well-known Dirac's quantization condition [6]. It what follows, we will consider the wave function as an ordinary function, without mentioning its relationship with a line bundle.

Making use of the expression for the Laplace operator in spherical coordinates

$$ \nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}, \tag{13} $$

and since the vector potential given in Eq. (11) satisfies

$$ \nabla\cdot\mathbf{A} = 0, \tag{14} $$

the Schrödinger equation (10) takes the form

$$-\frac{\hbar^2}{2M}\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}+\frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}e^{\mp iq\phi}\frac{\partial^2}{\partial\phi^2}e^{\pm iq\phi}\right.\right.$$
$$\left.\left.+\,2iq\frac{\cos\theta}{\sin^2\theta}e^{\mp iq\phi}\frac{\partial}{\partial\phi}e^{\pm iq\phi}-\frac{q^2}{\sin^2\theta}\right)+\frac{q^2}{r^2}\right]\Psi-\frac{Ze^2}{r}\Psi=E\Psi, \quad (15)$$

where we have introduced $q \equiv eg/\hbar c$ which, according to Dirac's quantization condition [Eq. (12)] can only take the values $q = n/2$, $n = 0, \pm 1, \pm 2, \ldots$

Making use of Eq. (4), Eq. (15) can be rewritten as

$$-\frac{\hbar^2}{2M}\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}+\frac{1}{r^2}(\bar{\eth}\eth-q)\right]e^{\pm iq\phi}\Psi-\left(\frac{Ze^2}{r}+E\right)e^{\pm iq\phi}\Psi=0, \quad (16)$$

provided we *assign* a spin weight q to the wave function Ψ. In order to solve Eq. (16), we look for a separable solution of the form

$$\Psi = R(r)e^{\mp iq\phi}\,{}_qY_{jm}(\theta,\phi) \quad (17)$$

with $j = |q|, |q|+1, |q|+2, \ldots$, and $-j \leq m \leq j$ [see Eqs. (6)]. Substituting the solution (17) into Eq. (16), with the aid of Eqs. (1) we obtain the radial equation

$$-\frac{\hbar^2}{2M}\left[\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}R(r)+\frac{1}{r^2}\left[q^2-j(j+1)\right]R(r)\right]-\left(\frac{Ze^2}{r}+E\right)R(r)=0. \quad (18)$$

Thus, the only effect on the radial equation of the presence of the magnetic monopole is to replace the factor $l(l+1)$, where l is the orbital quantum number, by $j(j+1)-q^2$, where, by contrast with the quantum number l, j can take half-integral values. It should be clear that a similar result applies if one considers any central potential in place of the Coulomb potential (*cf.* Refs. 2 and 3). (The meaning of the quantum numbers j and m will be given in the next section.) Hence, the solution of the radial equation (18) can be obtained from that corresponding to the hydrogen atom by simply replacing l by $-\frac{1}{2}+\sqrt{(j+1/2)^2-q^2}$ [which comes from the identification $l(l+1) = j(j+1)-q^2$]. In this manner (assuming $E < 0$) we conclude that (see, *e.g.*, Ref. 7)

$$(\rho) = \rho^{-\frac{1}{2}+\sqrt{(j+\frac{1}{2})^2-q^2}}e^{-\rho/2}L_{n_r}^{2\sqrt{(j+\frac{1}{2})^2-q^2}}(\rho), \quad (19)$$

where

$$\rho \equiv \left(\frac{8M|E|}{\hbar^2}\right)^{1/2} r \quad (20)$$

and L_n^p denotes the associated Laguerre polynomials (the subscript n corresponds to the degree of the polynomial L_n^p). The energy eigenvalues are given by

$$E = -\frac{MZ^2e^4}{2\hbar^2}\left[n_r+\tfrac{1}{2}+\sqrt{(j+\tfrac{1}{2})^2-q^2}\right]^{-2}, \quad (21)$$

with $n_r = 0, 1, 2, \ldots$. Thus, by contrast with the hydrogen atom, the degeneracy of each energy level is $2j+1$, since m does not enter into Eq. (21). In the case where q vanishes, Eq. (21) reduces to the well-known expression for the energy eigenvalues of the hydrogen atom, identifying $n_r + j + 1$ with the principal quantum number n and j with l.

4. Characterization of the separable solutions

The quantity $\mathbf{r} \times (M\mathbf{v}) - \frac{eg}{c}\frac{\mathbf{r}}{r}$ is a constant of the motion for a particle of mass M and electric charge e in a central force field superposed to the field of a magnetic monopole g at the origin; the term $-\frac{eg}{c}\frac{\mathbf{r}}{r}$ is the angular momentum of the electromagnetic field produced by the charges e and g (see, *e.g.*, Ref. 8). Taking into account the relation $\mathbf{p} = M\mathbf{v} + e\mathbf{A}/c$, the operator

$$\mathbf{J} = \frac{\hbar}{i}\mathbf{r} \times \nabla - \frac{e}{c}\mathbf{r} \times \mathbf{A} - \frac{eg}{c}\frac{\mathbf{r}}{r} \tag{22}$$

must commute with the Hamiltonian (*cf.* Ref. [3] and the references cited therein). In fact, making use of Eq. (11), one finds that

$$\begin{aligned} J_+ &\equiv J_1 + iJ_2 = \hbar e^{i\phi} e^{\mp iq\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi} - \frac{q}{\sin\theta}\right)e^{\pm iq\phi}, \\ J_- &\equiv J_1 - iJ_2 = \hbar e^{-i\phi} e^{\mp iq\phi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi} - \frac{q}{\sin\theta}\right)e^{\pm iq\phi}, \\ J_3 &= -i\hbar e^{\mp iq\phi}\frac{\partial}{\partial\phi}e^{\pm iq\phi}. \end{aligned} \tag{23}$$

These operators commute with $e^{\mp iq\phi}\bar{\eth}\eth e^{\pm iq\phi}$ and satisfy the commutation relations $[J_i, J_j] = i\hbar\varepsilon_{ijk}J_k$ (see, *e.g.*, Ref. [4]).

Similarly, it can be shown that the square of the operator $\mathbf{J}$ can be expressed as

$$J^2 = e^{\mp iq\phi}\left(L^2 - 2iq\hbar^2\frac{\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\phi} + \frac{q^2\hbar^2}{\sin^2\theta}\right)e^{\pm iq\phi}, \tag{24}$$

where $\mathbf{L} \equiv -i\hbar\mathbf{r} \times \nabla$. Hence, by virtue of Eq. (4), J^2 can also be written as

$$J^2 = e^{\mp iq\phi}\hbar^2\left[q(q+1) - \bar{\eth}\eth\right]e^{\pm iq\phi}, \tag{25}$$

assuming that this operator acts on functions with spin weight q.

Then, from Eqs. (1), (23) and (25), it follows that for the separable solution (17)

$$J^2\Psi = j(j+1)\hbar^2\Psi, \qquad J_3\Psi = m\hbar\Psi, \tag{26}$$

which means that the separable solution (17) is an eigenstate of the z component and of the square of the *total* angular momentum (which includes the angular momentum of the electromagnetic field) with eigenvalues $m\hbar$ and $j(j+1)\hbar^2$, respectively, which explains why j can take half-integral values.

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