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The Lagrangian for a causal curve

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ABSTRACT. In this paper the inverse problem of obtaining Lagrangians and Hamiltonians from a given a family of curves (which satisfy their classical equations of motion) with its initial conditions is considered. The application of this method to the damped harmonic oscillator is also considered and two known Hamiltonians are rederived.

RESUMEN. En este artículo se considera el problema inverso de obtener ecuaciones de evolución, lagrangianos y hamiltonianos a partir de una trayectoria dada con sus condiciones iniciales. La aplicación de éste método al oscilador armónico amortiguado es también considerado y se encuentran dos nuevos hamiltonianos.

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1. INTRODUCTION

For a long time, there has been an interest in the inverse problem for the Lagrangian method in classical mechanics: obtaining of Lagrangians and Hamiltonians from Newton's equations of motion. As is well known, the knowledge of the Lagrangian function for a given system is useful for the integration of the equations of motion, for the obtaining of conservation laws from its invariance properties, and, from the Hamiltonian derived from it, for the quantization of classical systems. First investigations of this question are due to Helmholtz [1] who studied the problem of given the set of equations

$$G_i(q, \dot{q}, \ddot{q}) = 0, \qquad i = 1, 2, \ldots, n,$$

under what conditions does there exist a function $L(q, \dot{q}, t)$ such that

$$G_i = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i},$$

i.e., that $G_i = 0$ are the Euler-Lagrange equations of a variational principle of the form $\delta \int L(q, \dot{q}, t)dt = 0$? Other authors have also made contributions to the answer to this question [2–6].

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The damped harmonic oscillator has playing an esscential role in order to understand the mentioned inverse problem. In particular, the works of Havas [6], Caldirola [7] and Kanai [8] are remarkable examples of it. Very recently, the oscillator with damping has been of special interest due to its relation to squeezed states in quantum mechanics [9].

In this paper, the inverse problem of obtaining Lagrangians and Hamiltonians from a given trajectory (which satisfy their classical equations of motion) with its initial conditions is considered. In Sect. 2, we introduce the concept of a *causal curve* which is a trajectory given in terms of its initial conditions. In Sect. 3, the relationship between the charts $(\dot{q}, q; t)$ and $(\dot{q}_0, q_0; t)$ is analyzed. In Sect. 4, embedding the causal curves into the Havas formalism [6], a partial differential equation for this causal curve is obtained and its relationship with the Lagrangian for the system is given. The example of the damped harmonic oscillator is considered in Sect. 5. Here using our results of Sect. 4, for a particular choice of certain functions, we rederive two versions of the damped oscillator Lagrangians (and their associated Hamiltonians). The first one appears familiar from [7–9] and the second one is shown to be equivalent to the Havas Lagrangian [6]. Finally, some concluding remarks are made.

2. CAUSAL CURVES

Consider the function of three variables $(\dot{q}_0, q_0, t) \in \Re^3$,

$$Q = Q(\dot{q}_0, q_0, t) \in \Re, \tag{1}$$

(of course $\dot{q}_0, q_0$ and t are independent coordinates) which, for simplicity is assumed to be smooth. Then the curve

$$q = q(t) := Q(\dot{q}_0, q_0, t), \qquad \dot{q}_0 = \text{const.}, \qquad q_0 = \text{const.}, \tag{2}$$

is called a *causal curve* if and only if the function Q comply with two properties: *(i)* the equations

$$q = Q(\dot{q}_0, q_0, t), \qquad \dot{q} = Q_{(t)}(\dot{q}_0, q_0, t), \tag{3}$$

are invertible with respect to $\dot{q}_0$ and q_0 (the subscript (t) denotes partial derivative with respect to t and $\dot{q}_0 = \text{const.}$, $q_0 = \text{const.}$); the condition for this is

$$J := Q_{(t)\dot{q}_0} Q_{q_0} - Q_{(t)q_0} Q_{\dot{q}_0} \neq 0, \tag{4}$$

and, *(ii)*

$$q_0 \equiv Q(\dot{q}_0, q_0, 0), \qquad \dot{q}_0 \equiv Q_{(t)}(\dot{q}_0, q_0, 0). \tag{5}$$

Granted the validity of (4), Eqs. (3) can be inverted in the form of

$$\dot{q}_0 = R(\dot{q}, q, t), \qquad q_0 = S(\dot{q}, q, t). \tag{6}$$

Notice that because of (5), the Jacobian defined in (4) as $J = J(\dot{q}_0, q_0, t)$ at $t = 0$ assumes the value

$$J(\dot{q}_0, q_0, 0) = 1. \tag{7}$$

The second derivative with respect to t of the causal curve,

$$\ddot{q} = Q_{(tt)}(\dot{q}_0, q_0, t), \tag{8}$$

appears in the ordinary differential equation that q obeys,

$$\ddot{q} = F(\dot{q}, q, t), \tag{9}$$

where

$$F = F(\dot{q}, q, t) := Q_{(tt)}(\dot{q}_0, q_0, t)\big|_{\dot{q}_0 = R,\, q_0 = S} . \tag{10}$$

Inversely, given a function $F(\dot{q}, q, t)$ "sufficiently smooth" such that existence and uniqueness theorems do apply—in terms of the initial conditions and via fixed point theorem—, the second order differential equation (9) determines a causal curve.

For our purposes, its more convenient to consider the concept of causal curve as more *fundamental* than Eq. (9), eliminating in this way possible inconveniences due to singularities of the ordinary differential equation (9). In addition, the concept of causal curve, as related to the uniquely determined evolution in time of the initial data, is *per se* of basic interest from the physical point of view.

3. THE CHART $(\dot{q}, q, t)$

The formulae

$$\dot{q} = Q_{(t)}(\dot{q}_0, q_0, t), \qquad q = Q(\dot{q}_0, q_0, t), \qquad t = t, \tag{11}$$

can be considered as the coordinate transformation which takes us from the chart $(\dot{q}_0, q_0, t)$ to the equivalent chart $(\dot{q}, q, t)$. Indeed, we have

$$d\dot{q}\, dq\, dt = J\; d\dot{q}_0\, dq_0\, dt \neq 0. \tag{12}$$

It is convenient to elaborate the relationships between the partial derivatives with respect to the chart $(\dot{q}_0, q_0, t)$, denoted as

$$(\partial_{\dot{q}_0}, \partial_{q_0}, \partial_{(t)}), \tag{13}$$

and the partial derivatives with respect to the chart $(\dot{q}, q, t)$ denoted by

$$(\partial_{\dot{q}}, \partial_q, \partial_t) . \tag{14}$$

After some algebra, we obtain

$$\begin{aligned}
\partial_q &= \frac{1}{J}\Big(Q_{(t)\dot{q}_0}\partial_{q_0} - Q_{(t)q_0}\partial_{\dot{q}_0}\Big),\\
\partial_{\dot{q}} &= -\frac{1}{J}\Big(Q_{\dot{q}_0}\partial_{q_0} - Q_{q_0}\partial_{\dot{q}_0}\Big),\\
\partial_t &= \frac{1}{J}\Big(Q_{(t)}Q_{(t)q_0} - Q_{q_0}Q_{(tt)}\Big)\partial_{\dot{q}_0} - \frac{1}{J}\Big(Q_{(t)}Q_{(t)\dot{q}_0} - Q_{\dot{q}_0}Q_{(tt)}\Big)\partial_{q_0} + \partial_{(t)},
\end{aligned} \tag{15}$$

the right hand side given explicitly in terms of the objects referred to the chart $(\dot{q}_0, q_0, t)$.

We notice that according to the definition of J given in Eq. (4),

$$J = -\Big(Q_{\dot{q}_0}\partial_{q_0} - Q_{q_0}\partial_{\dot{q}_0}\Big)Q_{(t)}. \tag{16}$$

As it should be, according to the second line of Eqs. (15), the above equality leads to the identity

$$1 = -\frac{1}{J}\Big(Q_{\dot{q}}\partial_{q_0} - Q_{q_0}\partial_{\dot{q}_0}\Big)Q_{(t)} = \partial_{\dot{q}}\dot{q} \equiv 1, \tag{17}$$

in the chart $(\dot{q}, q, t)$.

Consider now the quantity $F_{\dot{q}}$ expressed in the chart $(\dot{q}_0, q_0, t)$,

$$\begin{aligned}
F_{\dot{q}} &= -\frac{1}{J}\Big(Q_{\dot{q}_0}\partial_{q_0} - Q_{q_0}\partial_{\dot{q}_0}\Big)F\\
&= -\frac{1}{J}\Big(Q_{\dot{q}_0}\partial_{q_0} - Q_{q_0}\partial_{\dot{q}_0}\Big)Q_{(tt)}\\
&= -\frac{1}{J}\partial_{(t)}\Big(Q_{\dot{q}_0}Q_{(t)q_0} - Q_{q_0}Q_{(t)\dot{q}_0}\Big)\\
&= \frac{1}{J}\partial_{(t)}J = \partial_{(t)}\ln J.
\end{aligned} \tag{18}$$

Among other things, from this identity it follows that if the causal curve is such that the corresponding F is independent of $\dot{q}$, $F = F(q,t)$, then $\partial_{(t)}J = 0$, and then $J(\dot{q}_0, q_0, t) = 1$, according to Eq. (7).

4. Embedding of the causal curve into the canonical formalism

After the introduction of the causal curve, we now ask ourselves the question: is the causal curve embedded into the canonical formalism? *i.e.*, can the Lagrange function $L = L(\dot{q}, q, t)$ be constructed such that the corresponding Euler-Lagrange equations

$$\dot{(L_{\dot{q}})} - L_q = 0, \tag{19}$$

i.e.,

$$\ddot{q}L_{\dot{q}\dot{q}} + \dot{q}L_{\dot{q}q} + L_{\dot{q}t} - L_q = 0, \tag{20}$$

are satisfied *only* by a *fixed* causal curve $q = q(t) = Q(\dot{q}_0, q_0, t)$ with arbitrary constants $\dot{q}_0$ and q_0?

The problem as stated above, is a variant of the classical inverse problem of the variational calculus whose relevance in physics was emphatically stated in publications by Peter Havas [6].

Following the work of Havas, if $q = Q(\dot{q}_0, q_0, t)$, which satisfies $\ddot{q} = F(\dot{q}, q, t)$, also satisfies Eq. (20) then any $L(\dot{q}, q, t)$ which leads to this *unique* solution, must satisfy

$$F(\dot{q}, q, t)L_{\dot{q}\dot{q}} + \dot{q}L_{\dot{q}q} + L_{\dot{q}t} - L_q = 0, \tag{21}$$

which can be considered as a linear partial differential equation for the searched $L(\dot{q}, q, t)$ in the chart $(\dot{q}, q, t)$. Equation (21) is equivalent to Eq. (9) if and only if

$$K = K(\dot{q}, q, t) := L_{\dot{q}\dot{q}} \neq 0. \tag{22}$$

We will focus now in the determination of the most general solution to the partial differential equation (21) under the condition (22).

Acting with $\partial_{\dot{q}}$ on Eq. (21) we can establish the partial differential equation for K,

$$\partial_{\dot{q}}(FK) + \dot{q}K_q + K_t = 0. \tag{23}$$

4.1. Reduction to quadratures

Suppose that the solution $K = K(\dot{q}, q, t)$ to Eq. (23) is known, then

$$L_{\dot{q}} = \int_0^{\dot{q}} d\dot{q}'\, K(\dot{q}', q, t) + M_q(q, t), \tag{24}$$

which corresponds to the first quadrature of $L_{\dot{q}\dot{q}} = K$. Without any loss of generality, the lower limit of the integral can be selected as 0, and the arbitrary integration constant depending only on q and t can be represented as the derivative $M_q(q, t)$. The second quadrature of Eq. (24) is given by

$$L = \int_0^{\dot{q}} d\dot{q}'' \int_0^{\dot{q}''} d\dot{q}'\, K(\dot{q}', q, t) + \dot{q}M_q(q, t) + M_t + \int_0^q dq'\, N(q', t), \tag{25}$$

because the "integration constant" dependent on q and t surely can be always represented in the form of $M_t(q, t) + \int_0^q dq' N(q', t)$. Now, the change of the order of integration brings Eq. (25) to

$$L = \int_0^{\dot{q}} d\dot{q}'(\dot{q} - \dot{q}')\, K(\dot{q}', q, t) + \int_0^q dq'\, N(q', t) + \dot{q}M_q(q, t) + M_t(q, t). \tag{26}$$

Now, the substitution of L from Eq. (26) into Eq. (21) results in

$$F(\dot{q}, q, t)K(\dot{q}, q, t) + \int_0^{\dot{q}} d\dot{q}'\, [\dot{q}'K_q(\dot{q}', q, t) + K_t(\dot{q}', q, t)] - N(q, t) = 0. \tag{27}$$

However, K fulfills Eq. (23) and hence

$$\dot{q}' K_q(\dot{q}', q, t) + K_t(\dot{q}', q, t) = -\partial_{\dot{q}'} F(\dot{q}', q, t) K(\dot{q}', q, t). \tag{28}$$

This used in Eq. (27) yields the relationship

$$N(q, t) = F(0, q, t) K(0, q, t). \tag{29}$$

Consequently, the most general solution to Eq. (21) has the form of

$$L = \int_0^{\dot{q}} d\dot{q}' \,(\dot{q} - \dot{q}') K(\dot{q}', q, t) + \int_0^{q} dq' \, F(0, q', t) K(0, q', t) + \dot{q} M_q(q, t) + M_t(q, t), \tag{30}$$

where the two-variable function $M(q, t)$ is arbitrary, and the function $K(\dot{q}, q, t) \neq 0$ is the most general solution to Eq. (23). Note that, taken along $\dot{q} = (d/dt)q(t)$ and $q = q(t)$, the contribution from the second line of Eq. (30) amounts to $dM[q(t), t]/dt$, *i.e.*, to the standard ambiguity of the integrand in the action principle $\delta \int_0^t dt \, L(\dot{q}, q, t) = 0$.

Now, considering Eq. (23), it is legitimate to employ the coordinate transformation (11) and search for the solution $K = K(\dot{q}_0, q_0, t)$ in the chart $(\dot{q}_0, q_0, t)$. Replacing in Eq. (23) F by $Q_{(tt)}$, $\dot{q}$ by $Q_{(t)}$ and $\partial_{\dot{q}}$, ∂_q, ∂_t by the right hand sides of Eqs. (15), we have

$$\begin{aligned} &-\frac{1}{J}\Big(Q_{\dot{q}_0}\partial_{q_0} - Q_{q_0}\partial_{\dot{q}_0}\Big) Q_{(tt)} K + \frac{Q_{(t)}}{J}\Big(Q_{(t)\dot{q}_0}\partial_{q_0} - Q_{(t)q_0}\partial_{\dot{q}_0}\Big) K \\ &\quad + \frac{Q_{(t)}}{J}\Big(Q_{(t)q_0}\partial_{\dot{q}_0} - Q_{(t)\dot{q}_0}\partial_{q_0}\Big) K - \frac{Q_{(tt)}}{J}\Big(Q_{q_0}\partial_{\dot{q}_0} - Q_{\dot{q}_0}\partial_{q_0}\Big) K + \partial_{(t)} K = 0. \end{aligned} \tag{31}$$

After some cancellations, we find

$$\partial_{(t)} K - \frac{1}{J}\Big[(Q_{\dot{q}_0}\partial_{q_0} - Q_{q_0}\partial_{\dot{q}_0}) Q_{(tt)}\Big] K = 0. \tag{32}$$

On the other hand, acting with $\partial_{(t)}$ on Eq. (16) one obtains

$$\partial_{(t)} J = -(Q_{\dot{q}_0}\partial_{q_0} - Q_{q_0}\partial_{\dot{q}_0}) Q_{(tt)} \equiv \partial_{\dot{q}} F, \tag{33}$$

and then Eq. (32) reduces to

$$\partial_{(t)} K + \frac{\partial_{(t)} J}{J} K = 0, \tag{34}$$

or

$$\partial_{(t)} JK = 0. \tag{35}$$

Therefore, the solution to Eq. (23), expressed in the chart $(\dot{q}_0, q_0, t)$, is given by

$$K = K(\dot{q}_0, q_0, t) = \frac{1}{J} \mathcal{P}(\dot{q}_0, q_0), \tag{36}$$

where $\mathcal{P}(\dot{q}_0, q_0) \neq 0$ is an *arbitrary* function of two variables. The factor J^{-1} is determined uniquely by the causal curve, *i.e.*, in terms of the function $Q(\dot{q}_0, q_0, t)$ according to the definition (4).

The function K, expressed in the chart $(\dot{q}, q, t)$, is obtained by substitution of $\dot{q}_0$ and q_0 from Eq. (6) into Eq. (36),

$$K(\dot{q}, q, t) = \left(\frac{1}{J}\mathcal{P}(\dot{q}_0, q_0)\right)_{\dot{q}_0 = R(\dot{q},q,t),\, q_0 = S(\dot{q},q,t)} . \tag{37}$$

Corollary

In cases where the causal curve is such that the dependence of $F(\dot{q}, q, t)$ on $\dot{q}$ is simple, in order to determine $J(\dot{q}_0, q_0, t)$ it can be convenient to employ the identity $F_{\dot{q}} = \partial_{(t)} \ln J$ [see Eq. (18)] in the form of

$$F_{\dot{q}}\Big(Q_{(t)}(\dot{q}_0, q_0, t), Q(\dot{q}_0, q_0, t), t\Big) = \partial_{(t)} \ln J. \tag{38}$$

The quadrature of this, consistently with the initial condition for J, Eq. (7), amounts to

$$J = \exp\left[\int_0^t dt\, F_{\dot{q}}\Big(Q_{(t)}(\dot{q}_0, q_0, t), Q(\dot{q}_0, q_0, t), t\Big)\right]. \tag{39}$$

5. THE EXAMPLE OF THE DAMPED HARMONIC OSCILLATOR

The damped harmonic oscillator is of special interest due to its relation to squeezed states [7]. In order to illustrate how our general construction of $L(\dot{q}, q, t)$ works in practice, consider the causal curve defined by

$$q = Q := e^{-\beta t}\left[q_0 \cos\omega t + (\dot{q}_0 + \beta q_0)\frac{\sin\omega t}{\omega}\right], \tag{40}$$

where $\beta \geq 0$, $\omega \geq 0$ are two constants and the case when $\beta = 0$ or $\omega = 0$ is obtained via a limiting process. The time derivative is given by

$$\dot{q} = Q_{(t)} = e^{-\beta t}\left\{\dot{q}_0 \cos\omega t - \left[\beta\dot{q}_0 + \left(\beta^2 + \omega^2\right) q_0\right]\frac{\sin\omega t}{\omega}\right\}. \tag{41}$$

We first verify that Eq. (40) is indeed a causal curve. From Eq. (4) we find that $J = \exp(-2\beta t) \neq 0$ and then the conditions (5) are valid.

By solving Eqs. (40) and (41) for $\dot{q}_0$ and q_0 we obtain the functions R and S which appear in Eq. (6),

$$\begin{aligned} \dot{q}_0 = R &:= e^{\beta t}\left\{\dot{q}\cos\omega t + \left[\beta\dot{q} + \left(\beta^2 + \omega^2\right) q\right]\frac{\sin\omega t}{\omega}\right\}, \\ q_0 = S &:= e^{\beta t}\left[q\cos\omega t - (\dot{q} + \beta q)\frac{\sin\omega t}{\omega}\right]. \end{aligned} \tag{42}$$

Next, with Q given by Eq. (40), one easily finds that the function $F(\dot{q}, q, t)$ defined in Eq. (10) becomes

$$F = -2\beta\dot{q} - \left(\beta^2 + \omega^2\right) q. \tag{43}$$

Therefore, our causal curve $q = q(t) = Q(\dot{q}_0, q_0, t)$ with constants $\dot{q}_0$ and q_0 satisfies the equation

$$\ddot{q} + 2\beta\dot{q} + \left(\beta^2 + \omega^2\right) q = 0, \tag{44}$$

which is precisely the equation of the damped harmonic oscillator—with unit mass—written in the most convenient parametrization. Inversely, one easily sees that the $q = Q$ defined in Eq. (40) is the most general solution to Eq. (44) as determined by the arbitrary initial conditions.

In order to obtain the Lagrangian which corresponds to Eq. (44), we first observe that

$$K(\dot{q}_0, q_0, t) = e^{2\beta t}\mathcal{P}(\dot{q}_0, q_0), \tag{45}$$

with $\mathcal{P}(\dot{q}_0, q_0) \neq 0$, being an arbitrary function of two variables, therefore K understood as a function of $(\dot{q}, q, t)$ is

$$K = e^{2\beta t}\mathcal{P}\Big(e^{\beta t}\Big[\dot{q}\cos\omega t + \Big[\beta\dot{q} + \left(\beta^2 + \omega^2\right)q\Big]\frac{\sin\omega t}{\omega}\Big], \\ e^{\beta t}\Big(q\cos\omega t - (\dot{q} + \beta q)\frac{\sin\omega t}{\omega}\Big)\Big). \tag{46}$$

Then

$$K(0, q, t)\, F(0, q, t) \\ = -\left(\beta^2 + \omega^2\right)qe^{2\beta t}\mathcal{P}\Big[e^{\beta t}\left(\beta^2 + \omega^2\right)q\frac{\sin\omega t}{\omega}, e^{\beta t}\Big(\cos\omega t - \beta\frac{\sin\omega t}{\omega}\Big)q\Big]. \tag{47}$$

By substitution of Eqs. (46) and (47) into Eq. (30) we end up with an effective Lagrangian for the damped harmonic oscillator, whose structure depends *linearly* on the arbitrary function of two variables $\mathcal{P}(\dot{q}_0, q_0) \neq 0$. We will now investigate some cases of interest which correspond to various possible choices for the function $\mathcal{P}(\dot{q}_0, q_0)$.

The simplest case is when $\mathcal{P} = 1$, then

$$K = e^{2\beta t}, \\ K(0, q, t)\, F(0, q, t) = -\left(\beta^2 + \omega^2\right) qe^{2\beta t}. \tag{48}$$

The substitution of this into Eq. (30) leads to the Lagrangian

$$L = e^{2\beta t}\left[\int_0^{\dot{q}} d\dot{q}'\,(\dot{q} - \dot{q}') - \left(\beta^2 + \omega^2\right)\int_0^q dq' q'\right] = \tfrac{1}{2}e^{2\beta t}\Big[\dot{q}^2 - \left(\beta^2 + \omega^2\right)q^2\Big]. \tag{49}$$

This simple time dependent Lagrangian for the oscillator with damping, was considered by researchers like Caldirola [7], Kanai [8], etc. [9]. It has the merit that in the limit $\beta \to 0$ it reduces to the standard Lagrangian of the harmonic oscillator, and then in the limit $\omega \to 0$ to the standard Lagrangian for the free particle. With $p = L_{\dot{q}}$, and $H = p\dot{q} - L$, the corresponding Hamiltonian is

$$H = \tfrac{1}{2}\Big[e^{-2\beta t}p^2 + \Big(\beta^2 + \omega^2\Big)e^{2\beta t}q^2\Big]. \tag{50}$$

It is possible, however, with an alternative choice for $\mathcal{P}(\dot{q}_0, q_0)$, to construct a stationary Lagrangian $L(\dot{q}, q)$ independent on t for the oscillator with damping. We will derive a Lagrangian of this type first via a technique which does not employ the general result (30) accompanied by (37).

With the specific F from (43), which is independent on t, one can look for L from Eq. (21), with $L_t = 0$, confronting thus the two dimensional problem

$$-\Big[2\beta\dot{q} + \Big(\beta^2 + \omega^2\Big)q\Big]L_{\dot{q}\dot{q}} + \dot{q}L_{\dot{q}q} - L_q = 0. \tag{51}$$

Let now

$$\mathcal{H} := \mathcal{H}(\dot{q}, q) := \dot{q}L_{\dot{q}} - L \equiv \dot{q}^2\left(\frac{1}{\dot{q}}L\right)_{\dot{q}}, \tag{52}$$

then

$$\dot{q}L_{\dot{q}\dot{q}} = \mathcal{H}_{\dot{q}}. \tag{53}$$

One then easily sees that Eq. (51) multiplied by $\dot{q}$ reduces to

$$-\Big[2\beta\dot{q} + \Big(\beta^2 + \omega^2\Big)q\Big]\mathcal{H}_{\dot{q}} + \dot{q}\mathcal{H}_q = 0. \tag{54}$$

The Monge characteristic band for this partial differential equation is

$$-\frac{d\dot{q}}{2\beta\dot{q} + (\beta^2 + \omega^2)q} = \frac{dq}{\dot{q}}, \tag{55}$$

which is an equation of the Abel integrable type. Indeed, the substitution $\dot{q} = sq$ reduces (55) to

$$\frac{dq}{q} + \frac{s\,ds}{(s + \beta)^2 + \omega^2} = 0, \tag{56}$$

or

$$d\Big\{[\dot{q} + (\beta + i\omega)q]^{(1+\beta/i\omega)/2}\,[\dot{q} + (\beta - i\omega)q]^{(1-\beta/i\omega)/2}\Big\} = 0. \tag{57}$$

Consequently, with the real and positive

$$h := \tfrac{1}{2}[\dot{q} + (\beta + i\omega)q]^{(1+\beta/i\omega)}\,[\dot{q} + (\beta - i\omega)q]^{(1-\beta/i\omega)} \geq 0, \tag{58}$$

being constant along the characteristic band Eq. (55), the most general solution to Eq. (54) has the form of

$$\mathcal{H} = \mathcal{H}(h),$$

where $\mathcal{H}(h)$ is a real *arbitrary* function of one variable. Note that we have normalized h in such a manner that $\lim_{\beta\to 0} h = (\dot{q}^2 + \omega^2 q^2)/2$, which is the standard Hamiltonian for the harmonic oscillator. With $\mathcal{H}(\dot{q}, q)$ assumed to be known, the definition of $\mathcal{H}$ can be integrated for L in the form of [see Eq. (52)]

$$L = \dot{q}\int_0^{\dot{q}} \frac{d\dot{q}'}{\dot{q}'^{\,2}}\,\mathcal{H}(\dot{q}', q) + \dot{q}\, M_q(q), \tag{59}$$

with the arbitrary $M(q)$ being irrelevant from the point of view of the action principle. One easily verifies that the so constructed L satisfies Eq. (51)—for every $M(q)$—and that the crucial quantity $K = L_{\dot{q}\dot{q}}$ becomes [see Eq. (53)]

$$K = \frac{1}{\dot{q}}\,\mathcal{H}_{\dot{q}} \tag{60}$$

With the help of Eqs. (40) and (41), we now observe that

$$z := \dot{q} + (\beta + i\omega)q = [\dot{q}_0 + (\beta + i\omega)q_0]\, e^{(-\beta + i\omega)t}. \tag{61}$$

It follows that the argument function h can be expressed in terms of $\dot{q}_0$, q_0 and t in the form

$$h = \tfrac{1}{2}[\dot{q}_0 + (\beta + i\omega)q_0]^{1+\beta/i\omega}\,[\dot{q}_0 + (\beta - i\omega)q_0]^{1-\beta/i\omega}, \tag{62}$$

i.e., as dependent *only* on $\dot{q}_0$ and q_0, with the complex exponents in Eq. (58) doing the trick.

At this point we can easily determine the form of the specific function $\mathcal{P}(\dot{q}_0, q_0)$ which leads to the stationary Lagrangian (59). Indeed, spelling out Eq. (60) in terms of the chart $(\dot{q}_0, q_0, t)$ [see Eq. (36)]

$$\frac{1}{J}\mathcal{P}(\dot{q}_0, q_0) = -\frac{1}{JQ_{(t)}}\Big(Q_{\dot{q}_0}\partial_{q_0} - Q_{q_0}\partial_{\dot{q}_0}\Big)\mathcal{H}(h), \tag{63}$$

where now h is to be interpreted as given by Eq. (62). Multiplying this by $J \neq 0$, we thus have

$$\mathcal{P}(\dot{q}_0, q_0) = \frac{\mathcal{H}_h}{Q_{(t)}}(Q_{q_0}\partial_{\dot{q}_0} - Q_{\dot{q}_0}\partial_{q_0})h. \tag{64}$$

On the other hand, using the definition (40) of Q, we have

$$\frac{1}{Q_{(t)}}(Q_{q_0}\partial_{\dot{q}_0} - Q_{\dot{q}_0}\partial_{q_0})h = \left\{\dot{q}_0\cos\omega t - \frac{\sin\omega t}{\omega}\left[\beta\dot{q}_0 + \left(\beta^2 + \omega^2\right)q_0\right]\right\}^{-1}$$
$$\times\left[\left(\cos\omega t + \beta\,\frac{\sin\omega t}{\omega}\right)\partial_{\dot{q}_0} - \frac{\sin\omega t}{\omega}\,\partial_{q_0}\right]h. \tag{65}$$

From the explicit form of h, Eq. (62), we easily finds that

$$h_{\dot{q}_0} = \frac{2h}{(\dot{q}_0 + \beta q_0)^2 + \omega^2 q_0^2}\,\dot{q}_0,$$
$$h_{q_0} = \frac{2h}{(\dot{q}_0 + \beta q_0)^2 + \omega^2 q_0^2}\,\left[2\beta\dot{q}_0 + \left(\beta^2 + \omega^2\right)q_0\right]. \tag{66}$$

Substituting this into Eq. (65) leads to

$$\frac{1}{Q_{(t)}}(Q_{q_0}\partial_{\dot{q}_0} - Q_{\dot{q}_0}\partial_{q_0})h = \frac{2h}{(\dot{q}_0 + \beta q_0)^2 + \omega^2 q_0^2}. \tag{67}$$

This substituted into Eq. (64), with terms dependent on t cancelling out, leads to the simple result that

$$\mathcal{P}(\dot{q}_0, q_0) = \frac{2h\mathcal{H}_h}{(\dot{q}_0 + \beta q_0)^2 + \omega^2 q_0^2} \neq 0, \tag{68}$$

with the function $\mathcal{H} = \mathcal{H}(h)$ arbitrary, and h given by Eq. (62). This is the most general form of $\mathcal{P}(\dot{q}_0, q_0)$ which leads to the stationary Lagrangian for the damped harmonic oscillator Eq. (59).

5.1. The havas Lagrangian

There is an infinite number of choices for the stationary Lagrangian for the damped harmonic oscillator, Eq. (59), due to the ambiguity of the choice for $\mathcal{H} = \mathcal{H}(h)$, $\mathcal{H}_h \neq 0$. With this function being entirely arbitrary, the transition from the Lagrangian formalism to the Hamiltonian formalism encounters the algebraic difficulty concerned with solving effectively $p = L_{\dot{q}}(\dot{q}, q)$ in the form $\dot{q} = \dot{q}(p, q)$. There is, however, a special choice for $\mathcal{H}(h)$—detected already in Ref. 6,

$$\mathcal{H} = -\tfrac{1}{2}\ln(2h), \tag{69}$$

within which this difficulty disappears and the effective Hamiltonian $H = H(p, q)$ can be easily constructed. According to Eq. (68), this corresponds to the choice for $\mathcal{P}(\dot{q}_0, q_0)$

$$\mathcal{P}(\dot{q}_0, q_0) = -\frac{1}{(\dot{q}_0 + \beta q_0)^2 + \omega^2 q_0^2}. \tag{70}$$

Indeed, with this choice for $\mathcal{H}$, we have [see Eq. (52)]

$$\left(\frac{1}{\dot{q}}L\right)_{\dot{q}} = \frac{1}{\dot{q}^2}\mathcal{H} = -\left(\frac{1}{\dot{q}}\mathcal{H}\right)_{\dot{q}} - \frac{h_{\dot{q}}}{2\dot{q}h}. \tag{71}$$

But parallel to the first line of Eq. (66), $h_{\dot{q}} = 2\dot{q}h/\left[(\dot{q} + \beta q)^2 + \omega^2 q^2\right]$, so that

$$\left(\frac{1}{\dot{q}}L\right)_{\dot{q}} = \left[-\frac{\mathcal{H}}{\dot{q}} + \frac{1}{2i\omega q}\ln\frac{\dot{q} + (\beta + i\omega)q}{\dot{q} + (\beta - i\omega)q}\right]_{\dot{q}}. \tag{72}$$

It follows, using the change of variable $z := \dot{q} + (\beta + i\omega)q$, that

$$L = \frac{1}{2}\ln z^{1+\beta/i\omega}\,\bar{z}^{1-\beta/i\omega} + \frac{\dot{q}}{2i\omega q}(\ln z - \ln\bar{z}) + \dot{q}M_q(q), \tag{73}$$

where $M(q)$ is arbitrary and irrelevant from the point of view of $\delta \int_{t_0}^{t} dt\, L = 0$, so that without any loss of generality, it can be set equal to zero. The remaining terms are easily seen to amount to

$$L = \frac{1}{z - \bar{z}} \left(z \ln z - \bar{z} \ln \bar{z} \right),$$
$$z := \dot{q} + (\beta + i\omega) q \,. \tag{74}$$

This is essentially the Havas Lagrangian in a slightly different notation. In order to determine the corresponding Hamiltonian, we observe first that

$$p := L_{\dot{q}} = \frac{1}{z - \bar{z}} \ln \left(\frac{z}{\bar{z}} \right) = \frac{1}{2i\omega q} \ln \left(\frac{z}{\bar{z}} \right), \tag{75}$$

and then

$$\frac{z}{\bar{z}} = \frac{\dot{q} + (\beta + i\omega) q}{\dot{q} + (\beta - i\omega) q} = e^{2i\omega pq}. \tag{76}$$

From the second equality of Eq. (76) it easily follows that

$$\dot{q} + \beta q = \omega q \cot(\omega pq), \tag{77}$$

amounting to the effective inversion formula $\dot{q} = \dot{q}(p, q)$. With this established, we have

$$z = \frac{\omega q}{\sin(\omega pq)}\, e^{i\omega pq}, \tag{78}$$

consistently with $z - \bar{z} = 2i\omega q$. Therefore, the Hamiltonian expressed as the function of p and q is given by [see Eq. (58)]

$$\begin{aligned} H &= \mathcal{H} \\ &= -\tfrac{1}{2} \ln z^{1+\beta/i\omega}\, \bar{z}^{1-\beta/i\omega} \\ &= \ln \left[e^{-\beta pq} \frac{\sin(\omega pq)}{\omega q} \right], \end{aligned} \tag{79}$$

which is the Havas Hamiltonian for the damped harmonic oscillator.

5.2. Comparison between canonical equations for the damped harmonic oscillator

As we have seen, in the case of the causal line with $F = -2\beta\dot{q} - (\beta^2 + \omega^2)\, q$, the Hamiltonians

$$H = \tfrac{1}{2} \left[e^{-2\beta t} p^2 + \left(\beta^2 + \omega^2 \right) e^{2\beta t} q^2 \right], \tag{80}$$

$$H = \ln \left[e^{-\beta pq} \frac{\sin(\omega pq)}{\omega q} \right], \tag{81}$$

which correspond to the choices for the arbitrary $\mathcal{P}(\dot{q}_0, q_0)$

$$\mathcal{P} = 1, \tag{82}$$

$$\mathcal{P} = -\frac{1}{(\dot{q}_0 + \beta q_0)^2 + \omega^2 q_0^2}, \tag{83}$$

respectively, are *both* inducing the same ordinary differential equation

$$\ddot{q} + 2\beta\dot{q} + \left(\beta^2 + \omega^2\right) q = 0. \tag{84}$$

Being concerned with the implications of the ambiguity for the choice of $\mathcal{P}$—within the general theory—and its consequences on the level of the quantum mechanics, the comparison of the theories based on Hamiltonians (80) and (81) is of interest, in particular perhaps for the researchers interested in group-theoretical treatments with harmonic oscillators because in the limit $\beta \to 0$, the equivalent Hamiltonians (80) and (81), *i.e.*, in the case of the equation $\ddot{q} + \omega^2 q = 0$, amount to

$$H = \tfrac{1}{2}\left(p^2 + \omega^2 q^2\right), \tag{85}$$

$$H = \ln\left[\frac{\sin(\omega p q)}{\omega q}\right]. \tag{86}$$

Equation (86) is a rather not commonly known expression for the Hamiltonian of the standard harmonic oscillator. Some intuitions gathered by comparing the theories founded on the Hamiltonians (80) and (81) will be later useful on the level of the general theory of the causal curves embedded into the canonical formalism.

6. THE HAMILTONIAN $H = \frac{1}{2}[e^{-2\beta t}p^2 + (\beta^2 + \omega^2)\, e^{2\beta t} q^2]$

Consider the canonical equations $\dot{q} = H_p$, $\dot{p} = -H_q$, where H is given by Eq. (80),

$$\dot{q} = e^{-2\beta t} p, \qquad \dot{p} = -\left(\beta^2 + \omega^2\right) e^{2\beta t} q. \tag{87}$$

By differentiating we have

$$\ddot{q} + 2\beta\dot{q} + \left(\beta^2 + \omega^2\right) q = 0. \tag{88}$$

Similarly one finds that

$$\ddot{p} - 2\beta\dot{p} + \left(\beta^2 + \omega^2\right) p = 0. \tag{89}$$

The explicit solution to (87) in terms of initial conditions at $t = 0$, p_0 and q_0 is

$$q = e^{-\beta t}\left[q_0 \cos\omega t + (p_0 + \beta q_0)\frac{\sin\omega t}{\omega}\right], \tag{90}$$

$$p = e^{\beta t}\left\{p_0 \cos\omega t - \left[\beta p_0 + \left(\beta^2 + \omega^2\right) q_0\right]\frac{\sin\omega t}{\omega}\right\}, \tag{91}$$

It is important to stress that with this Hamiltonian the concept of canonically conjugated momentum is defined simply by

$$p = e^{2\beta t}\dot{q}\,. \tag{92}$$

6.1. THE HAMILTONIAN $H = \ln[\exp(-\beta pq)\sin(\omega pq)/\omega q]$

Here the canonical equations amount to

$$\begin{aligned} \dot{q} &= Gq, \\ \dot{p} &= -Gp + \frac{1}{q}, \\ G &:= -\beta + \omega\cot(\omega pq), \end{aligned} \tag{93}$$

and therefore the canonically conjugated momentum is given by

$$p = \frac{1}{\omega q}\cot^{-1}\left(\frac{\dot{q} + \beta q}{\omega q}\right). \tag{94}$$

From Eqs. (93) it follows that $\dot{(pq)} = 1$, and consequently $\dot{G} = -\omega^2/\sin^2(\omega pq)$. Then the second derivative of q with respect to time is given by

$$\begin{aligned} \ddot{q} &= -\frac{\omega^2 q}{\sin^2(\omega pq)} + [-\beta + \omega\cot(\omega pq)]\dot{q} \\ &= -2\beta\dot{q} - \left(\beta^2 + \omega^2\right)q. \end{aligned} \tag{95}$$

Therefore the Hamilton equations generated by the Hamiltonian (81) are such that q satisfies duly the equation of the damped oscillator.

The most general solution to Eq. (93) in terms of initial conditions p_0 and q_0 at $t = 0$ is

$$\begin{aligned} q &= \frac{\omega q_0}{\sin(\omega p_0 q_0)}\, e^{-\beta t}\, \frac{\sin\omega(t + p_0 q_0)}{\omega}, \\ p &= \frac{\sin(\omega p_0 q_0)}{\omega q_0}\, e^{\beta t}\, \frac{\omega(t + p_0 q_0)}{\sin\omega(t + p_0 q_0)}. \end{aligned} \tag{96}$$

7. CONCLUDING REMARKS

We have developed a method for generating Lagrangians and Hamiltonians for a given system based on the knowledge of its trajectories in phase space. The usefulness of this method has been shown with the generation of an infinite number of Lagrangians and Hamiltonians for the damped harmonic oscillator, included among them two well known ones. The possibility of obtaining an infinite number of equivalent Lagrangians and Hamiltonians for a given system is intriguing and worth exploring its consequences in future papers. Once the Lagrangian for a system is generated, a Hamiltonian can be determined and classical and quantum dynamics can be analyzed. One interesting application could be that of the determination of the Lagrangian of a system when one can only generate trajectories numerically or by experiment instead of being able to write a differential equation for it.

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