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On dynamical symmetry groups

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Abstract. The action of the groups SU(2) and SO(3) as dynamical symmetry groups of the two-dimensional isotropic harmonic oscillator and of the Kepler problem in two dimensions, respectively, is analyzed and the corresponding quantum problems are solved employing these groups.

Resumen. Se analiza la acción de los grupos SU(2) y SO(3) como grupos de simetría dinámica del oscilador armónico isótropo bidimensional y del problema de Kepler en dos dimensiones, respectivamente, y se resuelven los problemas cuánticos correspondientes empleando estos grupos.

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1. Introduction

The importance of the continuous symmetry groups in analytical mechanics is very well known; if the Lagrangian function of a mechanical system is invariant with respect to a one-parameter group of transformations on the configuration space, then there exists a corresponding constant of the motion. However, for some mechanical systems, there exist constants of the motion that are not related with symmetries of the configuration space; nevertheless, these constants are always related to symmetry groups that act on the phase space, leaving the Hamiltonian function of the system invariant. Such symmetries are called "hidden symmetries" or "dynamical symmetries." In quantum mechanics, a constant of the motion that does not depend explicitly on the time corresponds to an operator that commutes with the Hamiltonian, and the existence of such an operator explains partially the degeneracy of the energy levels; when the degeneracy of the energy levels is related to a hidden symmetry, the degeneracy is called "accidental" (see, *e.g.*, Refs. 1–3 and the references cited therein).

Two well-known examples of mechanical systems with hidden symmetries are the Kepler problem and the isotropic harmonic oscillator. In both cases the potential is spherically symmetric, which implies the conservation of the angular momentum, but there exist additional constants of the motion whose existence does not come from obvious geometrical symmetries (see, *e.g.*, Refs. 3, 4 and the references cited therein). For the Kepler problem, there exists a conserved vector—the Hermann-Bernoulli-Laplace-Runge-Lenz (HBLRL) vector—which lies on the plane of the orbit and points along its symmetry axis passing through the center of force. The analog of the HBLRL vector was employed by Pauli [5] to find the energy levels of the hydrogen atom, whose accidental degeneracy is accounted for by the existence of this vector (see also Refs. 2 and 3). In the case of the isotropic harmonic oscillator, one finds a conserved symmetric second-rank tensor whose eigenvectors determine the axes of the orbit (see, *e.g.*, Refs. 2–4).

The aim of this paper is to give an elementary analysis of the hidden symmetries of the isotropic harmonic oscillator and of the Kepler problem (for bounded orbits) in two dimensions and to show their application in finding the energy levels of the corresponding quantum analogs. In particular, we discuss in some detail the manner in which the group SU(2) acts as a dynamical symmetry group of the two-dimensional isotropic harmonic oscillator (TIHO), pointing out the erroneous arguments employed in Refs. 1 and 4 in order to identify this group from its infinitesimal action [see the discussion after Eq. (19)]. In Sect. 2 we show that the groups SU(2) and SO(3) are dynamical symmetry groups for the TIHO and the two-dimensional Kepler problem with negative energy, respectively. In Sect. 3, we show how these symmetry groups can be used to obtain the eigenvalues and eigenvectors of the corresponding Hamiltonian operators. A more detailed account on the application of the symmetry groups in the solution of the Schrödinger equation as well as several examples can be found in Ref. 3 and the references cited therein. This book includes a comprehensive list of references about hidden symmetries and the connections between the harmonic oscillator and the Kepler problem.

2. Examples of dynamical symmetries in classical mechanics

2.1. Dynamical symmetries of the two-dimensional isotropic harmonic oscillator

The Hamiltonian function of the two-dimensional isotropic harmonic oscillator (TIHO),

$$H = \frac{1}{2M}\left(p_x^2 + p_y^2\right) + \frac{1}{2}M\omega^2\left(x^2 + y^2\right), \tag{1}$$

can be expressed in the form

$$H = \frac{1}{2M}\psi^\dagger\psi, \tag{2}$$

where

$$\psi \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \equiv \begin{pmatrix} ip_x + M\omega x \\ ip_y + M\omega y \end{pmatrix}. \tag{3}$$

Equation (3) establishes a one-to-one correspondence between the points of the phase space and the two-component complex vectors ψ. Furthermore, we can use the complex variables ψ_1, ψ_2, in place of the canonical coordinates. With the Poisson bracket defined in such a way that $\{x, p_x\} = 1$, one finds that the Poisson brackets among the components of ψ and their complex conjugates are

$$\{\psi_A, \psi_B\} = 0 = \{\overline{\psi}_A, \overline{\psi}_B\}, \qquad \{\psi_A, \overline{\psi}_B\} = -2iM\omega\delta_{AB} \tag{4}$$

$(A, B, \ldots = 1, 2)$. Except for a constant factor, the components ψ_A are the analogs of the "destruction" or "lowering" operators employed in the usual treatment of the quantum harmonic oscillator.

Let α be a 2×2 matrix, the (scalar) function $f \equiv \psi^\dagger \alpha \psi$ is real if and only if α is hermitian, $\alpha^\dagger = \alpha$. In fact, $\bar{f} = f^\dagger = (\psi^\dagger \alpha \psi)^\dagger = \psi^\dagger \alpha^\dagger \psi$, which coincides with f if and only if $\alpha^\dagger = \alpha$. On the other hand, making use of Eq. (4), one finds that the Poisson bracket of the function $f \equiv \psi^\dagger \alpha \psi$ with $g \equiv \psi^\dagger \beta \psi$, where β is another 2×2 matrix, is given by

$$\{f, g\} = -2iM\omega\psi^\dagger[\alpha, \beta]\psi. \tag{5}$$

Since the Hamiltonian is given by $H = (1/2M)\psi^\dagger I \psi$, where I is the 2×2 unit matrix [Eq. (2)] and any matrix commutes with I, from Eq. (5) it follows that any function of the form $f = \psi^\dagger \alpha \psi$ is a constant of the motion:

$$\{f, H\} = -i\omega\psi^\dagger[\alpha, I]\psi = 0.$$

In particular, if α is a hermitian 2×2 matrix, $\psi^\dagger \alpha \psi$ is a real-valued constant of the motion.

The matrices

$$\sigma_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 \equiv \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_3 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{6}$$

together with the unit matrix, form a basis for the 2×2 hermitian matrices. The matrices in Eq. (6) are the Pauli matrices (though not in the usual representation) and they satisfy the commutation relations

$$[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k, \tag{7}$$

the indices $i, j, \ldots$ run from 1 to 3 and a repeated index implies summation. According to the results of the preceding paragraph, the three quantities

$$S_i \equiv \frac{1}{4M\omega}\psi^\dagger \sigma_i \psi \tag{8}$$

are real-valued constants of the motion which, by virtue of Eqs. (5), (7) and (8), satisfy

$$\{S_i, S_j\} = -2iM\omega \left(\frac{1}{4M\omega}\right)^2 \psi^\dagger [\sigma_i, \sigma_j]\psi = \varepsilon_{ijk} S_k. \tag{9}$$

Making use of Eqs. (3), (6) and (8) one finds the explicit expressions

$$\begin{aligned} S_1 &= \frac{1}{2M\omega} p_x p_y + \frac{M\omega}{2} xy, \\ S_2 &= \frac{1}{4M\omega}\left(p_y^2 - p_x^2\right) + \frac{M\omega}{4}\left(y^2 - x^2\right), \\ S_3 &= \tfrac{1}{2}(x p_y - y p_x). \end{aligned} \tag{10}$$

The representation (6) was chosen so that the expressions (10) coincide with the definitions given in Ref. 4, Sect. 9.7.

The conservation of S_3, which is one half of the angular momentum with respect to the origin, is a consequence of the invariance of the system under rotations on the x-y plane about the origin. However, the conservation of S_1 and S_2 is not related to symmetries in the configuration space, which corresponds to the fact that S_1 and S_2 are not homogeneous functions of first degree in the momenta. Nevertheless, since all the S_i are constants of the motion, each S_i is the generating function of a one-parameter group of canonical transformations that leave the Hamiltonian invariant.

In order to find explicitly the canonical transformations generated by S_k, *i.e.*, the symmetry of H leading to the conservation of S_k, we recall that any function of the coordinates and momenta, G, is the generating function of a one-parameter group of transformations, parameterized by a variable s, in such a way that the rate of change of an arbitrary function f under the transformations generated by G is given by

$$\frac{df}{ds} = \{f, G\}. \tag{11}$$

Hence, from Eqs. (4), (8) and (11), it follows that under the transformations generated by S_k,

$$\frac{d\psi}{ds} = -\tfrac{1}{2} i\sigma_k \psi\,, \tag{12}$$

whose solution can be written as

$$\psi(s) = \exp(-\tfrac{1}{2} is\sigma_k)\psi(0) \tag{13}$$

or, since $\sigma_k^2 = I$, we also have

$$\psi(s) = \{[\cos(s/2)]\, I - i\,[\sin(s/2)]\,\sigma_k\}\,\psi(0). \tag{14}$$

Since σ_k is hermitian and traceless, $U \equiv \exp(-\tfrac{1}{2}is\sigma_k)$ is a unitary matrix with determinant $+1$, *i.e.*, U belongs to the group SU(2). It is easy to see directly that if $U \in \mathrm{SU}(2)$, the transformation $\psi \mapsto U\psi$ leaves the Hamiltonian (2) invariant. As we have shown, by construction, the transformation (13) is canonical. [It can also be verified directly that the transformation $\psi \mapsto U\psi$, for $U \in \mathrm{SU}(2)$, is canonical since preserves the Poisson brackets (4)].

Making use of Eqs. (3), (6) and (14), the symmetry transformations generated by the S_i can be expressed in terms of the coordinates x, y, p_x, and p_y; one finds that under the transformations generated by S_1 and S_2 the coordinates and momenta are mixed. It should be noticed that the right-hand side of Eq. (14) is a periodic function of s, with period 4π, and that in order to obtain all the different transformations generated by each S_k, the parameter s must range over an interval of length 4π.

By substituting Eq. (14) into Eq. (8), making use of the fact that $\sigma_i\sigma_j = \delta_{ij} I + i\varepsilon_{ijk}\sigma_k$, one can find the transformations generated by S_k on the S_i themselves (see, *e.g.*, Ref. 6).

In this manner we find that under the transformations generated by S_1, the functions S_i transform according to

$$\begin{aligned} S_1(s) &= S_1(0), \\ S_2(s) &= \cos(s)\, S_2(0) - \sin(s)\, S_3(0), \\ S_3(s) &= \sin(s)\, S_2(0) + \cos(s)\, S_3(0), \end{aligned} \tag{15}$$

which represents a rotation through an angle s about the S_1 axis in the (S_1, S_2, S_3) space. By permuting cyclicly the subscripts appearing in Eqs. (15) one obtains the effect of the transformations generated by S_2 and S_3. Note that for $s = 2\pi$, the transformation given by Eqs. (15) is the identity, but the sign of ψ is inverted [see Eq. (14)].

It should be noticed that Eqs. (15) follow directly from Eqs. (9). In fact, Eq. (11) with $G = S_1$ yields

$$\begin{aligned} \frac{dS_1}{ds} &= \{S_1, S_1\} = 0, \\ \frac{dS_2}{ds} &= \{S_2, S_1\} = -S_3, \\ \frac{dS_3}{ds} &= \{S_3, S_1\} = S_2, \end{aligned} \tag{16}$$

which leads to Eqs. (15). However, if we consider the functions S_i only, it would seem that all the different transformations generated by S_1 are obtained by restricting s to the interval $0 \leq s < 2\pi$, which, in the present case, would be wrong.

The values of the constants of the motion S_1, S_2 and S_3 label the solutions of the dynamical system under consideration; in other words, through each point of the phase space (which has dimension four), there passes one (and only one) curve that represents the time evolution of the system; along each of these curves, S_1, S_2 and S_3 are constant and there exists a one-to-one correspondence between these curves and the points (S_1, S_2, S_3). The orbits of the TIHO in the configuration space are ellipses centered at the origin. The constants of the motion S_1, S_2, S_3 can be expressed in terms of the semi-axes of the ellipse, a, b ($a \geq b \geq 0$), and of the angle γ formed by the major axis with the x axis:

$$\begin{aligned} S_1 &= \tfrac{1}{4} M\omega(a^2 - b^2) \sin 2\gamma, \\ S_2 &= -\tfrac{1}{4} M\omega(a^2 - b^2) \cos 2\gamma, \\ S_3 &= \pm\tfrac{1}{2} M\omega ab. \end{aligned} \tag{17}$$

The value of S_3 is positive or negative according to whether the ellipse is traversed counterclockwise or clockwise, respectively. It may be noticed that, due to the symmetry of the ellipse, the angle γ is defined up to an integral multiple of π; but this ambiguity does not affect the value of the constants (17).

Conversely, the parameters of the ellipse traced by the TIHO in the configuration space can be expressed in terms of S_1, S_2 and S_3. From Eqs. (17) one finds that

$$
\begin{aligned}
a^2 &= \frac{2}{M\omega}\left(\sqrt{S_1^2+S_2^2+S_3^2}+\sqrt{S_1^2+S_2^2}\right),\\
b^2 &= \frac{2}{M\omega}\left(\sqrt{S_1^2+S_2^2+S_3^2}-\sqrt{S_1^2+S_2^2}\right),\\
\gamma = -\frac{1}{2}\arctan\frac{S_1}{S_2}, &\qquad e^2 = \frac{2\sqrt{S_1^2+S_2^2}}{\sqrt{S_1^2+S_2^2+S_3^2}+\sqrt{S_1^2+S_2^2}},
\end{aligned}
\tag{18}
$$

where e is the eccentricity.

Under the action of the transformations (14) (and, hence, of any composition of them) each point of the phase space is mapped onto another point corresponding, in general, to a distinct orbit in the configuration space (or to motion in the opposite direction), but with the same energy, therefore a^2+b^2 is invariant [note that $H=\frac{1}{2}M\omega^2(a^2+b^2)$]. Making use of Eq. (18) we can easily find the effect of the transformations generated by S_k on the orbit parameters. For instance, substituting Eq. (15) into Eq. (18), one finds that under the transformations generated by S_1 the axes of the ellipse rotate and the ellipse parameters change according to

$$
\begin{aligned}
a^2 &= \frac{2}{M\omega}\left(\sqrt{S_1^2+S_2^2+S_3^2}+\sqrt{S_1^2+[\cos(s)S_2-\sin(s)S_3]^2}\right),\\
b^2 &= \frac{2}{M\omega}\left(\sqrt{S_1^2+S_2^2+S_3^2}-\sqrt{S_1^2+[\cos(s)S_2-\sin(s)S_3]^2}\right),\\
e^2 &= \frac{2\sqrt{S_1^2+[\cos(s)S_2-\sin(s)S_3]^2}}{\sqrt{S_1^2+S_2^2+S_3^2}+\sqrt{S_1^2+[\cos(s)S_2-\sin(s)S_3]^2}}.
\end{aligned}
\tag{19}
$$

These expressions are periodic functions of s, with period π; therefore, for a given set of values of a and b, there are *at least four* (in most cases, eight) distinct rotations generated by S_1 leading to orbits with the same set of parameters and not only two, as claimed without proof in Ref. 4, where such an assumed double-valuedness is employed to conclude that the group generated by the S_i is SU(2) and not SO(3), the group of rotations in three dimensions. Similarly, in Ref. 1 the effect of the transformations generated by $K \equiv 2S_1$ on the orbit parameters is considered and it is asserted that it takes a 4π rotation to bring a given orbit into itself; however, all the different transformations generated by $2S_1$ are obtained if the corresponding parameter takes values in an interval of length 2π (note that $2S_1$ generates the rotations (15) with s replaced by $2s$ [see Eqs. (16)]) and there are two different rotations in the phase space that produce the same values of S_i and, hence, the same orbit. Thus, it would take a π rotation generated by $2S_1$ to bring a given orbit into itself.

We close this subsection with the following remarks. The preceding results can be easily extended to the isotropic harmonic oscillator in n dimensions. It can be readily seen that Eqs. (2), (4) and (5) hold if ψ is a n-component complex vector defined in a similar way to Eq. (3). Then, if the matrices λ_a $(a = 1, \ldots, n^2 - 1)$ form a basis for the traceless hermitian $n \times n$ matrices, the scalar functions $S_a \equiv \psi^\dagger \lambda_a \psi/(4M\omega)$ are constants of the motion and satisfy the Poisson bracket relations

$$\{S_a, S_b\} = f_{abc}\, S_c,$$

where f_{abc} are real constants such that $[\lambda_a, \lambda_b] = 2if_{abc}\lambda_c$ [*cf.* Eqs. (7–9)]. This simple procedure to construct constants of the motion satisfying the relations obeyed by the matrices λ_a contrasts with the cumbersome indirect method followed, *e.g.*, in Refs. 2 and 3. The group of canonical transformations generated by the S_a is SU(n) and, under these transformations, the S_a transform according to the adjoint representation of SU(n).

2.2. Dynamical symmetries of the Kepler problem in two dimensions

The Hamiltonian function corresponding to the Kepler problem in two dimensions is given by

$$H = \frac{1}{2M}\left(p_x^2 + p_y^2\right) - \frac{k}{r}, \tag{20}$$

where $r \equiv \sqrt{x^2 + y^2}$ and k is a positive constant. As in the case of the TIHO Hamiltonian (1), the only obvious continuous symmetries of the Hamiltonian function (20) (that are, at the same time, canonical transformations) are the rotations about the origin. Owing to the form of the Hamiltonian (20), we shall follow a different approach to that employed in the preceding subsection in order to find additional symmetries.

The invariance of H under the rotations about the origin implies that the angular momentum

$$L_z = xp_y - yp_x, \tag{21}$$

is a constant of the motion. As is well known, the Hermann-Bernoulli-Laplace-Runge-Lenz (HBLRL) vector is a conserved vector that lies in the plane of the orbit in the direction of the point of closest approach of the particle to the origin (see, *e.g.*, Refs. 3 and 4). The components of the HBLRL vector are given by

$$A_x = p_y L_z - Mkx/r, \qquad A_y = -p_x L_z - Mky/r \tag{22}$$

and it is easy to see that their Poisson brackets with the Hamiltonian (20) indeed vanish and that

$$\{A_x, A_y\} = -2MHL_z, \qquad \{L_z, A_x\} = A_y, \qquad \{L_z, A_y\} = -A_x. \tag{23}$$

Therefore, considering only bounded orbits, for which $H < 0$, from Eq. (23) it follows that

$$S_1 \equiv \frac{A_x}{\sqrt{-2MH}}, \qquad S_2 \equiv \frac{A_y}{\sqrt{-2MH}}, \qquad S_3 \equiv L_z, \tag{24}$$

satisfy the Poisson bracket relations of the angular momentum [Eq. (9)]

$$\{S_i, S_j\} = \varepsilon_{ijk} S_k. \tag{25}$$

Since the S_i are constants of the motion, they are generating functions of canonical transformations that leave the Hamiltonian invariant; however, the use of Eq. (11) to find the explicit form of the transformations generated by S_1 and S_2 leads to sets of differential equations that are difficult to solve. Nevertheless, the effect of the transformations generated by the S_i on the S_i themselves is already known since, as we have seen, from Eq. (25) it follows that in the (S_1, S_2, S_3) space the transformations generated by S_k correspond to rotations about the S_k axis, which are given explicitly by Eq. (15) and similar expressions obtained by the cyclic permutation of the indices. In the present case, S_3 coincides with the angular momentum [*cf.* Eq. (10)] which is the generating function of rotations about the origin, under which (S_1, S_2) transforms as an ordinary vector; this implies that, by contrast with the preceding case, the parameter s appearing in Eq. (15) and those obtained by cyclic permutation of the indices now ranges over an interval of length 2π and that the group generated by the constants S_i is SO(3).

As in the case of the TIHO, the constants of the motion S_i are related to the orbit parameters and there is a one-to-one relation between orbits and points of the (S_1, S_2, S_3) space. In the present case, the orbits are ellipses with a focus at the origin. Since the magnitude of the HBLRL vector amounts to Mke, where e is the eccentricity, from Eq. (24) one finds that

$$S_1 = \sqrt{Mka}\,(e\cos\gamma), \quad S_2 = \sqrt{Mka}\,(e\sin\gamma), \quad S_3 = \pm\sqrt{Mka(1-e^2)}, \tag{26}$$

where a is the semimajor axis, γ is the angle formed by the HBLRL vector with the x axis and we have made use of the relations $H = -k/(2a)$ and $L_z^2 = Mka(1-e^2)$ (see, *e.g.*, Ref. 4). The sign of S_3 is positive or negative according to whether the particle moves counterclockwise or clockwise, respectively. The inverse relations to Eq. (26) are

$$a = \frac{S_1^2 + S_2^2 + S_3^2}{Mk}, \qquad \gamma = \arctan\frac{S_2}{S_1}, \qquad e = \frac{\sqrt{S_1^2 + S_2^2}}{\sqrt{S_1^2 + S_2^2 + S_3^2}} \tag{27}$$

[*cf.* Eq. (18)]. From these expressions it is clear that the semimajor axis, a, is left unchanged by all the transformations generated by the S_i (which is also apparent from the relation $H = -k/(2a)$, since all these transformations leave H invariant); the transformations generated by S_1 and S_2 change the eccentricity, rotate the major axis of the ellipse and change the direction of motion, while the transformations generated by S_3 only produce (rigid) rotations of the orbits, as expected.

3. Solution of the Schrödinger equation by means of the dynamical symmetry groups

3.1. Energy eigenvalues and eigenvectors of the quantum TIHO

Now we shall consider the quantum TIHO and we shall find its energy eigenvalues and eigenvectors making use of the dynamical symmetry group discussed in Sect. 2.1. By considering the variables x, y, p_x and p_y appearing in Eqs. (1) and (10) as operators in the usual manner, it follows that the operators S_i commute with the Hamiltonian and satisfy the commutation relations [Eq. (9)]

$$[S_i, S_j] = i\hbar\varepsilon_{ijk}S_k, \tag{28}$$

which are the commutation relations of the angular momentum, even though, in the present case, only S_3 is related to the angular momentum (in fact, S_3 corresponds to one half of the angular momentum). As is well known, the commutation relations (28) imply that each operator S_i commutes with $S^2 \equiv S_1^2 + S_2^2 + S_3^2$ and, therefore, there exist common eigenvectors of S^2 and S_3, ψ_{sm_s}, such that

$$S^2\psi_{sm_s} = s(s+1)\hbar^2\psi_{sm_s}, \qquad S_3\psi_{sm_s} = m_s\hbar\psi_{sm_s}, \tag{29}$$

with $s = 0, \frac{1}{2}, 1, \ldots$ and $m_s = -s, -s+1, \ldots, s$. The half-integral values of the quantum number s appear only in the representations of SU(2) which, as we have seen, is a symmetry group of the TIHO. Alternatively, we can see that the half-integral values of s and m_s are allowed because the eigenvalues of the orbital angular momentum L_z are integral multiples of $\hbar$ and $S_3 = \frac{1}{2}L_z$. The normalized eigenvectors ψ_{sm_s} can be chosen in such a way that

$$S_\pm\psi_{sm_s} = \sqrt{s(s+1) - m_s(m_s \pm 1)}\;\hbar\psi_{s,m_s\pm 1} \tag{30}$$

where, following the usual notation,

$$S_\pm \equiv S_1 \pm iS_2 \tag{31}$$

[see, *e.g.*, Refs. 2 and 7].

Making use of Eqs. (1) and (10) and the commutation relations for the position and momentum operators, one finds that

$$H^2 = \omega^2(4S^2 + \hbar^2) \tag{32}$$

and since H commutes with S_3 and S^2, from Eqs. (29) it follows that

$$H\psi_{sm_s} = (2s+1)\hbar\omega\psi_{sm_s}, \tag{33}$$

which means that the energy eigenvalues are of the form $(2s+1)\hbar\omega$ and, since m_s can take the $2s+1$ values $-s, -s+1, \ldots, s$, the energy level $(2s+1)\hbar\omega$ has degeneracy $2s+1$.

From Eq. (10) and (31) one finds that

$$S_\pm = \frac{\mp i}{4M\omega}\left[(p_x \pm ip_y)^2 + M^2\omega^2(x \pm iy)^2\right] \tag{34}$$

hence, introducing the dimensionless complex variable

$$z \equiv \sqrt{\frac{M\omega}{\hbar}}\,(x + iy) \tag{35}$$

and making use of the expressions $p_x = -i\hbar\partial/\partial x$, $p_y = -i\hbar\partial/\partial y$, we have

$$S_+ = i\hbar\left(\frac{\partial^2}{\partial\overline{z}^2} - \frac{1}{4}z^2\right), \qquad S_- = -i\hbar\left(\frac{\partial^2}{\partial z^2} - \frac{1}{4}\overline{z}^2\right). \tag{36}$$

According to Eq. (30), the wavefunction ψ_{ss} must obey $S_+\psi_{ss} = 0$, *i.e.*, $\partial^2\psi_{ss}/\partial\overline{z}^2 = \frac{1}{4}z^2\psi_{ss}$; hence,

$$\psi_{ss} = C_s e^{-z\overline{z}/2} f(z), \tag{37}$$

where C_s is a normalization constant and $f(z)$ is a function of z only. We exclude solutions of the form $e^{z\overline{z}/2}f(z)$, since they are not square-integrable.

On the other hand, one finds that

$$S_3 = \frac{\hbar}{2}\left(z\frac{\partial}{\partial z} - \overline{z}\frac{\partial}{\partial\overline{z}}\right) \tag{38}$$

therefore, from the condition $S_3\psi_{ss} = s\hbar\psi_{ss}$, making use of Eq. (37) and (38), one obtains $z\partial f/\partial z = 2sf$, which yields $f(z) = z^{2s}$, thus

$$\psi_{ss} = C_s e^{-z\overline{z}/2} z^{2s} = C_s\left(\frac{M\omega}{\hbar}\right)^s e^{-M\omega(x^2+y^2)/2\hbar}(x+iy)^{2s}. \tag{39}$$

It is easy to see that $|C_s|^2 = M\omega/[\hbar\pi(2s)!]$ and making use repeatedly of Eq. (30) one finds

$$\psi_{sm_s} = \hbar^{m_s-s}\sqrt{\frac{(s+m_s)!}{(2s)!\,(s-m_s)!}}\,S_-^{s-m_s}\psi_{ss}. \tag{40}$$

Using Eq. (30), (36) and (39), one can prove by induction that, for $m_s \geq 0$,

$$\psi_{sm_s} = e^{-z\overline{z}/2} z^{2m_s}\mathcal{P}_{s-m_s}(z\overline{z}), \tag{41}$$

where $\mathcal{P}_k$ is a polynomial of degree k (which turns out to be an associated Laguerre polynomial). Note that, since m_s takes integral or half-integral values, the wavefunction (41) is single-valued.

3.2. THE QUANTUM TWO-DIMENSIONAL KEPLER PROBLEM

In the case of the quantum-mechanical Kepler problem in two dimensions, owing to the fact that L_z does not commute with p_x and p_y, one has to replace the constants of the motion A_x and A_y [Eqs. (22)] by the operators

$$A_x = \frac{1}{2}(p_y L_z + L_z p_y) - \frac{Mkx}{r}, \qquad A_y = -\frac{1}{2}(p_x L_z + L_z p_x) - \frac{Mky}{r}, \tag{42}$$

where $L_z = xp_y - yp_x$. Then one can verify that A_x, A_y and L_z commute with the Hamiltonian and satisfy the commutation relations

$$[A_x, A_y] = -i\hbar 2MHL_z, \qquad [L_z, A_x] = i\hbar A_y, \qquad [L_z, A_y] = -i\hbar A_x, \tag{43}$$

which are analogous to the Poisson bracket relations (23). In order to define the analogs of S_1 and S_2 [Eqs. (24)], one replaces the Hamiltonian operator appearing in Eq. (43) by one of its eigenvalues E; this amounts to restrict Eq. (43) to the subspace formed by the eigenvectors of H with eigenvalue E. Then, assuming $E < 0$ (bound states), we define

$$S_1 \equiv \frac{A_x}{\sqrt{-2ME}}, \qquad S_2 \equiv \frac{A_y}{\sqrt{-2ME}}, \qquad S_3 \equiv L_z. \tag{44}$$

These operators satisfy the commutation relations of the angular momentum [Eqs. (28)] and, therefore, there exist eigenvectors of H with eigenvalue E, ψ_{sm_s}, that are eigenvectors of S^2 and S_3 satisfying Eq. (29), where $s = 0, 1, 2, \ldots$ and $m_s = -s, -s+1, \ldots, s$, (integral values only) since the eigenvalues of the orbital angular momentum $L_z = S_3$ must be integral multiples of $\hbar$. This restriction on the values of s is equivalent to the fact that the group generated by the operators S_i is not SU(2) but SO(3).

Using the canonical commutation relations one finds that on the subspace of eigenvectors of H with eigenvalue E,

$$S^2 = -\frac{Mk^2}{2E} - \frac{\hbar^2}{4}, \tag{45}$$

hence, the energy eigenvalues are given by

$$E = -\frac{2Mk^2}{\hbar^2(2s+1)^2}, \tag{46}$$

and have degeneracy $2s + 1$.

The explicit expression of the energy eigenfunctions can be obtained as in Sect. 3.1. In terms of the polar coordinates we have

$$L_z = -i\hbar\frac{\partial}{\partial\theta}, \qquad A_x \pm iA_y = e^{\pm i\theta}\left[\hbar^2\left(\pm i\frac{\partial}{\partial r} - \frac{1}{r}\frac{\partial}{\partial\theta}\right)\left(\frac{\partial}{\partial\theta} \pm \frac{i}{2}\right) - Mk\right] \tag{47}$$

or, introducing the dimensionless variable

$$\rho \equiv \frac{Mk}{\hbar^2}r, \tag{48}$$

we obtain

$$A_x \pm iA_y = Mke^{\pm i\theta}\left[\left(\pm i\frac{\partial}{\partial\rho} - \frac{1}{\rho}\frac{\partial}{\partial\theta}\right)\left(\frac{\partial}{\partial\theta} \pm \frac{i}{2}\right) - 1\right], \tag{49}$$

therefore, writing $\psi_{ss}(r, \theta) = f(r)e^{is\theta}$, the condition $S_+\psi_{ss} = 0$ is equivalent to

$$\frac{df}{d\rho} + \left(\frac{2}{2s+1} - \frac{s}{\rho}\right)f = 0$$

thus, $f(\rho) = C_s\rho^s e^{-2\rho/(2s+1)}$, where C_s is a normalization constant. The remaining wavefunctions can be expressed as in Eq. (40). One finds, by induction, that for $m_s \geq 0$,

$$\psi_{sm_s} = e^{-2\rho/(2s+1)}\rho^{m_s}\mathcal{P}_{s-m_s}(\rho)e^{im_s\theta}, \tag{50}$$

where $\mathcal{P}_k$ is a polynomial of degree k. (It can be shown that, also in this case, $\mathcal{P}_k$ obeys the associated Laguerre equation.)

4. CONCLUDING REMARKS

As pointed out in Ref. 1 and the references cited therein, the two-dimensional isotropic harmonic oscillator is an interesting example because of the appearance of the Pauli matrices, or of half-integral angular momentum quantum numbers, even though the spin angular momentum is not taken into account.

The examples considered here are particularly convenient in order to show the role of the dynamical symmetry groups, owing to the presence of quantities that satisfy the Poisson bracket (or commutation) relations of the angular momentum, since the representations of operators satisfying such relations are relatively simple and well-known.

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