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On homogenization and effective coefficients in laminated piezocomposite materials

R.R. RAMOS, J.A. OTERO, AND J.B. CASTILLERO
Institute of Cybernetics, Mathematics and Physics (ICIMAF)
Acoustic Group, Calle D No. 353 e/t 15 y 17, Vedado, Habana, Cuba

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ABSTRACT. The general expressions to compute effective elastic, piezoelectric and dielectric characteristics of a laminated piezocomposite are derived by means of an asymptotic expansion method. The local problems used to determine the so-called local auxiliary functions are shown. We compute explicitly the effective material characteristics for two examples of such layered medium. Finally, we apply these results to a piezocomposite material and obtain new piezoelectrics with better global properties.

RESUMEN. Mediante el método de promediación asintótica se obtienen las expresiones generales para el cálculo de los coeficientes efectivos elásticos, piezoeléctricos y dieléctricos de un laminado piezocompuesto. Se muestran los problemas locales utilizados para determinar las denominadas funciones locales auxiliares. Se calculan explícitamente las características efectivas del material para dos ejemplos de tales medios laminados. Finalmente se aplican estos resultados a un material piezocompuesto, obteniéndose un nuevo material piezoeléctrico con mejores propiedades globales.

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1. INTRODUCTION

When some crystals (such as quartz, tourmaline, seignette salt) are under stresses, an electric momentum is produced. This is the so-called simple piezoelectric effect. Besides of the simple piezoelectric effect there also occurs the inverse piezoelectric effect, in which the electric potential produces a deformation. Materials with both properties are called piezoelectric materials.

Linear piezoelectricity, which is the linear coupling between electric vectorial quantities and strains or stresses, in case of homogeneous materials is a source of research and technological applications [1, 2]. The question naturally arises of the estimate of global homogenized properties for composite materials involving one or several linear piezoelectric constituents. By varying the proportion of various constituents (*e.g.*, a piezoelectric inactive polymer matrix-piezoelectric ceramic fibres, as in Refs. 3–6) one may obtain effective macroscopic properties for the homogenized macroscopic material. For instance, in Sect. 7 of this paper we consider a two phase piezocomposite laminated and show how the composite's properties vary with the volume fraction of piezoceramic and what are the consequences for medical ultrasonic imaging transducer.

During the last two decades an increasing amount of research has been conducted to develop method and procedures for improving the description of macroproperties for given micro-inhomogeneous structure of media. The two scale expansion method [7] associated to the Energy Convergence was applied previously by many authors to compute macrobehaviors such as, for instance, thermoelastic fields [8–10], thermopiezoelectricity in solids [11], magnetoelasticity in solids [12], plastic fields [13], flow in porous media [14, 15], etc. The variational method called Γ-convergence was used to obtain the effective moduli of a piezoelectric composite with fine periodic structure in Ref. 16. In Ref. 17 these results were extended for investigating the dynamical behavior and the Method of Bloch expansions is used. The method we used in this paper is the asymptotic expansion method and was developed, for instance, in Ref. 7, 18–20. Our goal is to show, by using a simple model of a laminated piezocomposite, their importance for ultrasonic transducer design.

First, in Sect. 2 we recall the fundamental relations of linear piezoelectric theory and the boundary value problem associated to the displacement field $\vec{u}$ and to the electrical potential φ. In Sect. 3 we seek the solution of a statical piezoelectric problem for an heterogeneous periodic medium in the form of an asymptotic expansion. Afterwards we obtain a sequence of recurrent boundary value problems with constant coefficients. In Sect. 4 we obtain the so-called "local functions" for a general piezoelectric composite. In Sect. 5 the local problems are used to determine the local functions of the first order and the effective coefficients for a laminated piezoelectric medium. We determine, in Sect. 6, the effective coefficients for two laminated piezoelectric composites with hexagonal symmetry. In Sect. 7, we consider a two phase laminated piezocomposite with several examples for different choices of their phases (polymer matrix-ceramic fibres) and volume fraction. We present both our model's predictions and their implications for medical ultrasonic imaging transducers. Section 8 is devoted to some general concluding remarks.

2. Fundamental relations of linear piezoelectricity

All subscripts appearing in the text take values 1, 2 and 3. The summation convention is consequently used throughout this paper.

We start our considerations by formulating the constitutive equations. We assume that the solid under consideration undergoes a deformation due to an external loading electromagnetic field, which may vary with the time. We assume also that there are no heat sources in the body and no heat conduction (*i.e.*, that the process is adiabatic). Applying to an arbitrary region V of the solid bounded by a surface A the principle of energy conservation [21] leads to

$$\frac{\partial}{\partial t}\int_V \left(\tfrac{1}{2}\rho v_i v_i + U\right) dV = \textstyle\int_V X_i v_i \, dV + \int_A p_i v_i \, dA + \int_A E_i D_i \, dA \,. \tag{1}$$

Here ρ is the mass density, $v_i = \partial u_i/\partial t$ the time derivative of displacement, U the internal (mechanical and electromagnetic) energy, X_i the volumic external forces, $p_i = \sigma_{ij} n_j$ the contact forces, σ_{ij} the components of the stress tensor, n_j the components of the outward unit normal vector, E_i the components of the electric field vector and D_i the components of the electric displacement vector.

Transforming Eq. (1) using the motion equations of continuous media

$$\sigma_{ij,j} + X_i = \rho \frac{\partial v_i}{\partial t} \tag{2}$$

(the comma denote partial differentiation), we obtain the local form of energy balance

$$H = \sigma_{ij} \frac{\partial \varepsilon_{ij}}{\partial t} - \frac{\partial E_i}{\partial t} D_i \,, \tag{3}$$

where ε_{ij} are the strain tensor components

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \,, \qquad H = U - E_i D_i \,, \tag{4}$$

and $H = H(\varepsilon_{ij}, E_i)$ is the electric entalpy. Hence we obtain

$$\left(\sigma_{ij} - \frac{\partial H}{\partial \varepsilon_{ij}}\right) \frac{\partial \varepsilon_{ij}}{\partial t} - \left(D_i + \frac{\partial H}{\partial E_i}\right) \frac{\partial E_i}{\partial t} = 0 \,. \tag{5}$$

This equation should hold for any values of $\partial \varepsilon_{ij}/\partial t, \partial E_i/\partial t$, hence

$$\sigma_{ij} = \frac{\partial H}{\partial \varepsilon_{ij}} \,, \qquad D_i = -\frac{\partial H}{\partial E_i} \,. \tag{6}$$

These relations will be employed for deriving the constitutive equations.

Expanding the electric entalpy $H = H(\varepsilon_{ij}, E_i)$ into Maclaurin's series in the vicinity of the natural state ($\varepsilon_{ij} = 0,\ E_i = 0$), neglecting terms of higher order than two, we obtain, for a homogeneous anisotropic body the following expressions:

$$H\left(\varepsilon_{ij}, E_i\right) = \tfrac{1}{2} C^E_{ijkl}\, \varepsilon_{ij}\, \varepsilon_{kl} - e_{kij}\, \varepsilon_{ij}\, E_k - \tfrac{1}{2}\, \epsilon^{\varepsilon}_{ij}\, E_i\, E_j \,. \tag{7}$$

Here C^E_{ijkl} is the elastic stiffness; e_{kij} are the piezoelectric coefficients and $\epsilon^{\varepsilon}_{ij}$ the dielectric permittivity (dielectric constants). Furthermore, superscripts E and ε designate values of coefficients at constant electric field and strain, respectively.

From thermodynamic considerations and the momentum balance equation we have symmetry of stress and strain tensors,

$$C_{ijkl} = C_{klij} = C_{jikl} = C_{ijlk}, \qquad e_{kij} = e_{kji}, \qquad \epsilon_{ij} = \epsilon_{ji}. \tag{8}$$

By introducing (7) into (6) we obtain the constitutive equations for piezoelectric materials:

$$\sigma_{ij} = C_{ijkl}\, \varepsilon_{kl} - e_{kij}\, E_k \,, \tag{9}$$

$$D_i = e_{ikl}\, \varepsilon_{kl} + \epsilon_{ik}\, E_k \,. \tag{10}$$

The above mentioned relations present the material law and the mathematical model of quasi-static piezoelectricity in the classical Voigt's theory [22]. Stresses σ_{ij} and electric displacements D_i are linear functions of strains ε_{ij} and components of vector E_i.

Substituting (9) into (2) and using the geometrical relations, *i.e.*, the first equations of (4), we get

$$C_{ijkl}\, u_{k,lj} + e_{mij}\, E_{m,j} + X_i = \rho \frac{\partial^2 u_i}{\partial t^2}\,. \tag{11}$$

When the current flow and free electric charges are not present, the electromagnetic field in the piezoelectric medium is studied by the Maxwell's equations:

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t}, \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \qquad \nabla \cdot \vec{B} = 0, \qquad \nabla \cdot \vec{D} = 0\,, \tag{12}$$

where $\vec{H}$ is the magnetic field vector, $\vec{D}$ the electric displacement vector, $\vec{E}$ the electric field vector and $\vec{B}$ the magnetic induction vector.

The solutions of the motion Eq. (11) and Maxwell's equations. (12) are described by coupled elastic-electromagnetic waves, *i.e.* the elastic wave and it is interaction with the electric field and electromagnetic waves together with the deformations of the medium. Let the expansion velocity of the elastic wave be V, the corresponding velocity of electromagnetic wave v will have an order of $10^5 V$ and therefore when we study the elastic wave, the magnetic field can be neglected. Hence, in the majority of the problems related to piezoelectric materials, the electroacoustic waves are considered to have not magnetic effects ($\vec{H} = \vec{0}$, $\vec{B} = \vec{0}$) and using the quasistatic approach for the electric field, we have

$$\nabla \times \vec{E} = \vec{0}\,, \qquad \nabla \cdot \vec{D} = 0\,. \tag{13}$$

Then, the electric field vector $\vec{E}$ is given by the electric potential φ:

$$E_i = -\varphi_{,i}\,. \tag{14}$$

Introducing into the second equation of (13) the relation (10) and taking into account (14), we obtain

$$e_{nij}\, u_{i,jn} - \epsilon_{,in}\, \varphi_{,in} = 0\,. \tag{15}$$

From (11), (14) and (15) we get

$$C_{ijkl}\, u_{k,lj} + e_{mij}\, \varphi_{,mj} + X_i = \rho \frac{\partial^2 u_i}{\partial t^2}\,, \qquad e_{nij}\, u_{i,jn} - \epsilon_{in}\, \varphi_{,in} = 0\,. \tag{16}$$

Thus, we have finally four linear partial differential equations with four unknowns, the three components of the displacement vector $\vec{u}$ and the electric potential φ. Obviously, this system has to be completed with boundary and initial conditions, see, for instance, Refs. 2 or 22.

Finally we consider the linear static piezoelectric boundary value problem for heterogeneous media inside the domain $\Omega \subset R^3$ with boundary $\Gamma = \partial\Omega$:

$$(C_{ijkl}\, u_{k,l} + e_{mij}\, \varphi_{,m})_{,j} + X_i = 0\,, \qquad (e_{iml}\, u_{m,l} - \epsilon_{im}\, \varphi_{,m})_{,i} = 0\,. \tag{17}$$

$$u_i|_{\Gamma_0} = 0\,; \qquad \varphi|_{\Gamma_2} = \varphi_0\,; \qquad \sigma_{ij}n_j|_{\Gamma_1} = S_i^o\,; \qquad D_i n_i|_{\Gamma_3} = 0\,. \tag{18}$$

Where φ_o, $\vec{S}$ are the electric potential on Γ_2, the mechanical load on Γ_1, respectively. $\Gamma = \bar{\Gamma}_0 \cup \bar{\Gamma}_1$, $\Gamma_0 \cap \Gamma_1 = \emptyset$, $\Gamma = \bar{\Gamma}_2 \cup \bar{\Gamma}_3$, $\Gamma_2 \cap \Gamma_3 = \emptyset$. The moduli: C_{ijkl} (elastic), e_{mij} (piezoelectric) and ϵ_{im} (dielectric) satisfy the usual symmetry conditions like in (8).

We make the following assumption:

$$\exists\, \eta > 0 \qquad \forall \quad \varepsilon \in E_s^3 \qquad C_{ijkl}(\vec{x})\,\varepsilon_{ij}\,\varepsilon_{kl} \geq \eta|\varepsilon|^2,$$
$$\exists\, \eta_1 > 0 \qquad \forall \quad \vec{a} \in R^3 \qquad \epsilon_{ij}(\vec{x})\,a_i\,a_j \geq \eta_1|\vec{a}|^2$$

for almost every $\vec{x} \in \Omega$. Here above E_s^3 is the space of symmetric matrices of third order.

3. Homogenization

Let the material functions C_{ijkl}, e_{mij}, ϵ_{im} be Y-periodic functions. As usual, Y is the typical periodic cell, say $Y = (0, Y_1) \times (0, Y_2) \times (0, Y_3)$. We set $C_{ijkl} = C_{ijkl}(\vec{\xi})$, $e_{ijm} = e_{ijm}(\vec{\xi})$ and $\epsilon_{im} = \epsilon_{im}(\vec{\xi})$. Here $\vec{\xi} = (\xi_1, \xi_2, \xi_3)$ is the local coordinate (or fast coordinate) and $\vec{x} = (x_1, x_2, x_3)$ is the global (or slow) coordinate; $\vec{\xi} = \vec{x}/\alpha$, and $\alpha = l/L$ is a small parameter, which represents the ratio between the characteristic length, l, of the periodic cell Y, and the characteristic length L of the whole domain.

The asymptotic expansion for the solution of the problem (17), (18), (with periodic functions) is sought in the form

$$u_i(\vec{x}) = u_i^0(\vec{x}, \vec{\xi}\,) + \alpha u_i^1(\vec{x}, \vec{\xi}\,) + \alpha^2 u_i^2(\vec{x}, \vec{\xi}\,) + \cdots\,,$$
$$\varphi(\vec{x}) = \varphi^0(\vec{x}, \vec{\xi}\,) + \alpha\varphi^1(\vec{x}, \vec{\xi}\,) + \alpha^2\varphi^2(\vec{x}, \vec{\xi}\,) + \cdots\,.$$

As usual in this kind of problem, the functions u_i^0 and φ^0 do not depend on $\vec{\xi}$. Due to the linearity of this problem and assuming both regularity of the inclusions shapes and smoothness in variation of the coefficients, we have (like in Refs. 7, 18 and 19):

$$u_i^1(\vec{x}, \vec{\xi}\,) = N_{ijp}(\vec{\xi}\,)\frac{\partial u_j^0}{\partial x_p} + \Phi_{ip}(\vec{\xi}\,)\frac{\partial \varphi^0}{\partial x_p},$$
$$\varphi^1(\vec{x}, \vec{\xi}\,) = M_{np}(\vec{\xi}\,)\frac{\partial u_n^0}{\partial x_p} + P_p(\vec{\xi}\,)\frac{\partial \varphi^0}{\partial x_p}.$$

This leads to seek an asymptotic expansion of the solution in the following form:

$$u_i = \sum_{q=0}^{\infty} \alpha^q \left[N^{(q)}_{ijk_1\ldots k_q}(\vec{\xi}\,)V_{j,k_1\ldots k_q}(\vec{x}) + \Phi^{(q)}_{ik_1\ldots k_q}(\vec{\xi}\,)S_{,k_1\ldots k_q}(\vec{x})\right],$$
$$\varphi = \sum_{q=0}^{\infty} \alpha^q \left[M^{(q)}_{nk_1\ldots k_q}(\vec{\xi}\,)V_{n,k_1\ldots k_q}(\vec{x}) + P^{(q)}_{k_1\ldots k_q}(\vec{\xi}\,)S_{,k_1\ldots k_q}(\vec{x})\right]. \tag{19}$$

The functions $\underline{N}^{(q)}$, $\underline{M}^{(q)}$, $\underline{\Phi}^{(q)}$ and $\underline{P}^{(q)}$ are local auxiliary Y-periodic functions, independent of $\vec{x}$ and satisfying the following conditions: $N_{ij}^{(0)} = \delta_{ij}$ (the Kronecker symbol), $\underline{P}^{(0)} = 1$, $\underline{M}^{(0)} = \underline{\Phi}^{(0)} = 0$. $\underline{N}^{(q)}$, $\underline{M}^{(q)}$, $\underline{\Phi}^{(q)}$, $\underline{P}^{(q)}$ are equal to zero for $q < 0$. Moreover, for unicity we require for the local auxiliary functions:

$$\left\langle \underline{N}^{(q)} \right\rangle = 0, \qquad \left\langle \underline{\mathrm{M}}^{(q)} \right\rangle = 0, \qquad \left\langle \underline{\Phi}^{(q)} \right\rangle = 0, \qquad \left\langle \underline{\mathrm{P}}^{(q)} \right\rangle = 0, \qquad q > 0\,, \tag{20}$$

where $\langle f \rangle$ stands for $1/|Y| \int_Y f\, dY$. The periodic conditions are

$$[\![\underline{N}^{(q)}]\!] = 0, \qquad [\![\underline{\mathrm{M}}^{(q)}]\!] = 0, \qquad [\![\underline{\Phi}^{(q)}]\!] = 0, \qquad [\![\underline{P}^{(q)}]\!] = 0 \tag{21}$$

and

$$\begin{aligned}
[\![C_{ijml} N^{(q)}_{mnk_1\ldots k_q,l} + e_{mij} M^{(q)}_{nk_1\ldots k_q,m} + C_{ijmk_q} N^{(q-1)}_{mnk_1\ldots k_{q-1}} + e_{k_q ij} M^{(q-1)}_{nk_1\ldots k_{q-1}}]\!] = 0\,,\\
[\![e_{iml} N^{(q)}_{mnk_1\ldots k_q,l} - \epsilon_{im} M^{(q)}_{nk_1\ldots k_q,m} + e_{imk_q} N^{(q-1)}_{mnk_1\ldots k_{q-1}} - \epsilon_{ik_q} M^{(q-1)}_{nk_1\ldots k_{q-1}}]\!] = 0\,,\\
[\![C_{ijml} \Phi^{(q)}_{mk_1\ldots k_q,l} + e_{mij} P^{(q)}_{k_1\ldots k_q,m} + C_{ijmk_q} \Phi^{(q-1)}_{mk_1\ldots k_{q-1}} + e_{k_q ij} P^{(q-1)}_{k_1\ldots k_{q-1}}]\!] = 0\,,\\
[\![\epsilon_{in} P^{(q)}_{k_1\ldots k_q,n} - e_{iml} \Phi^{(q)}_{mk_1\ldots k_q,l} + \epsilon_{ik_q} P^{(q-1)}_{k_1\ldots k_{q-1}} - e_{imk_q} \Phi^{(q-1)}_{mk_1\ldots k_{q-1}}]\!] = 0\,.
\end{aligned}$$

where $[\![u]\!]$ means the difference of values of the function "u" on opposite sides of Y. $V_n(\vec{x}) = \langle u_n(\vec{x}, \vec{\xi}) \rangle$ is called the averaged mechanical vector and $S(\vec{x}) = \langle \varphi(\vec{x}, \vec{\xi}) \rangle$ the averaged electrical potential.

We now substitute the expansions (19) into Eqs. (17), (18) and we collect the terms of same order α^q and after some manipulations, we obtain the following boundary value problems:

$$\begin{aligned}
&\sum_{q=0}^{\infty} \alpha^q \left[h^{(q)}_{ijnmk_1\ldots k_q} V_{n,mk_1\ldots k_q j}(\vec{x}) + r^{(q)}_{mijk_1\ldots k_q} S_{,mk_1\ldots k_q j}(\vec{x}) \right] + X_i = 0\,,\\
&\sum_{q=0}^{\infty} \alpha^q \left[t^{(q)}_{ijmk_1\ldots k_q} V_{j,mk_1\ldots k_q i}(\vec{x}) - s^{(q)}_{imk_1\ldots k_q} S_{,mk_1\ldots k_q i}(\vec{x}) \right] = 0\,.
\end{aligned} \tag{22}$$

$$\begin{aligned}
&\sum_{q=0}^{\infty} \alpha^q \left[N^{(q)}_{ijk_1\ldots k_q}(\vec{\xi}) V_{j,k_1\ldots k_q}(\vec{x}) + \Phi^{(q)}_{ik_1\ldots k_q}(\vec{\xi}) S_{,k_1\ldots k_q}(\vec{x}) \right] \big|_{\Gamma_0} = 0\,,\\
&\sum_{q=0}^{\infty} \alpha^q \left[M^{(q)}_{nk_1\ldots k_q}(\vec{\xi}) V_{n,k_1\ldots k_q}(\vec{x}) + P^{(q)}_{k_1\ldots k_q}(\vec{\xi}) S_{,k_1\ldots k_q}(\vec{x}) \right] \big|_{\Gamma_2} = \varphi^o\,,\\
&\sum_{q=0}^{\infty} \alpha^q \left[h^{(q)}_{ijnmk_1\ldots k_q} V_{n,mk_1\ldots k_q}(\vec{x}) + r^{(q)}_{mijk_1\ldots k_q} S_{,mk_1\ldots k_q}(\vec{x}) \right] n_j \big|_{\Gamma_1} = S_i^o\,,\\
&\sum_{q=0}^{\infty} \alpha^q \left[t^{(q)}_{ijmk_1\ldots k_q} V_{j,mk_1\ldots k_q}(\vec{x}) - s^{(q)}_{imk_1\ldots k_q} S_{,mk_1\ldots k_q}(\vec{x}) \right] n_i \big|_{\Gamma_3} = 0\,,
\end{aligned} \tag{23}$$

where $\underline{h}^{(q)}$, $\underline{t}^{(q)}$, $\underline{r}^{(q)}$ and $\underline{s}^{(q)}$ are constant tensorial functions (vanishing for $q < 0$). The expressions to determine these constants will be given in the next section.

To find the functions V_n and S, we seek for the solution of (22), (23) in the asymptotic expansions form:

$$V_n = \sum_{p=0}^{\infty} \alpha^p w_n^{\{p\}}, \qquad S = \sum_{p=0}^{\infty} \alpha^p y^{\{p\}} . \tag{24}$$

Putting (24) in (22) and (23) we obtain the following sequence of recurrent periodic boundary value problems with constant coefficients:

$$\begin{aligned} h^{(0)}_{ijmn} w^{\{p\}}_{n,mj} + r^{(0)}_{mij} y^{\{p\}}_{,mj} + X^{\{p\}}_i &= 0 , \\ t^{(0)}_{iml} w^{\{p\}}_{m,li} - s^{(0)}_{im} y^{\{p\}}_{,mi} + Y^{\{p\}} &= 0 , \qquad p = 0,1,2,\ldots \end{aligned} \tag{25}$$

$$\begin{aligned} w_i^{\{p\}}\big|_{\Gamma_0} &= u_i^{0\{p\}} , \\ y^{\{p\}}\big|_{\Gamma_2} &= \varphi^{\{p\}} , \\ (h^{(0)}_{ijnm} w^{\{p\}}_{n,m} + r^{(0)}_{mij} y^{\{p\}}_{,m}) n_j\big|_{\Gamma_1} &= S_i^{0\{p\}} , \\ (t^{(0)}_{iml} w^{\{p\}}_{m,l} - s^{(0)}_{im} y^{\{p\}}_{,m}) n_i\big|_{\Gamma_3} &= q^{0\{p\}} , \qquad p = 0,1,2,\ldots \end{aligned} \tag{26}$$

where

$$X_i^{\{p\}} = \sum_{q=1}^{p} \left[h^{(q)}_{ijmnk_1\ldots k_q} w^{\{p-q\}}_{m,nk_1\ldots k_q j} + r^{(q)}_{mijk_1\ldots k_q} y^{\{p-q\}}_{,mk_1\ldots k_q j} \right],$$

$$Y^{\{p\}} = \sum_{q=1}^{p} \left[t^{(q)}_{ijmk_1\ldots k_q} w^{\{p-q\}}_{j,mk_1\ldots k_q i} - s^{(q)}_{imk_1\ldots k_q} y^{\{p-q\}}_{,mk_1\ldots k_q i} \right],$$

$$u_i^{0\{p\}} = -\sum_{q=1}^{p} \left[N^{(q)}_{ijk_1\ldots k_q} w^{\{p-q\}}_{j,k_1\ldots k_q} + \Phi^{(q)}_{ik_1\ldots k_q} y^{\{p-q\}}_{,k_1\ldots k_q} \right] \big|_{\Gamma_0} ,$$

$$\varphi^{0\{p\}} = -\sum_{q=1}^{p} \left[M^{(q)}_{jk_1\ldots k_q} w^{\{p-q\}}_{j,k_1\ldots k_q} + P^{(q)}_{k_1\ldots k_q} y^{\{p-q\}}_{,k_1\ldots k_q} \right] \big|_{\Gamma_2} ,$$

$$S_i^{0\{p\}} = -\sum_{q=1}^{p} \left[h^{(q)}_{ijnmk_1\ldots k_q} w^{\{p-q\}}_{n,mk_1\ldots k_q} + r^{(q)}_{mijk_1\ldots k_q} y^{\{p-q\}}_{,mk_1\ldots k_q} \right] n_j\big|_{\Gamma_1} ,$$

$$q^{0\{p\}} = -\sum_{q=1}^{p} \left[t^{(q)}_{ijmk_1\ldots k_q} w^{\{p-q\}}_{j,mk_1\ldots k_q} - s^{(q)}_{ijk_1\ldots k_q} y^{\{p-q\}}_{,jk_1\ldots k_q} \right] n_j\big|_{\Gamma_3} ,$$

with $p > 0$, and

$$\begin{aligned} X_i^{\{0\}} &\equiv X_i , & Y^{\{0\}} &\equiv 0 , & u_i^{0\{0\}} &\equiv 0 , \\ \varphi^{0\{0\}} &\equiv \varphi^0 , & S_i^{0\{0\}} &\equiv S_i^0 , & q^{0\{0\}} &\equiv 0 . \end{aligned}$$

Finally, with the solution of the problems (25) and (26) the procedure of constructing the formal asymptotic solution of problem (17) and (18) is complete and we only need now, to find the auxiliary local functions and the constant tensors.

4. Computation of the constant tensors and of the local auxiliary functions

To obtain the constants $\underline{h}^{(q)}$, $\underline{t}^{(q)}$, $\underline{r}^{(q)}$ and $\underline{s}^{(q)}$ and the functions $\underline{N}^{(q+1)}$, $\underline{M}^{(q+1)}$, $\underline{\Phi}^{(q+1)}$ and $\underline{P}^{(q+1)}$ we solve the following problems with periodic boundary values

Problems $P_I^{(q+1,q)}$:

$$\begin{aligned}
&\left(C_{ijml}N^{(q+2)}_{mnk_1\ldots k_{q+2},l} + e_{mij}M^{(q+2)}_{nk_1\ldots k_{q+2},m}\right)_{,j} + \left(C_{ijmk_{q+2}}N^{(q+1)}_{mnk_1\ldots k_{q+1}} + e_{k_{q+2}ij}M^{(q+1)}_{nk_1\ldots k_{q+1}}\right)_{,j}\\
&\quad + C_{ik_{q+2}ml}N^{(q+1)}_{mnk_1\ldots k_{q+1},l} + e_{mik_{q+2}}M^{(q+1)}_{nk_1\ldots k_{q+1},m}\\
&\qquad + C_{ik_{q+2}mk_{q+1}}N^{(q)}_{mnk_1\ldots k_q} + e_{k_{q+1}ik_{q+2}}M^{(q)}_{nk_1\ldots k_q} = h^{(q)}_{ik_{q+2}nk_1\ldots k_{q+1}},\\
&\left(e_{iml}N^{(q+2)}_{mnk_1\ldots k_{q+2},l} - \epsilon_{im}M^{(q+2)}_{nk_1\ldots k_{q+2},m}\right)_{,i} + \left(e_{imk_{q+2}}N^{(q+1)}_{mnk_1\ldots k_{q+1}} - \epsilon_{ik_{q+2}}M^{(q+1)}_{nk_1\ldots k_{q+1}}\right)_{,i}\\
&\quad + e_{k_{q+2}ml}N^{(q+1)}_{mnk_1\ldots k_{q+1},l} - \epsilon_{k_{q+2}m}M^{(q+1)}_{nk_1\ldots k_{q+1},m} + e_{k_{q+2}mk_{q+1}}N^{(q)}_{mnk_1\ldots k_q}\\
&\qquad - \epsilon_{k_{q+2}k_{q+1}}M^{(q)}_{nk_1\ldots k_q} = t^{(q)}_{k_{q+2}nk_1\ldots k_{q+1}}, \qquad q=-1,0,1,\ldots
\end{aligned} \tag{27}$$

$$\begin{aligned}
h^{(q)}_{ik_{q+2}nk_1\ldots k_{q+1}} &= \langle C_{ik_{q+2}ml}N^{(q+1)}_{mnk_1\ldots k_{q+1},l} + C_{ik_{q+2}mk_{q+1}}N^{(q)}_{mnk_1\ldots k_q}\rangle\\
&\quad + \langle e_{mik_{q+2}}M^{(q+1)}_{nk_1\ldots k_{q+1},m} + e_{k_{q+1}ik_{q+2}}M^{(q)}_{nk_1\ldots k_q}\rangle,\\
t^{(q)}_{k_{q+2}nk_1\ldots k_{q+1}} &= \langle e_{k_{q+2}ml}N^{(q+1)}_{mnk_1\ldots k_{q+1},l} + e_{k_{q+2}mk_{q+1}}N^{(q)}_{mnk_1\ldots k_q}\rangle\\
&\quad - \langle \epsilon_{k_{q+2}l}M^{(q+1)}_{nk_1\ldots k_{q+1},l} + \epsilon_{k_{q+2}k_{q+1}}M^{(q)}_{nk_1\ldots k_q}\rangle, \qquad q=0,1,2\ldots
\end{aligned} \tag{28}$$

Problems $P_{II}^{(q+1,q)}$:

$$\begin{aligned}
&\left(C_{ijml}\Phi^{(q+2)}_{mk_1\ldots k_{q+2},l} + e_{mij}P^{(q+2)}_{k_1\ldots k_{q+2},m}\right)_{,j} + \left(C_{ijmk_{q+2}}\Phi^{(q+1)}_{mk_1\ldots k_{q+1}} + e_{k_{q+2}ij}P^{(q+1)}_{k_1\ldots k_{q+1}}\right)_{,j}\\
&\quad + C_{ik_{q+2}ml}\Phi^{(q+1)}_{mk_1\ldots k_{q+1},l} + e_{mik_{q+2}}P^{(q+1)}_{k_1\ldots k_{q+1},m}\\
&\qquad + C_{ik_{q+2}mk_{q+1}}\Phi^{(q)}_{mk_1\ldots k_q} + e_{k_{q+1}ik_{q+2}}P^{(q)}_{k_1\ldots k_q} = r^{(q)}_{k_{q+1}ik_{q+2}k_1\ldots k_q},
\end{aligned}$$

$$\left(\epsilon_{in}P^{(q+2)}_{k_1\ldots k_{q+2},n} - e_{iml}\Phi^{(q+2)}_{mk_1\ldots k_{q+2},l}\right)_{,i} + \left(\epsilon_{ik_{q+2}}P^{(q+1)}_{k_1\ldots k_{q+1}} - e_{imk_{q+2}}\Phi^{(q+1)}_{mk_1\ldots k_{q+1}}\right)_{,i}$$
$$+ \epsilon_{k_{q+2}n}P^{(q+1)}_{k_1\ldots k_{q+1},n} - e_{k_{q+2}ml}\Phi^{(q+1)}_{mk_1\ldots k_{q+1},l} + \epsilon_{k_{q+2}k_{q+1}}P^{(q)}_{k_1\ldots k_q}$$
$$- e_{k_{q+2}mk_{q+1}}\Phi^{(q)}_{mk_1\ldots k_q} = s^{(q)}_{k_{q+2}k_1\ldots k_{q+1}}, \qquad q = -1, 0, 1, \ldots \quad (29)$$

$$r^{(q)}_{k_{q+1}ik_{q+2}k_1\ldots k_q} = \langle C_{ik_{q+2}ml}\Phi^{(q+1)}_{mk_1\ldots k_{q+1},l} + C_{ik_{q+2}mk_{q+1}}\Phi^{(q)}_{mk_1\ldots k_q}\rangle$$
$$+ \langle e_{mik_{q+2}}P^{(q+1)}_{k_1\ldots k_{q+1},m} + e_{k_{q+1}ik_{q+2}}P^{(q)}_{k_1\ldots k_q}\rangle,$$
$$s^{(q)}_{k_{q+2}k_1\ldots k_{q+1}} = \langle \epsilon_{k_{q+2}l}P^{(q+1)}_{k_1\ldots k_{q+1},l} + \epsilon_{k_{q+2}k_{q+1}}P^{(q)}_{k_1\ldots k_q}\rangle$$
$$- \langle e_{k_{q+2}ml}\Phi^{(q+1)}_{mk_1\ldots k_{q+1},l} + e_{k_{q+2}mk_{q+1}}\Phi^{(q)}_{mk_1\ldots k_q}\rangle. \qquad q = 0, 1, 2\ldots \quad (30)$$

To obtain $N^{(1)}_{pmn}$, $M^{(1)}_{mn}$ and $h^{(0)}_{ijmn}$, $t^{(0)}_{imn}$ we start to solve the first problem $P_I^{(1,0)}$, *i.e.*, from (27) and (28) we consider:

Problems $P_I^{(1,0)}$*:*

$$\left(C_{ijmn} + C_{ijpq}N^{(1)}_{pmn,q} + e_{pij}M^{(1)}_{mn,p}\right)_{,j} = 0,$$
$$\left(e_{imn} + e_{ipq}N^{(1)}_{pmn,q} - \epsilon_{ip}M^{(1)}_{mn,p}\right)_{,i} = 0. \quad (31)$$

$$h^{(0)}_{ijmn} = \langle C_{ijmn} + C_{ijpq}N^{(1)}_{pmn,q} + e_{pij}M^{(1)}_{mn,p}\rangle,$$
$$t^{(0)}_{imn} = \langle e_{imn} + e_{ipq}N^{(1)}_{pmn,q} - \epsilon_{ip}M^{(1)}_{mn,p}\rangle. \quad (32)$$

Analogously $\Phi^{(1)}_{mn}$, $P^{(1)}_n$ and $r^{(0)}_{nij}$, $s^{(0)}_{in}$ are computed by means of the periodic problem $P_{II}^{(1,0)}$; *i.e.*, from (29) and (30) we have:

Problems $P_{II}^{(1,0)}$*:*

$$\left(e_{nij} + C_{ijpq}\Phi^{(1)}_{pn,q} + e_{pij}P^{(1)}_{n,p}\right)_{,j} = 0, \qquad \left(\epsilon_{in} - e_{ipq}\Phi^{(1)}_{pn,q} + \epsilon_{ip}P^{(1)}_{n,p}\right)_{,i} = 0. \quad (33)$$

$$r^{(0)}_{nij} = \langle e_{nij} + e_{pij}P^{(1)}_{n,p} + C_{ijpq}\Phi^{(1)}_{pn,q}\rangle, \qquad s^{(0)}_{in} = \langle \epsilon_{in} + \epsilon_{ip}P^{(1)}_{n,p} - e_{ipq}\Phi^{(1)}_{pn,q}\rangle. \quad (34)$$

Equations (31) and (33) give respectively, the system of equations for finding $N^{(1)}_{pmn}$ and $M^{(1)}_{mn}, \Phi^{(1)}_{pn}$ and $P^{(1)}_n$ taking in account (20) and (21). These problems are strong formulations of the local problems, and are meaningful if the periodic solutions are smooth

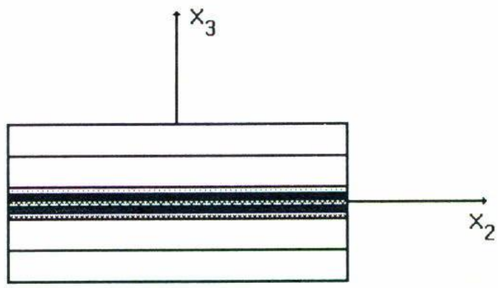

FIGURE 1. Series connection.

enough. However, this regularity may be significantly weakened provided that one use a weak or variational formulations, as in Refs. 7 or 19.

Indeed, in the case of laminate composite with axis of symmetry in the direction normal to the layers, the periodic local functions $\underline{N}^{(q)}$, $\underline{M}^{(q)}$, $\underline{\Phi}^{(q)}$, $\underline{P}^{(q)}$ and the material functions C_{ijmn}, e_{imn} and ϵ_{in} will only depend on one variable. For this kind of media we prove $\underline{t}^{(0)} = \underline{r}^{(0)}$. Consequently, the first problem ($p = 0$) of the recurrent sequence of boundary value problems (25) and (26) is a typical boundary value problem for linear piezoelectricity in a homogeneous medium and has the form

$$C^h_{ijmn}\, w^{(0)}_{n,mj} + e^h_{mij}\, y^{(0)}_{,mj} + X_i = 0, \qquad e^h_{iml}\, w^{(0)}_{m,li} - \epsilon^h_{im}\, y^{(0)}_{,mi} = 0, \tag{35}$$

$$w^{(0)}_i\big|_{\Gamma_0} = 0, \qquad y^{(0)}\big|_{\Gamma_2} = \varphi^{(0)}, \qquad \sigma^h_{ij} n_j\big|_{\Gamma_1} = S^0_i, \qquad D^h_i n_i\big|_{\Gamma_3} = 0. \tag{36}$$

where

$$\sigma^h_{ij} = C^h_{ijnm}\, w^{(0)}_{n,m}(\vec{x}) + e^h_{mij}\, y^{(0)}_{,m}(\vec{x}), \qquad D^h_i = e^h_{ijm}\, w^{(0)}_{j,m}(\vec{x}) - \epsilon^h_{im}\, y^{(0)}_{,m}(\vec{x})$$

$$C^h_{ijmn} = h^{(0)}_{ijmn}, \qquad e^h_{mij} = t^{(0)}_{mij} = r^{(0)}_{mij}, \qquad \epsilon^h_{im} = s^{(0)}_{im},$$

where the effective constants coefficients: C^h_{ijmn} (elastic), e^h_{mij} (piezoelectric) and ϵ^h_{im} (dielectric) are given by (32) and (34).

5. LOCAL PROBLEMS IN A LAMINATED MEDIUM

Let us now particularize our study to a laminated piezoelectric composite, *i.e.*, made of cells which are periodically along the axis x_3 (Fig. 1), the axis x_1 is in the direction perpendicular to the plane of the drawing. Each cell may be made of piezoelectric laminates. For our problem the elasticity modulus tensor $\underline{C}$, the piezoelectric modulus tensor $\underline{e}$, and the dielectric modulus tensor $\underline{\epsilon}$ are periodic functions of the coordinate x_3 and they don't depend on x_1 and x_2.

We then introduce the fast variable in the following form:

$$\xi \equiv \xi_3 = \frac{x_3}{\alpha}, \tag{37}$$

where α is the small parameter, representing the ratio between the characteristic length of the periodic cell l and the characteristic length of the body L. In the engineering literature these kinds of layer's distribution (Fig. 1) are known as "connectivity in series" and the case corresponding to $\xi \equiv \xi_\beta = x_\beta/\alpha$ where $\beta = 1$ or 2 is called "connectivity in parallel", see for instance Ref. 24.

Using the results obtained in the above sections we seek the solution of this heterogeneous problem in the following form:

$$u_i = \sum_{q=0}^{\infty} \alpha^q \sum_{p=0}^{q} \left[N^{(p)}_{ijk_1\ldots k_p}(\xi)\, w^{\{q-p\}}_{j,k_1\ldots k_p}(\vec{x}) + \Phi^{(p)}_{ik_1\ldots k_p}(\xi)\, y^{\{q-p\}}_{,k_1\ldots k_p}(\vec{x}) \right],$$

$$\varphi = \sum_{q=0}^{\infty} \alpha^q \sum_{p=0}^{q} \left[M^{(p)}_{mk_1\ldots k_p}(\xi)\, w^{\{q-p\}}_{m,k_1\ldots k_q}(\vec{x}) + P^{(p)}_{k_1\ldots k_p}(\xi)\, y^{\{q-p\}}_{,k_1\ldots k_p}(\vec{x}) \right]. \tag{38}$$

We require the periodic local auxiliary functions to verify

$$\left\langle \underline{N}^{(q)} \right\rangle = 0, \qquad \left\langle \underline{\mathrm{M}}^{(q)} \right\rangle = 0, \qquad \left\langle \underline{\Phi}^{(q)} \right\rangle = 0, \qquad \left\langle \underline{\mathrm{P}}^{(q)} \right\rangle = 0, \qquad q > 0, \tag{39}$$

where $\langle f \rangle = \int_0^1 f(\xi)d\xi$. Moreover, $\underline{N}^{(q)}(\xi)$, $\underline{M}^{(q)}(\xi)$, $\underline{\Phi}^{(q)}(\xi)$ and $\underline{P}^{(q)}(\xi)$ are 1-periodic in ξ. And

$$\left[\!\left[C_{i3m3}\left(N^{(q)}_{mnk_1\ldots k_q}\right)' + e_{3i3}\left(M^{(q)}_{nk_1\ldots k_q}\right)' + C_{i3mk_q} N^{(q-1)}_{mnk_1\ldots k_{q-1}} + e_{k_q i3} M^{(q-1)}_{nk_1\ldots k_{q-1}} \right]\!\right] = 0\,,$$

$$\left[\!\left[e_{3m3}\left(N^{(q)}_{mnk_1\ldots k_q}\right)' - \epsilon_{3m}\left(M^{(q)}_{nk_1\ldots k_q}\right)' + e_{3mk_q} N^{(q-1)}_{mnk_1\ldots k_{q-1}} - \epsilon_{3k_q} M^{(q-1)}_{nk_1\ldots k_{q-1}} \right]\!\right] = 0\,,$$

$$\left[\!\left[C_{i3m3}\left(\Phi^{(q)}_{mk_1\ldots k_q}\right)' + e_{3i3}\left(P^{(q)}_{k_1\ldots k_q}\right)' + C_{i3mk_q} \Phi^{(q-1)}_{mk_1\ldots k_{q-1}} + e_{k_q i3} P^{(q-1)}_{k_1\ldots k_{q-1}} \right]\!\right] = 0\,,$$

$$\left[\!\left[C_{i3m3}\left(\Phi^{(q)}_{mk_1\ldots k_q}\right)' + e_{3i3}\left(P^{(q)}_{k_1\ldots k_q}\right)' + C_{i3mk_q} \Phi^{(q-1)}_{mk_1\ldots k_{q-1}} + e_{k_q i3} P^{(q-1)}_{k_1\ldots k_{q-1}} \right]\!\right] = 0\,.$$

Understanding $(.)'$ as $d(.)/d\xi$ and $[\![u]\!] = 0$ as $u(0) = u(1)$.

In order to obtain the corresponding $N^{(1)}_{pmn}$, $M^{(1)}_{mn}$ and $h^{(0)}_{ijmn}$, $t^{(0)}_{imn}$ we have to solve the system of problems $P_I^{(1,0)}$.

Problems $P_I^{(1,0)}$:

$$\left(C_{i3mn} + C_{i3p3}\left(N^{(1)}_{pmn}\right)' + e_{3i3}\left(M^{(1)}_{mn}\right)' \right)' = 0\,,$$

$$\left(e_{3mn} + e_{3p3}\left(N^{(1)}_{pmn}\right)' - \epsilon_{33}\left(M^{(1)}_{mn}\right)' \right)' = 0\,. \tag{40}$$

$$C^h_{ijmn} = \left\langle C_{ijmn} + C_{ijp3}\left(N^{(1)}_{pmn}\right)' + e_{3ij}\left(M^{(1)}_{mn}\right)' \right\rangle,$$

$$e^h_{imn} = \left\langle e_{imn} + e_{ip3}\left(N^{(1)}_{pmn}\right)' - \epsilon_{i3}\left(M_{mn}\right)' \right\rangle. \tag{41}$$

Analogously $\Phi^{(1)}_{mn}$, $P^{(1)}_n$ and $s^{(0)}_{in}$ are obtained from the problems $P_{II}^{(1,0)}$.

Problems $P_{II}^{(1,0)}$:

$$\left(e_{ni3} + C_{i3p3}\left(\Phi_{pn}^{(1)}\right)' + e_{3i3}\left(P_n^{(1)}\right)'\right)' = 0,$$
$$\left(\epsilon_{3n} - e_{3p3}\left(\Phi_{pn}^{(1)}\right)' + \epsilon_{33}\left(P_n^{(1)}\right)'\right)' = 0. \tag{42}$$

$$\epsilon_{in}^h = \left\langle \epsilon_{in} + \epsilon_{i3}\left(P_n^{(1)}\right)' - e_{ip3}\left(\Phi_{pn}^{(1)}\right)'\right\rangle, \tag{43}$$

where, $C_{ijmn}^h = h_{ijmn}^{(0)}$ (elastic), $e_{mij}^h = t_{mij}^{(0)} = r_{mij}^{(0)}$ (piezoelectric), $\epsilon_{im}^h = s_{im}^{(0)}$ (dielectric) are the effective moduli.

By solving the system of ordinary differential equations (40) and (42) we obtain, respectively,

$$\left(N_{mnk}^{(1)}\right)' = \bar{C}_{i3m3}^{-1}\left(-C_{i3nk} + A_{i3nk} + e_{3i3}\epsilon_{33}^{-1}B_{3nk} - e_{3i3}\epsilon_{33}^{-1}e_{3nk}\right),$$
$$\left(M_{nk}^{(1)}\right)' = \bar{\epsilon}_{33}^{-1}\left(e_{3nk} + e_{3m3}C_{m3i3}^{-1}A_{i3nk} - e_{3m3}C_{m3i3}^{-1}C_{i3nk} - B_{3nk}\right), \tag{44}$$
$$\left(\phi_{mk}^1\right)' = \bar{C}_{i3m3}^{-1}\left[-e_{ki3} + G_{ki3} + e_{3i3}\,\epsilon_{33}^{-1}(\epsilon_{3k} + H_{3k})\right],$$
$$\left(P_k^{(1)}\right)' = \bar{\epsilon}_{33}^{-1}\left[e_{3m3}C_{m3i3}^{-1}(-e_{ki3} + G_{ki3}) - \epsilon_{3k} - H_{3k}\right], \tag{45}$$

where

$$\bar{C}_{i3mp} = C_{i3mp} + e_{3i3}\,\epsilon_{33}^{-1}e_{3mp}\,, \qquad \bar{\epsilon}_{pq} = \epsilon_{pq} + e_{pm3}C_{m3i3}^{-1}e_{qi3}\,.$$

Taking the averages of the Eqs. (44) and (45) and considering that $\langle (N_{mnk}^{(1)})'\rangle = 0$, $\langle (M_{nk}^{(1)})'\rangle = 0$, $\langle (\phi_{mk}^{(1)})'\rangle = 0$ and $\langle (P_k^{(1)})'\rangle = 0$, we obtain

$$A_{i3nk} = \left\{\langle \bar{C}_{i3m3}^{-1}\rangle + \langle \bar{C}_{q3m3}^{-1}e_{3q3}\epsilon_{33}^{-1}\rangle\langle \bar{\epsilon}_{33}^{-1}\rangle^{-1}\langle C_{p3i3}^{-1}e_{3p3}\bar{\epsilon}_{33}^{-1}\rangle\right\}^{-1}$$
$$\times\left\{\langle \bar{C}_{i3m3}^{-1}e_{3q3}\epsilon_{33}^{-1}\rangle\langle \bar{\epsilon}_{33}^{-1}\rangle^{-1}\langle \bar{\epsilon}_{33}^{-1}(e_{3p3}C_{p3q3}^{-1}C_{q3nk} - e_{3nk})\rangle + \langle \bar{C}_{q3m3}^{-1}\bar{C}_{q3nk}\rangle\right\}, \tag{46}$$

$$B_{3nk} = \left\{\langle \bar{\epsilon}_{33}^{-1}\rangle + \langle C_{p3i3}^{-1}e_{3p3}\bar{\epsilon}_{33}^{-1}\rangle\langle \bar{C}_{i3m3}^{-1}\rangle^{-1}\langle \bar{C}_{q3m3}^{-1}e_{3q3}\epsilon_{33}^{-1}\rangle\right\}^{-1}$$
$$\times\left\{\langle \bar{\epsilon}_{33}^{-1}(-e_{3p3}C_{p3q3}C_{q3nk} + e_{3nk})\rangle + \langle C_{p3i3}^{-1}e_{3p3}\bar{\epsilon}_{33}^{-1}\rangle\langle \bar{C}_{i3m3}^{-1}\rangle^{-1}\langle \bar{C}_{q3m3}^{-1}\bar{C}_{q3nk}\rangle\right\}, \tag{47}$$

$$G_{ki3} = \left\{\langle \bar{C}_{i3m3}^{-1}\rangle + \langle \bar{C}_{q3m3}^{-1}e_{3q3}\epsilon_{33}^{-1}\rangle\langle \bar{\epsilon}_{33}^{-1}\rangle^{-1}\langle C_{p3i3}^{-1}e_{3p3}\bar{\epsilon}_{33}^{-1}\rangle\right\}^{-1}$$
$$\times\left\{\langle \bar{C}_{i3m3}^{-1}(e_{ki3} - e_{3i3}\epsilon_{33}^{-1}\epsilon_{3k})\rangle + \langle \bar{C}_{q3m3}^{-1}e_{3q3}\epsilon_{33}^{-1}\rangle\langle \bar{\epsilon}_{33}^{-1}\rangle^{-1}\langle \bar{\epsilon}_{33}^{-1}\bar{\epsilon}_{3k}\rangle\right\}, \tag{48}$$

$$H_{3k} = \Big\{ \langle \bar{\epsilon}_{33}^{-1} \rangle + \langle C_{p3i3}^{-1} e_{3p3} \bar{\epsilon}_{33}^{-1} \rangle \langle \bar{C}_{i3m3}^{-1} \rangle^{-1} \langle \bar{C}_{q3m3}^{-1} e_{3q3} \epsilon_{33}^{-1} \rangle \Big\}^{-1}$$
$$\times \Big\{ \langle \bar{\epsilon}_{33}^{-1} \bar{\epsilon}_{3k} \rangle + \langle C_{i3q3}^{-1} e_{3p3} \bar{\epsilon}_{33}^{-1} \rangle \langle \bar{C}_{i3m3}^{-1} \rangle^{-1} \langle \bar{C}_{q3m3}^{-1} (-e_{ki3} + e_{3i3} \epsilon_{33}^{-1} \epsilon_{3k}) \rangle \Big\}, \quad (49)$$

and supposing now that $\langle N_{mnk}^{(1)} \rangle = 0$, $\langle M_{nk}^{(1)} \rangle = 0$, $\langle \phi_{mk}^{(1)} \rangle = 0$ and $\langle P_k^{(1)} \rangle = 0$, the local functions $N^{(1)}$, $M^{(1)}$, $\phi^{(1)}$ and $P^{(1)}$ are given by

$$N_{mnk}^{(1)} = D_{mnk}(\xi) - \langle D_{mnk} \rangle, \qquad M_{nk}^{(1)} = T_{nk}(\xi) - \langle T_{nk} \rangle, \quad (50)$$

$$\phi_{mk}^{(1)} = R_{mk}(\xi) - \langle R_{mk} \rangle, \qquad P_k^{(1)} = Q_k(\xi) - \langle Q_k \rangle, \quad (51)$$

where

$$D_{mnk}(\xi) = \int_0^\xi d\eta\, \bar{C}_{i3m3}^{-1}(\eta) \Big\{ - C_{i3nk}(\eta) + A_{i3nk} + e_{3i3}(\eta) \epsilon_{33}^{-1}(\eta) B_{3nk}$$
$$- e_{3i3}(\eta) \epsilon_{33}^{-1} e_{3nk}(\eta) \Big\}, \quad (52)$$

$$T_{nk}(\xi) = \int_0^\xi d\eta\, \bar{\epsilon}_{33}^{-1}(\eta) \Big\{ e_{3nk}(\eta) + e_{3m3}(\eta) C_{m3i3}^{-1}(\eta) A_{i3nk}$$
$$- e_{3m3}(\eta) C_{m3i3}^{-1}(\eta) C_{i3nk}(\eta) - B_{3nk} \Big\}, \quad (53)$$

$$R_{mk}(\xi) = \int_0^\xi \bar{C}_{i3m3}^{-1}(\eta) \Big\{ e_{3i3}(\eta) \epsilon_{33}^{-1}(\eta) [\epsilon_{3k}(\eta) + H_{3k}] - e_{ki3}(\eta) + G_{ki3} \Big\}\, d\eta, \quad (54)$$

$$Q_k(\xi) = \int_0^\xi \bar{\epsilon}_{33}^{-1}(\eta) e \Big\{_{3m3}(\eta) C_{m3i3}^{-1}(\eta) - [e_{ki3}(\eta) + G_{ki3} -] \epsilon_{3k}(\eta) - H_{3k} \Big\}\, d\eta. \quad (55)$$

Finally, we obtain the elastic, piezoelectric and dielectric effective coefficients by the following expressions

$$C_{ijnk}^h = \Big\langle C_{ijnk} + C_{ijp3} \bar{C}_{q3p3}^{-1} \Big[- C_{q3nk} + A_{q3nk} + e_{3q3} \epsilon_{33}^{-1} (B_{3nk} - e_{3nk}) \Big]$$
$$+ e_{3ij} \bar{\epsilon}_{33}^{-1} \Big[e_{3nk} + e_{3p3} C_{p3q3}^{-1} (A_{i3nk} - C_{q3nk}) - B_{3nk} \Big] \Big\rangle, \quad (56)$$

$$e_{ink}^h = \Big\langle e_{ink} + e_{ip3} \bar{C}_{q3p3}^{-1} \Big[- C_{q3nk} + A_{q3nk} + e_{3q3} \epsilon_{33}^{-1} (B_{3nk} - e_{3nk}) \Big]$$
$$- \epsilon_{i3} \bar{\epsilon}_{33}^{-1} \Big[e_{3nk} + e_{3p3} C_{p3q3}^{-1} (A_{i3nk} - C_{q3nk}) - B_{3nk} \Big] \Big\rangle, \quad (57)$$

$$\epsilon_{ik}^h = \Big\langle \epsilon_{ik} - e_{im3} \bar{C}_{p3m3}^{-1} [- e_{kp3} + G_{kp3} + e_{3p3} \epsilon_{33}^{-1} (\epsilon_{3k} - H_{3k})]$$
$$+ \epsilon_{i3} \bar{\epsilon}_{33}^{-1} [e_{3m3} C_{m3p3}^{-1} (-e_{kp3} + G_{kp3}) - \epsilon_{3k} + H_{3k}] \Big\rangle. \quad (58)$$

We see that $C_{ijkl}^h = C_{klij}^h = C_{jikl}^h = C_{ijlk}^h$, $\quad e_{kij}^h = e_{kji}^h$, $\quad \epsilon_{ij}^h = \epsilon_{ji}^h$.

6. Piezoelectric laminates with hexagonal symmetry

Connectivity in series

Let's consider a piezoelectric laminated in which the periodic cell is composed by piezoelectric layers with hexagonal symmetry (6mm); such symmetry class involves crystals with one sixfold axis of symmetry, taken as x_3 axis, and six symmetry plane parallel to the axis, see for instance Ref. 25. These materials are characterized by the following independent constants: five elastic constants $[C_{1111} = C_{2222}, C_{1122}, C_{1133} = C_{2233}, C_{3333}, C_{2323} = C_{1313}, C_{1212} = \frac{1}{2}C_{1111} - C_{1122})]$, three piezoelectric constants $(e_{311} = e_{322}, e_{333}, e_{113} = e_{223})$, and two dielectric constants $(\epsilon_{11} = \epsilon_{22}, \epsilon_{33})$.

Using the expressions (56)–(58) the effective coefficients for this material are:

Elastic effective constants

$$\begin{aligned}
C^h_{1111} = C^h_{2222} &= \langle C_{1111}\rangle + \langle \bar{C}^{-1}_{3333}(C_{1133} + e_{311}\,\epsilon^{-1}_{33} e_{333})(A_{1133} - C_{1133})\rangle \\
&\quad + \langle \bar{\epsilon}^{-1}_{33}(C_{1133}C^{-1}_{3333}\, e_{333} - e_{311})(B_{311} - e_{311})\rangle, \\
C^h_{1122} &= \langle C_{1122}\rangle + \langle \bar{C}^{-1}_{3333}(C_{1133} + e_{311}\,\epsilon^{-1}_{33}\, e_{333})(A_{1133} - C_{1133})\rangle \\
&\quad + \langle \bar{\epsilon}^{-1}_{33}(C_{1133}C^{-1}_{3333}\, e_{333} - e_{311})(B_{311} - e_{311})\rangle, \\
C^h_{1133} &= \langle C_{1133}\rangle + \langle \bar{C}^{-1}_{3333}(C_{1133} + e_{311}\,\epsilon^{-1}_{33}\, e_{333})(A_{3333} - C_{3333})\rangle \\
&\quad + \langle \bar{\epsilon}^{-1}_{33}(C_{1133}C^{-1}_{3333}\, e_{333} - e_{311})(B_{333} - e_{333})\rangle, \\
C^h_{3333} &= A_{3333}, \qquad C^h_{1313} = C^h_{2323} = \langle C^{-1}_{2323}\rangle^{-1}, \qquad C^h_{1212} = \frac{1}{2}(C^h_{1111} - C^h_{1122}).
\end{aligned} \tag{59}$$

Piezoelectric effective constants

$$e^h_{113} = e^h_{223} = \langle C^{-1}_{1313}\rangle^{-1}\langle e_{113}C^{-1}_{1313}\rangle, \qquad e^h_{311} = e^h_{322} = B_{311}, \qquad e^h_{333} = B_{333}. \tag{60}$$

Dielectric effective constants

$$\epsilon^h_{11} = \epsilon^h_{22} = \langle \epsilon_{11}\rangle + \langle e_{113}\bar{C}^{-1}_{1313}(G_{113} - e_{113})\rangle, \qquad \epsilon^h_{33} = H_{33}. \tag{61}$$

where A_{3311}, A_{3333}, B_{311}, B_{333}, G_{113} and H_{33} can be determined by using of Eqs. (46)–(49) respectively. As we can see, there exist five independent elastic effective constants $[C^h_{1111} = C^h_{2222}, C^h_{1122}, C^h_{1133} = C^h_{2233}, C^h_{3333}, C^h_{2323} = C^h_{1313}, C^h_{1212} = \frac{1}{2}(C^h_{1111} - C^h_{1122})]$ given by (59), three piezoelectric effective constants $(e^h_{311} = e^h_{322}, e^h_{333}, e^h_{113} = e^h_{223})$ given by (60), and two dielectric effective constants $(\epsilon^h_{11} = \epsilon^h_{22}, \epsilon^h_{33})$ given by (61). Therefore, we conclude that the symmetry of piezocomposite laminated materials with periodic cells in series connection is conserved in the homogenized piezoelectric medium.

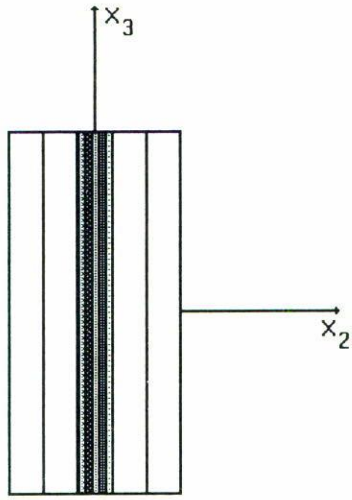

FIGURE 2. Parallel connection.

Connectivity in parallel

We assume that the laminated medium possesses the same periodic properties with hexagonal symmetry (6 mm) as in the above example, but the cells distribution are periodically along the axis x_2. The axis symmetry of each layer are parallel to each other and the x_2-axis perpendicular to layering as in Fig. 2. Then by using the formulae (56)–(58) (exchanging in these expressions the subscripts 3 by 2) we obtain the following effective coefficients:

Elastic effective constants

$$\begin{aligned}
C^h_{1111} &= \langle C_{1111}\rangle - \langle C^2_{1122}C^{-1}_{2222}\rangle + \langle C_{1122}C^{-1}_{2222}\rangle^2\langle C^{-1}_{2222}\rangle^{-1},\\
C^h_{1122} &= \langle C_{1122}C^{-1}_{2222}\rangle\langle C^{-1}_{2222}\rangle^{-1},\\
C^h_{1133} &= \langle C_{1133}\rangle - \langle C_{1122}C^{-1}_{2222}C_{2233}\rangle + \langle C_{1122}C^{-1}_{2222}\rangle\langle C^{-1}_{2222}\rangle^{-1}\langle C^{-1}_{2222}C_{2233}\rangle,\\
C^h_{2222} &= \langle C^{-1}_{2222}\rangle^{-1},\\
C^h_{2233} &= \langle C^{-1}_{2222}\rangle^{-1}\langle C_{2233}C^{-1}_{2222}\rangle,\\
C^h_{3333} &= \langle C_{3333}\rangle - \langle C^2_{2233}C^{-1}_{2222}\rangle + \langle C_{2233}C^{-1}_{2222}\rangle^2\langle C^{-1}_{2222}\rangle^{-1},\\
C^h_{2323} &= \langle C^{-1}_{2323}\rangle^{-1} + \langle \epsilon^{-1}_{22}e^2_{223}\rangle - \langle \epsilon^{-1}_{22}e^2_{223}\rangle^2,\\
C^h_{1313} &= \langle C_{1313}\rangle,\\
C^h_{1212} &= \langle C^{-1}_{1212}\rangle^{-1}. \qquad (62)
\end{aligned}$$

Piezoelectric effective constants

$$
\begin{aligned}
e^h_{113} &= \langle e_{113}\rangle,\\
e^h_{311} &= \langle e_{311}\rangle + \langle e_{322}C^{-1}_{2222}\rangle\langle C^{-1}_{2222}\rangle^{-1}\langle C^{-1}_{2222}C_{2211}\rangle - \langle e_{322}C^{-1}_{2222}C_{2211}\rangle,\\
e^h_{333} &= \langle e_{333}\rangle - \langle e_{322}C^{-1}_{2222}\rangle\langle C^{-1}_{2222}\rangle^{-1}\langle C^{-1}_{2222}C_{2211}\rangle - \langle e_{322}C^{-1}_{2222}C_{2211}\rangle,\\
e^h_{322} &= \langle e_{322}C^{-1}_{2222}\rangle\langle C^{-1}_{2222}\rangle^{-1},\\
e^h_{223} &= \langle e_{223}\epsilon^{-1}_{22}\rangle\langle \epsilon^{-1}_{22}\rangle^{-1}.
\end{aligned}
\tag{63}
$$

Dielectric effective constants

$$
\begin{aligned}
&\epsilon^h_{11} = \langle \epsilon_{11}\rangle, \qquad \epsilon^h_{22} = \langle \epsilon^{-1}_{22}\rangle^{-1},\\
&\epsilon^h_{33} = \langle \epsilon_{33}\rangle + \langle e^2_{322}C^{-1}_{2222}\rangle - \langle e_{322}C^{-1}_{2222}\rangle^2\langle C^{-1}_{2222}\rangle^{-1}.
\end{aligned}
\tag{64}
$$

As we can see, there exist nine independent elastic effective constants given by (59), five piezoelectric effective constants (60), and three dielectric effective constants (61). Therefore we conclude, taking into account the general classification for homogeneous piezoelectric materials; see for instance in Refs. 1 or 24, that if we have periodic cells composed by piezoelectric materials' layers with hexagonal symmetry (6 mm), but connected in parallel, the corresponding homogenized material will like a piezoelectric material with orthorhombic symmetry (2 mm).

7. Applications to transducers. Improvement of physical characteristics

Piezoelectric ceramic/polymer composites have become attractive candidates for use in transducer for underwater and biomedical imaging applications. Their low stiffness and low density lead to better matching of the acoustic impedance with the water, than in PZT ceramics.

In order to show an application of these piezocomposite materials we will consider the case of both connectivity, *i.e.*, parallel and series connection where each periodic cell consists only of two different homogeneous phases. The ceramic phase is a piezoelectric with hexagonal symmetry and the polymer phase is an isotropic homogeneous medium which is piezoelectrically inactive. The elastic and dielectric constants of the ceramic phase will be distinguished from those of the polymer phase by the superscripts E and S, respectively. Moreover, by utilizing the following mapping of adjacent indices:

$$
\begin{aligned}
&(11) \to 1, && (22) \to 2, && (33) \to 3,\\
&(23) = (32) \to 4, && (31) = (13) \to 5, && (12) = (21) \to 6,
\end{aligned}
$$

we express the elastic and piezoelectric coefficients briefly as: $C_{\alpha\beta} = C_{ijkl}, e_{i\beta} = e_{ikl}$, where $(ij) \to \alpha$ and $(kl) \to \beta$. Taking into account the above notations we have the following expressions for the effective coefficients in the case of parallel connection.

Elastic effective constants

$$
\begin{aligned}
C_{11}^{h} = {} & \chi\Big[C_{11}^{E} - \Big(C_{12}^{E}\Big)^{2}\Big(C_{11}^{E}\Big)^{-1}\Big] + (1-\chi)\Big[C_{11} - C_{12}(C_{11})^{-1}\Big] \\
& + \Big[\chi C_{12}^{E}\Big(C_{11}^{E}\Big)^{-1} + (1-\chi)C_{12}(C_{11})^{-1}\Big]^{2}\Big[\chi\Big(C_{11}^{E}\Big)^{-1} + (1-\chi)(C_{11})^{-1}\Big]^{-1},
\end{aligned}
$$

$$
C_{12}^{h} = \Big[\chi C_{12}^{E}\Big(C_{11}^{E}\Big)^{-1} + (1-\chi)C_{12}(C_{11})^{-1}\Big]\Big[\chi\Big(C_{11}^{E}\Big)^{-1} + (1-\chi)(C_{11})^{-1}\Big]^{-1},
$$

$$
\begin{aligned}
C_{13}^{h} = {} & \chi C_{13}^{E}\Big[1 - C_{12}^{E}\Big(C_{11}^{E}\Big)^{-1}\Big] + (1-\chi)C_{12}\Big[1 - C_{12}(C_{11})^{-1}\Big] \\
& + \Big[\chi C_{12}^{E}\Big(C_{11}^{E}\Big)^{-1} + (1-\chi)C_{12}(C_{11})^{-1}\Big]\Big[\chi\Big(C_{11}^{E}\Big)^{-1} + (1-\chi)(C_{11})^{-1}\Big]^{-1} \\
& \times \Big[\chi C_{13}^{E}\Big(C_{11}^{E}\Big)^{-1} + (1-\chi)C_{12}(C_{11})^{-1}\Big],
\end{aligned}
$$

$$
C_{22}^{h} = \Big[\chi\Big(C_{11}^{E}\Big)^{-1} + (1-\chi)(C_{11})^{-1}\Big]^{-1},
$$

$$
C_{23}^{h} = \Big[\chi C_{13}^{E}\Big(C_{11}^{E}\Big)^{-1} + (1-\chi)C_{12}(C_{11})^{-1}\Big]\Big[\chi\Big(C_{11}^{E}\Big)^{-1} + (1-\chi)(C_{11})^{-1}\Big]^{-1},
$$

$$
\begin{aligned}
C_{33}^{h} = {} & \chi\Big[C_{33}^{E} - \Big(C_{13}^{E}\Big)^{2}\Big(C_{11}^{E}\Big)^{-1}\Big] + (1-\chi)\Big[C_{11} - (C_{12})^{2}(C_{11})^{-1}\Big] \\
& + \Big[\chi C_{13}^{E}\Big(C_{11}^{E}\Big)^{-1} + (1-\chi)C_{12}(C_{11})^{-1}\Big]^{2}\Big[\chi\Big(C_{11}^{E}\Big)^{-1} + (1-\chi)(C_{11})^{-1}\Big]^{-1},
\end{aligned}
$$

$$
\begin{aligned}
C_{44}^{h} = {} & \Big[\chi\Big(C_{44}^{E}\Big)^{-1} + (1-\chi)(C_{44})^{-1}\Big]^{-1} + \chi\Big(\epsilon_{22}^{S}\Big)^{-1}(e_{15})^{2} \\
& - \chi\Big(\epsilon_{22}^{S}\Big)^{-1}(e_{15})^{2}(\chi\Big(\epsilon_{22}^{S}\Big)^{-1} + (1-\chi)\Big(\epsilon_{22}^{-1}\Big)-1,
\end{aligned}
$$

$$
\begin{aligned}
C_{55}^{h} &= \chi C_{44}^{E} + (1-\chi)C_{44}, \\
C_{66}^{h} &= \Big[\chi\Big(C_{66}^{E}\Big)^{-1} + (1-\chi)(C_{44})^{-1}\Big]^{-1}. \qquad (65)
\end{aligned}
$$

Piezoelectric effective constants

$$
\begin{aligned}
e_{15}^h &= \chi e_{15},\\
e_{24}^h &= \left[\chi\left(\epsilon_{22}^S\right)^{-1} + (1-\chi)(\epsilon_{22})^{-1}\right]^{-1}\left[\chi(\epsilon_{22}^S)^{-1}e_{24}\right],\\
e_{32}^h &= \left[\chi\left(C_{22}^E\right)^{-1} + (1-\chi)(C_{22})^{-1}\right]^{-1}\left[\chi\left(C_{22}^E\right)^{-1}e_{32}\right],\\
e_{31}^h &= \chi\left[e_{31} - e_{32}\left(C_{22}^E\right)^{-1}C_{12}^E\right] + \left[\chi\left(C_{22}^E\right)^{-1} + (1-\chi)(C_{22})^{-1}\right]^{-1}\\
&\quad \times \left[\chi\left(C_{22}^E\right)^{-1}C_{21}^E + (1-\chi)(C_{22})^{-1}]C_{21}\right]\left[\chi\left(C_{22}^E\right)^{-1}e_{32}\right],\\
e_{33}^h &= \chi\left[e_{33} - e_{32}\left(C_{22}^E\right)^{-1}C_{23}^E\right] + \left[\chi\left(C_{22}^E\right)^{-1} + (1-\chi)(C_{22}\right]^{-1})^{-1}\\
&\quad \times \left[\chi\left(C_{22}^E\right)^{-1}C_{23}^E + (1-\chi)(C_{22})^{-1}C_{23}\right]\left[\chi\left(C_{22}^E\right)^{-1}e_{32}\right],
\end{aligned}
\tag{66}
$$

Dielectric effective constants

$$
\begin{aligned}
\epsilon_{11}^h &= \chi\epsilon_{11}^S + (1-\chi)\epsilon_{11},\\
\epsilon_{22}^h &= \left[\chi\left(\epsilon_{22}^S\right)^{-1} + (1-\chi)\epsilon_{22} - 1\right]^{-1},\\
\epsilon_{33}^h &= \chi\left[\epsilon_{33}^S + (e_{32})^2\left(C_{22}^E\right)^{-1}\right] + (1-\chi)\epsilon_{33}\\
&\quad - \left[\chi\left(C_{22}^E\right)^{-1}e_{32}\right]^2\left[\chi\left(C_{22}^E\right)^{-1} + (1-\chi)(C_{22})^{-1}\right]^{-1},
\end{aligned}
\tag{67}
$$

where χ is the ceramic's volume fraction.

Using effective coefficients as computed above we can determine characteristic physical parameters, such that: electromechanical piezoelectric coupling coefficients K^h, specific acoustic impedance Z^h, and the longitudinal velocity V_l^h. They are defined given by the following formulae [25]:

Piezoelectric lateral and longitudinal coupling factors

$$
K_{31}^h = \frac{\bar{d}_{31}}{\sqrt{\bar{\epsilon}_{33}^T\,\bar{S}_{11}^E}}, \qquad K_{33}^h = \frac{\bar{d}_{33}}{\sqrt{\bar{\epsilon}_{33}^T\,\bar{S}_{33}^E}},
\tag{68}
$$

Piezoelectric thickness and planar coupling factors

$$K_t^h = \sqrt{1 - \frac{\bar{C}_{33}}{\bar{C}_{33}^D}}, \qquad K_p^h = \sqrt{\frac{2}{1-\sigma^E}} K_{31}^h \tag{69}$$

where,

$$\begin{aligned}
\bar{C}_{33}^D &= \bar{C}_{33} + \bar{e}_{33}^2\,\bar{\epsilon}_{33}^{-1}, & \bar{C}_{33} &\equiv C_{33}^h, \\
\bar{e}_{33} &\equiv e_{33}^h, & \bar{\epsilon}_{33} &\equiv \epsilon_{33}^h. \\
\sigma^E &= -\frac{\bar{S}_{12}^E}{\bar{S}_{11}^E}, & \bar{S}_{ij}^E &= (-1)^{i+j}\frac{\triangle_{ij}}{\triangle}, \\
\bar{d}_{mi} &= \bar{e}_{mj}\bar{S}_{ji}^E, & \bar{\epsilon}_{mn}^T &= \bar{d}_{mp}\,\bar{e}_{np} + \bar{\epsilon}_{mn}^S, \qquad i,j = 1,2,\ldots,6.
\end{aligned} \tag{70}$$

Longitudinal velocity

$$V_l^h = \left(\frac{\bar{C}_{33}^D}{\bar{\rho}}\right)^{1/2}. \tag{71}$$

Specific acoustic impedance

$$Z^h = \bar{\rho}\, V_l^h, \tag{72}$$

where, $\bar{\rho} = \chi\rho^c + (1-\chi\rho^p)$ is the averaged mass density, ρ^c (ρ^p) is the ceramic's (polymer) mass density, σ^E is the Poisson ratio, $\triangle$ is the determinant of the $\bar{C}_{ij}$ matrix and $\triangle_{ij}$ is the minor obtained by excluding the i-th row and j-th column.

Analogously, from Eqs. (56)–(58) we can writte down, the effective moduli for a binary layered in series connection. We show only the most important coefficients for computing the above mentioned physical parameters, *i.e.*,

$$\begin{aligned}
C_{33}^h &= \left\{ \frac{\chi}{C_{33}^D} + \frac{1-\chi}{C_{11}} + \left(\frac{\chi}{C_{33}^D}\cdot\frac{e_{33}}{\epsilon_{33}^S}\right)^2 \left(\frac{\chi}{\epsilon_{33}^D} + \frac{1-\chi}{\epsilon_{11}}\right)^{-1} \right\}^{-1}, \\
e_{33}^h &= \left\{ \left[\left(\frac{\chi}{\epsilon_{33}^D} + \frac{1-\chi}{\epsilon_{11}}\right)\left(\frac{\chi}{C_{33}^D} + \frac{1-\chi}{C_{11}}\right) + \left(\frac{\chi e_{33}}{C_{33}^D\epsilon_{33}^S}\right)^2 \right] \frac{\chi e_{33}}{C_{33}^D\epsilon_{33}^S} \right\}^{-1}, \\
\epsilon_{33}^h &= \left\{ \frac{\chi}{\epsilon_{33}^D} + \frac{1-\chi}{\epsilon_{11}} + \left(\frac{\chi}{C_{33}^D}\cdot\frac{e_{33}}{\epsilon_{33}^S}\right)^2 \left(\frac{\chi}{C_{33}^D} + \frac{1-\chi}{C_{11}}\right)^{-1} \right\}^{-1}
\end{aligned}$$

where

$$C_{33}^D = C_{33}^E + \frac{e_{33}^2}{\epsilon_{33}^S}, \qquad \epsilon_{33}^D = \epsilon_{33}^S + \frac{e_{33}^2}{C_{33}^E}.$$

TABLE I. Electroelastic material properties.

Material parameters of PZT-5A			
C_{11}^E $(10^{10} N/m^2)$	12.10	$\epsilon_{11}^S/\epsilon_0$	916.0
$C_{12}^E (10^{10} N/m^2)$	7.54	ϵ_0 $(10^{-12} F/m)$	8.85
C_{13}^E $(10^{10} N/m^2)$	7.52	$\rho(10^3 Kg/m^3)$	7.75
$C_{33}^E (10^{10} N/m^2)$	11.10	K_t	0.48
$C_{44}^E (10^{10} N/m^2)$	2.11	K_{33}	0.70
$e_{33}^E (C/m^2)$	15.8	K_p	0.60
$e_{31}^E (C/m^2)$	-5.4	K_{31}	-0.34
$e_{15}^E (C/m^2)$	12.3	$V_l(m/s)$	4350.0
$\epsilon_{11}^E/\epsilon_0$	830.0	$Z(MRayls)$	33.052

Material Parameters of Polymers

Polymer	Araldite	Eccothane	Araldite D
$C_{11}(10^{10} N/m^2)$	0.546	0.164	0.800
$C_{12}(10^{10} N/m^2)$	0.294	0.157	0.440
$\rho(10^3 Kg/m^3)$	1.17	1.13	1.15
ϵ_{11}/ϵ_0	7.0	5.4	4.0
$V_l(m/s)$	2160.25	1204.7	2603.0

The basic requirements of a piezoelectric transducer for ultrasonic diagnostic imaging are: 1) the piezoelectric material should have a high electromechanical coupling coefficient for high sensitivity; 2) the acoustic impedance of the transducer should match the load to minimize reflection losses at the interface.

Later, we will illustrate several examples for different choices of polymer, ceramic and volume fraction. We present the change of the composite's properties with respect to the volume fraction in parallel and series connection and their implications for ultrasonic transducers. To illustrate how the composite material parameters vary with volume fraction of piezoelectric ceramic, the material parameters of PZT-5A and different choices of polymer (Araldite, Eccothane, Araldite D) are used in the calculus. They are listed in Table I.

Figure 3 shows the variation in the basic material parameters, ρ^h, ϵ_{33}^h, C_{33}^h and e_{33}^h *versus* the ceramic's volume fraction for piezocomposites in series and parallel connection made from PZT-5A ceramic and Eccothane. These quantities vary essentially linearly with the volume fraction over most of the range. But, as the volume fraction become larger, the lateral clamping of the layers by the polymer has greater effect on the elastic and piezoelectric behavior. The elastic stiffness, C_{33}^h, increases and the piezoelectric strain constants, e_{33}^h decreases and this lateral clamping of the layers also reduce the

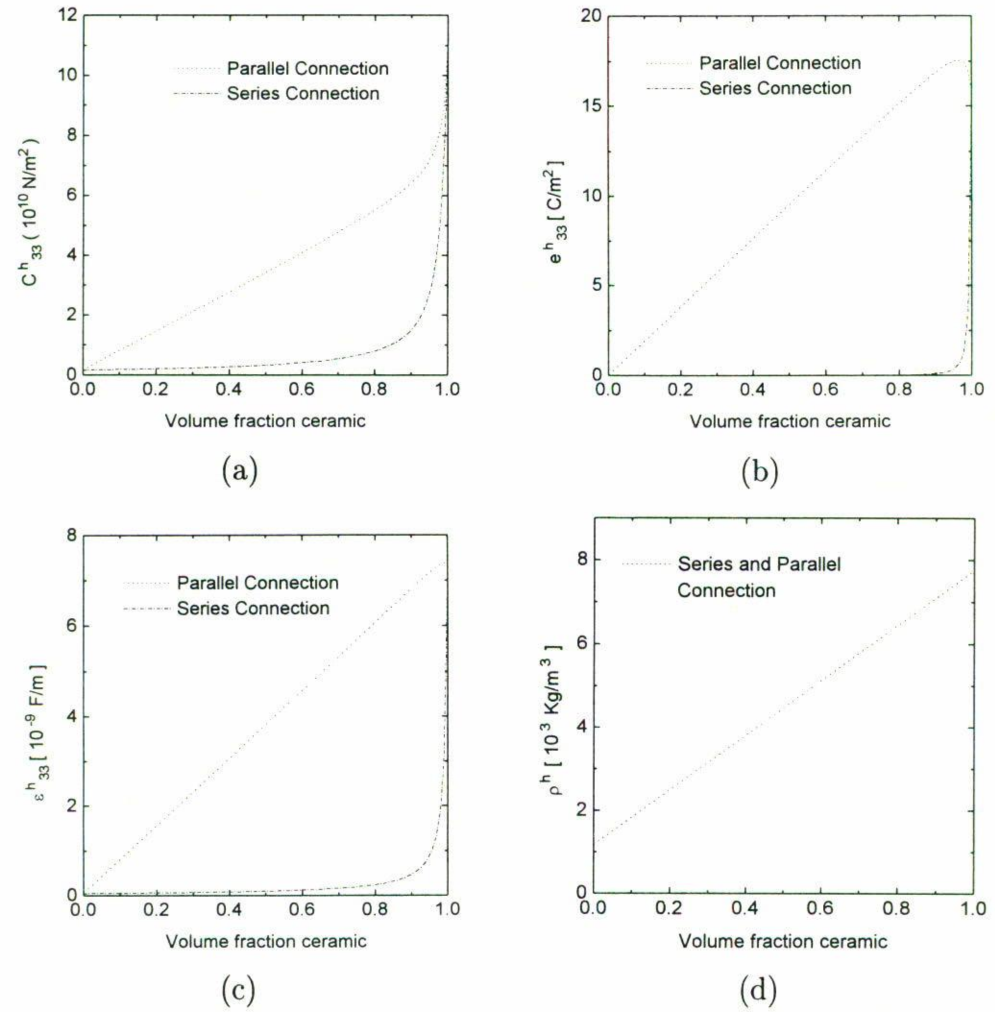

FIGURE 3. (a) Variation of stiffness C^h_{33} with volume fraction for a laminated composite (parallel and series connection) made from PZT-5A and Eccothane. (b) Variation of piezoelectric constant e^h_{33} with volume fraction for a laminated composite (parallel and series connection) made from PZT-5A and Eccothane. (c) Variation of dielectric constant ε^h_{33} with volume fraction for a laminated composite (parallel and series connection) made from PZT-5A and Eccothane. (d) Variation of density ρ with volume fraction for a laminated composite (parallel and series connection) made from PZT-5A and Eccothane.

dielectric constant in this range. It should be notice that the non linear effects are only of order of few percents. The behavior of these parameters is analogous to parameters for another type of composite, (for instance as shown in Ref. 4, 5 and 26) where the composite is made from PZT rods in a polymer matrix and piezoelectric ceramic rods in piezoelectric polymer matrix. For the case corresponding to series connection, the parameters C^h_{33}, e^h_{33} and ϵ^h_{33} are almost equal to zero for a large range of the volume fraction, only increase near the high volume fractions. These results are an immediate consequence

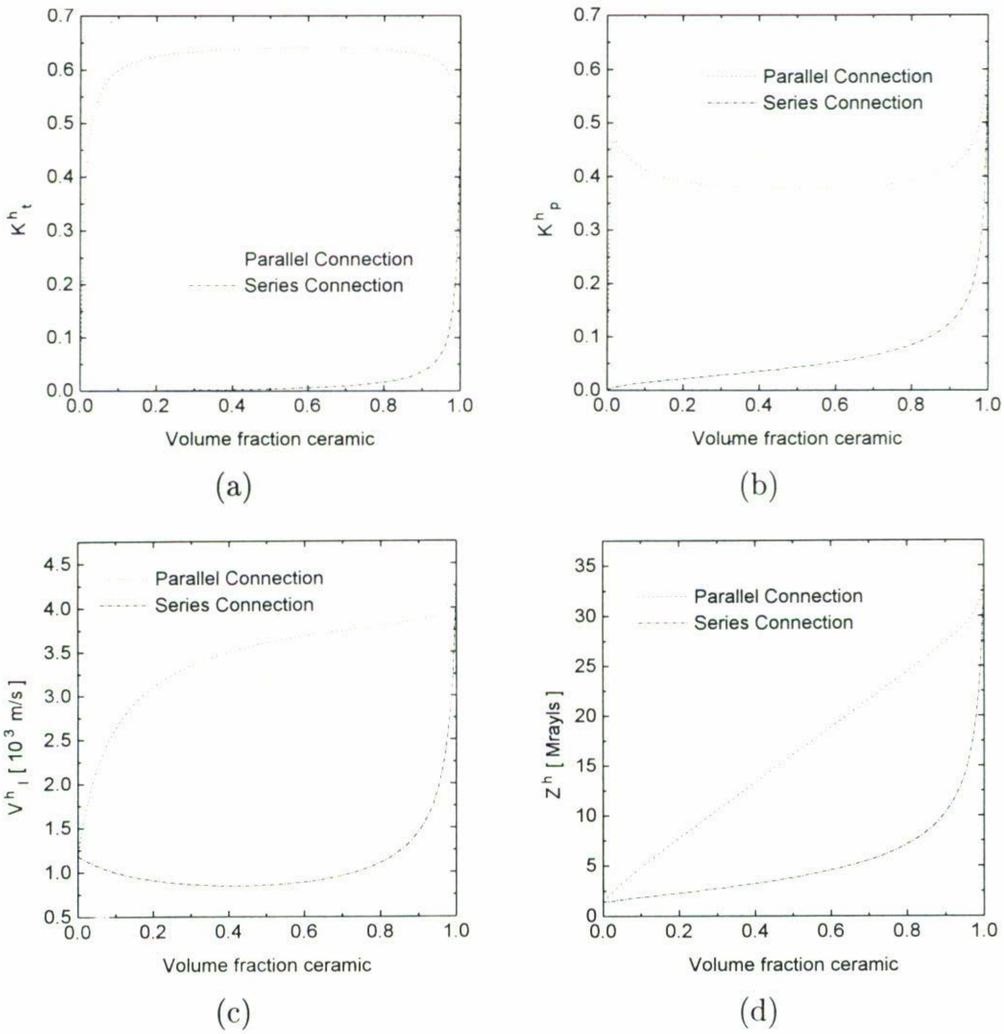

FIGURE 4. (a) Variation of thickness coupling constant K_t^h with volume fraction for a laminated composite (parallel and series connection) made from PZT-5A and Eccothane. (b) Variation of planar coupling constant K_p^h with volume fraction for a laminated composite (parallel and series connection) made from PZT-5A and Eccothane. (c) Variation of longitudinal velocity V_l^h with volume fraction for a laminated composite (parallel and series connection) made from PZT-5A and Eccothane. (d) Variation of acoustic impedance Z^h with volume fraction for a laminated composite (parallel and series connection) made from PZT-5A and Eccothane.

of the piezoelectric ceramic discontinuity in the direction of wave propagation, *i.e.*, the vertical direction. As a possible application in medical pulse-echo ultrasonic transducers, the composite in parallel connection has better properties, which will be shown in the following figures.

Figure 4 is devoted to the comparison for both connection types of basic physical parameters such as, thickness and planar electromechanical coupling factors, longitudinal velocity and acoustic impedance denoted by K_t^h, K_p^h, V_l^h, Z^h respectively, *versus* the ceramic's volume fraction for piezocomposites made from PZT-5A ceramic and Eccothane. Due to the above mentioned characteristics of the magnitudes ρ^h, ϵ_{33}^h, C_{33}^h and e_{33}^h shown in Fig. 3, the behavior of the parameters K_t^h, K_p^h, V_l^h, Z^h, in the series connection case, gives inferior results than in parallel connection, for instance the Fig. 4a shows that K_t^h (in series connection) decreases rapidly for high ceramic's volume fraction being almost zero for a large range of volume fraction, however, for the other curve, K_t^h is higher than K_t of the piezoelectric ceramic over almost all the range of ceramic's variation except for small ceramic's volume fraction. The above mentioned indicates that the parallel connection is more efficient in the design of pulse-echo ultrasonic transducers for obtaining medical image with good qualities.

Figure 5 shows the behavior of the acoustic impedance, the longitudinal velocity, the thickness and planar electromechanical coupling factors in parallel connection. These variations with the volume fraction follow directly from those of the basic material parameters. Essentially, the acoustic impedance Z^h increases linearly with χ except at large χ where the clamping of the ceramic's layers causes it to increase more rapidly to take the value of the ceramic's acoustic impedance Z. The longitudinal velocity V_l^h also increase rapidly in this zone due to the stiffening of the layers by lateral forces from the polymer. The presence of ceramic's layers has a stiffening effect and causes the velocity to increase quite rapidly for small ceramic's volume fraction. For intermediate values of χ the velocity increases only slowly. The thickness electromechanical coupling coefficient K_t^h for low volume fraction it increases rapidly and at the end decreases rapidly down to the value of the ceramic's coupling constant K_t. It is possible to note that, for all cases, the composite coupling factor K_t^h is higher than the ceramic's K_t for a large range of volume fraction. The planar coupling coefficient K_p^h stay almost constant around 0.40, which is much lower than K_p of the ceramic (0.60). A low K_p^h is an advantage in using composite in medical beam transducer construction. For the ceramic PZT-5A is known that K_p and K_t have the same order. It provokes that the radiation field is composed by a central lobe and differents representative side lobes. For obtaining better quality medical image it is necessary to reduce the side lobes for which the value of K_t must be increased and the magnitude of K_p decreased [27]. We would conclude, that the constituent materials of the composite play an important role for the improvement of the global properties, for instance, the combination of PZT-5A with the Eccothane polymer in this case is better, since we obtain greater K_t^h, lower Z^h and K_p^h. It satisfies the above mentioned requirements.

To make a sensitive, broadband ultrasonic transducer, one wants a piezoelectric with low acoustic impedance ($Z^h < 7.5$ Mrayl) and high electromechanical coupling ($K_t^h = 0.60$ to 0.70). These calculations show that composite piezoelectrics can be superior to solid ceramic piezoelectrics in both respects. The optimum material can be achieved by adjusting the volume fraction of piezoceramic. Lowering the volume fraction always lowers the acoustic impedance but eventually causes a deterioration in the electromechanical coupling. A trade-off then must be made between minimizing the impedance and maximizing the coupling, as illustrated in Fig. 6 for the parallel connection.

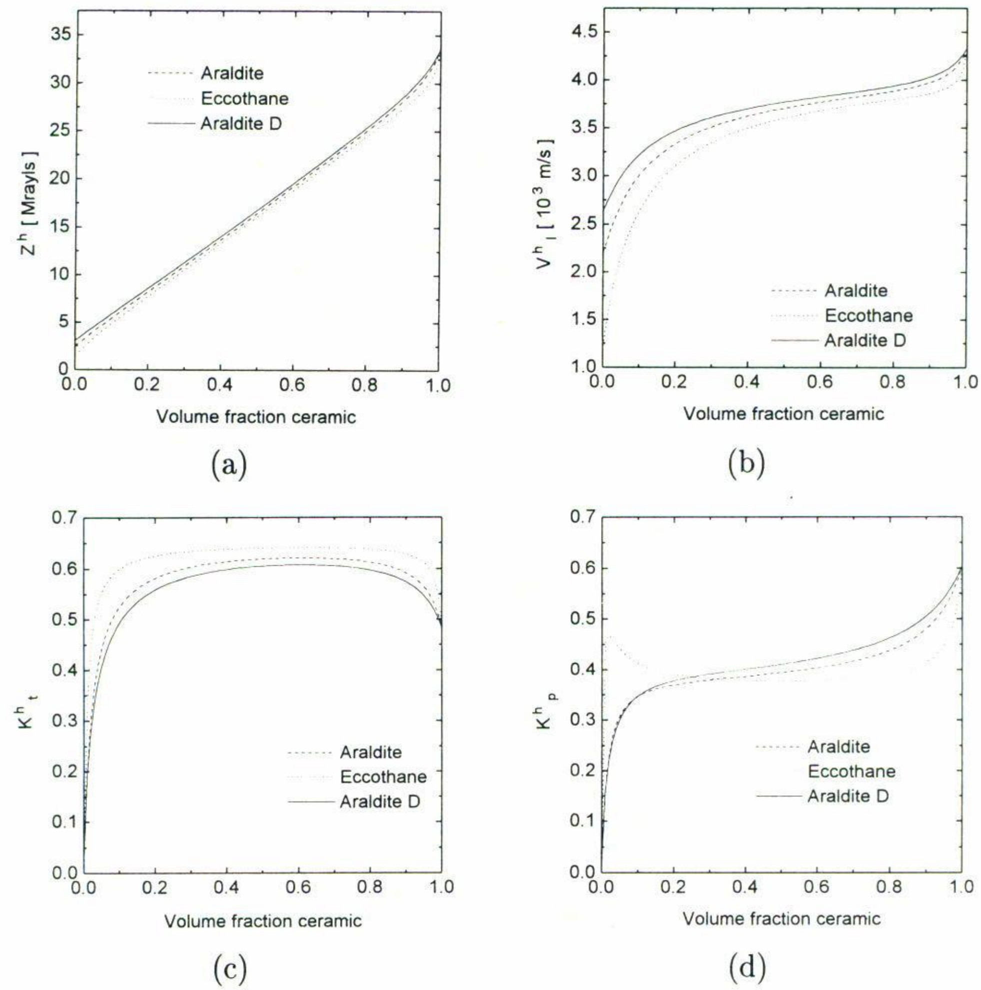

FIGURE 5. (a) Variation of acoustic impedance Z^h with volume fraction for a laminated composite, in parallel connection, made from PZT-5A and three different polymers. (b) Variation of longitudinal velocity V_l^h with volume fraction for a laminated composite, in parallel connection, made from PZT-5A and three different polymers. (c) Variation of thickness coupling constant K_t^h with volume fraction for a laminated composite, in parallel connection, made from PZT-5A and three different polymers. (d) Variation of planar coupling constant K_p^h with volume fraction for a laminated composite, in parallel connection, made from PZT-5A and three different polymers.

8. CONCLUSIONS

In this paper the procedure of constructing the formal asymptotic solution of linear static piezoelectric equations for a periodically heterogeneous medium is developed by means of Asymptotic Two Scale Expansion. The original boundary value problem with variable coefficients is transformed in a recurrent sequence of boundary value problems with constant coefficients. Actually, this asymptotic analysis leads to the solution of two recurrent sequence of problems. The first of these problems (problem $B(p)$, $p = 0, 1, \dots$)

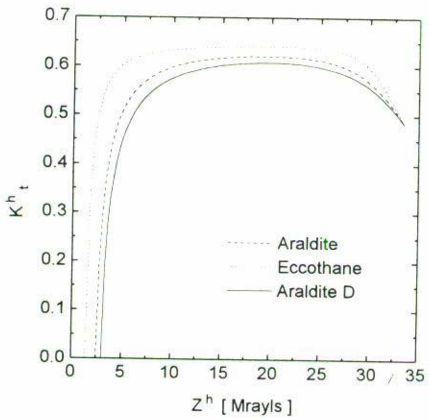

FIGURE 6. Trade-off between high electromechanical coupling and low acoustic impe dance for a laminated composite, in parallel connection, made from PZT-5A and three different polymers.

consists in the solution to multiple boundary value problems (25) and (26). For solving the problem B(p) it is necessary to solve the problems $B(r), r = 0, 1, \ldots, p-1$. The solution to each one of these problems permits to find the functions $w_n^{\{p\}}$ and $y^{\{p\}}$. Then by Eq. (24) it is possible to determine the averaged functions V_n and S. The solution to the original problem (17) and (18) is finally obtained from V_n and S by (19). Local auxiliary periodic functions $\underline{N}^{(q)}$, $\underline{M}^{(q)}$, $\underline{\Phi}^{(q)}$, $\underline{P}^{(q)}$ are included in these formulae, whose computation of solutions leads the second recurrent sequence of problems which is made up of $P_I^{(q+1,q)}$ [Eqs. (27) and (28)], $P_{II}^{(q+1,q)}$ [Eqs. (29) and (30)]. For a fixed value of q, Eqs. (27) and (29) represent respectively, a system for finding $\underline{N}^{(q+1)}$ and $\underline{M}^{(q+1)}$, $\underline{\Phi}^{(q+1)}$ and $\underline{P}^{(q+1)}$ taking into account (20) and (21). After that, by using (28), (30) the constant tensors $\underline{h}^{(q)}$, $\underline{t}^{(q)}$, $\underline{r}^{(q)}$ and $\underline{s}^{(q)}$ are obtained.

Based on the general problems (27), (28) and (29), (30) and the local problems (40), (41) and (42), (43) the global behavior of a laminated composite with axis of symmetry in the direction normal to the layers (Fig. 1) is computed by finding the general expressions for the effective coefficients (56), (57) and (58).

The above expressions (56), (57) and (58) were used to show analytically that not only the ratio of each component phase of piezocomposite materials have influence over the global or homogenized properties, but also the way of coupling the components. For this first, we considered a laminated structure connected in series (Fig. 1) and secondly in parallel (Fig. 2). Each periodic cell was composed by piezoelectric layers made of the hexagonal symmetry (6 mm). In the case of series connection we obtained as a result of the homogenization a material with the same hexagonal symmetry (6 mm) and in the case of parallel connection, the corresponding homogenized material behaves as a piezoelectric material with orthorhombic symmetry (2 mm). Moreover, such general expressions obtained for the effective coefficients were also applied to a binary layered medium and an application of these piezocomposite materials for the design of better ultrasonic transducers was shown in Sect. 7 where we proved that the parallel connection is more efficient for the design of pulse-echo ultrasonic transducers and the series connection composite is practically an inactive piezoelectric for this purpose.

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