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A new definition of the logarithmic temperature scale based on the triple point of water

JULIO PELLICER, MARÍA-JESÚS HERNÁNDEZ, AND MANUEL DOLZ
Departament de Termodinàmica. Facultat de Física. Universitat de València.
46100 Burjassot (Valencia) Spain

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ABSTRACT. Using the triple point of water as a fixed point, a logarithmic temperature scale is defined. After establishing the relation between this scale and the thermodynamic scale, its scientific and academic advantages at low temperature are stated.

RESUMEN. Se define una escala logarítmica de temperatura usando el punto triple del agua como punto fijo. Una vez establecida la relación entre esta escala y la escala termodinámica, se indican sus ventajas científicas y académicas a bajas temperaturas.

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1. INTRODUCTION

The unattainability of absolute zero of temperature is considered to be a consequence of the third law of thermodynamics [1]. Nevertheless, temperatures of 170 nanokelvin (billionths of a degree above absolute zero) have been reached in practice [2] and it might be argued that temperatures of this order of magnitude are already so small that an assertion to the effect that $T = 0$ K can not be reached becomes effectively empty. This argument rests upon the inherent assumption that temperatures as low as these are physically not distinct from $T = 0$ K. This, however, is not true. What we find is that the thermodynamic temperature scale is not altogether appropriate to describe the region of very low temperature [3]. In fact, as such scale is linear and also we are familiar with properties of substances at room temperature (*i.e.*, 300 K), we could think that a 1 K interval is equally meaningful at any temperature. If this were so, the difference in temperature between 10^{-3} K and 10^{-5} K would become insignificant. However, there are two orders of magnitude between these temperatures, as there is between 3 K and 300 K, yet in the latter the difference in temperature does not seem insignificant. Therefore, it is more meaningful to say that two pairs of temperatures have equivalent differences if their ratios have the same value. That is, the difference between 10^{-3} K and 10^{-5} K is equivalent to that between 3 K and 300 K.

The non-uniformity of the thermodynamic temperature scale is also evident in the Carathéodory formulation of the second law of thermodynamics. It is well known that [4]

$$T(\theta,\theta_0) = C\exp\left(\int_{\theta_0}^{\theta} g(\theta)d\theta\right), \tag{1}$$

where $g(\theta)$ is a universal function of the empirical temperature θ; θ_0 is a standard empirical temperature which is assumed to be the same for all physical systems, and C is also a constant with dimensions of temperature. It is easy to prove that from expression (1) we can derive

$$T(\theta_2, \theta_0) - T(\theta_1, \theta_0) \neq T(\theta_2, \theta_1). \tag{2}$$

Moreover, from the same expression (1), we can also get

$$\log \frac{T(\theta_2, \theta_0)}{C} - \log \frac{T(\theta_1, \theta_0)}{C} = \log \frac{T(\theta_2, \theta_1)}{C}. \tag{3}$$

It is clear from the expressions (2) and (3) that it is the ratio and not the difference of temperatures which provides a more realistic measure of temperature intervals.

The present paper emphasizes, from the scientific and academic points of view, the old idea that a logarithmic rather than a linear temperature scale might be more useful, especially at low temperatures. In fact, one uses often a logarithmic temperature scale without explicitly stating it, as evidenced by discussions in text books. For instance, we see that in order to evaluate entropy graphically, c_p (specific heat at constant pressure) is plotted as a function of $\log T$. Here the logarithmic temperature scale is defined similarly to the thermodynamic temperature scale. The only fixed point selected is the triple point of water, in accordance with the idea suggested by Lord Kelvin in 1854 [5], adopted one hundreds years later in the 10th General Conference on Weights and Measures.

2. THE LOGARITHMIC TEMPERATURE SCALE

It has already been stated (see, for example, Ref. 6) that one of the ways in which the thermodynamic temperature scale may be realized is to apply a constant volume gas thermometer using the pressure of the gas as the thermometric parameter (ideal gas scale). If p and p_3 are the gas pressures when the bulb of gas is immersed in a system whose temperature is going to be measured and in a system being at temperature $T_3 = 273.16$ K (the triple point of water), respectively, the ideal gas scale temperature is determined from the expression

$$T = T_3 \lim_{p_3 \to 0} \frac{p}{p_3}. \tag{4}$$

The limit for low pressures is taken because the readings of two thermometers with different thermometric substances or with different amounts of the same gas will never be coincident. This arbitrariness is removed if sufficient rarefied (ideal) gases are used as the thermometric substance. In this case, all gas thermometers will give the same temperature irrespective of the gas used.

Taking into account that in the logarithmic scale the argument of the logarithm must be adimensional, the temperature T^{L} in the new scale can be expressed as

$$T^{\mathrm{L}} = m \log \left(\lim_{p_3 \to 0} \frac{p}{p_0} \right), \tag{5}$$

where m and p_0 are two parameters with dimensions of temperature and pressure, respectively. If we assume that in the triple point of water $T^{\mathrm{L}} = T_3^{\mathrm{L}} = 273.16\,^{\circ}L$ (logarithmic degrees), the Eq. (5) is converted into the following one:

$$T^{\mathrm{L}} = T_3^{\mathrm{L}} \frac{\log\left(\lim_{p_3\to 0} \frac{p}{p_0}\right)}{\log\left(\lim_{p_3\to 0} \frac{p_3}{p_0}\right)}. \tag{6}$$

We should point out that the logarithmic scale (6) is defined in terms of p_0, and therefore the value of T^{L} depends on the choice of p_0.

Now we are going to find out the equation relating temperatures T and T^{L}. From Eq. (4) we have

$$\lim_{p_3\to 0} \frac{p}{p_0} = \frac{T}{T_3} \lim_{p_3\to 0} \frac{p_3}{p_0}. \tag{7}$$

Then, if we introduce a temperature T_0, so that

$$\lim_{p_3\to 0} \frac{p_3}{p_0} = \frac{T_3}{T_0}, \tag{8}$$

equation (7) becomes

$$\lim_{p_3\to 0} \frac{p}{p_0} = \frac{T}{T_0}. \tag{9}$$

Finally, combining Eqs. (6), (8) and (9), one has

$$T^{\mathrm{L}} = T_3^{\mathrm{L}} \frac{\log\left(T/T_0\right)}{\log\left(T_3/T_0\right)}. \tag{10}$$

From expression (10), $T^{\mathrm{L}} = 0$ results for $T = T_0$. That means T_0 is the temperature of thermodynamic scale corresponding to zero in the logarithmic scale.

It is obvious that if Eq. (10) is applied to temperatures T^{L} and $T^{\mathrm{L}} - 1$ (one degree interval), one obtains

$$\log \frac{T_2}{T_1} = \frac{1}{T_3^{\mathrm{L}}} \log \frac{T_3}{T_0}. \tag{11}$$

Therefore, ratio T_2/T_1 is constant and independent of T^{L}. Hence, all degrees have the same value, *i.e.*, a Carnot engine operating between heat reservoirs at T^{L} and $T^{\mathrm{L}} - 1$ would have the same efficiency whatever the value of T^{L}, as in Kelvin's first proposal of a scale of temperature [5].

3. SCIENTIFIC AND ACADEMIC ADVANTAGES

As indicated in Sect. 1, two pairs of temperatures have equivalent differences if their ratios have the same value. This statement makes sense when a logarithmic scale is used. In fact, from expression (10) we can easily get

$$T_2^{\mathrm{L}} - T_1^{\mathrm{L}} = T_3^{\mathrm{L}} \frac{\log\left(T_2/T_1\right)}{\log\left(T_3/T_0\right)}. \tag{12}$$

Therefore, if two pairs of thermodynamic temperatures have the same ratio T_2/T_1, then their differences in logarithmic scale, $T_2^{\mathrm{L}} - T_1^{\mathrm{L}}$, are identical. (This is what happens, for instance, with temperature intervals in the ranges 10^{-3}–10^{-5} K and 3–300 K).

It has been pointed out [7] that is a mistake to think of the difference between, say, $T_2 = 10^{-2}$ K and $T_1 = 10^{-3}$ K as being phenomenally in any way "comparable" with the difference between, say, $T_2' = 300.009$ K and $T_1' = 300.000$ K. In fact, if we use the expression (12) twice, one obtains

$$\frac{\Delta T^{\mathrm{L}}}{\Delta T^{\mathrm{L}'}} = \frac{\log (T_2/T_1)}{\log (T_2'/T_1')} = 76\,754, \quad i.e., \quad \Delta T^{\mathrm{L}} \gg \Delta T^{\mathrm{L}'}, \tag{13}$$

where $\Delta T^{\mathrm{L}} = T_2^{\mathrm{L}} - T_1^{\mathrm{L}}$ and $\Delta T^{\mathrm{L}'} = T_2^{\mathrm{L}'} - T_1^{\mathrm{L}'}$. However, in both cases, $\Delta T = \Delta T' = 0.009$ K. Consequently, the logarithmic scale has the advantage of being much more sensitive to low temperatures than the thermodynamic scale.

It can also be observed that if in Eq. (12) it is assumed that T_1 corresponds to a any initial temperature, while T_2 is identified with a absolute zero, then, the interval between both temperatures is infinite. Using the logarithmic temperature scale it seems unnecessary to state that $T = 0$ is unattainable, since the latter is separated by an infinite interval from any initial temperature. Thus, the "unattainability" of absolute zero becomes self-evident [8].

It is very didactic to show the correspondence between some meaningful values of temperature in both scales and also in the logarithmic temperature scale proposed by Kelvin. The latter is based in the specification of temperature at two fixed points—the ice (i) and steam (s) points for water—instead of only one (the triple point of water). The relation between thermodynamic (T) and Kelvin logarithmic (T_K^{L}) temperatures is analogous to (10) [5]

$$T_K^{\mathrm{L}} = 100 \frac{\log (T/T_i)}{\log (T_s/T_i)}, \tag{14}$$

where $T_i = 273.15$ K and $T_s = 373.15$ K. Table I shows the values of temperatures T_K^{L} and T^{L} corresponding to the values of T indicated. As the thermodynamic scale used in this paper is not completely defined with one value only, the values 273.15 K and 1 K have been considered for T_0.

4. CONCLUSION

As a conclusion, we want to point out that defining a logarithmic temperature scale is both possible and also convenient in many cases (though not always, as in systems with negative thermodynamic temperature, for example, this scale would be completely meaningless). In particular, this scale could be more useful than the thermodynamic scale in the case of low temperatures. The idea of the unattainability of absolute zero is implied by Table I since the lowest experimental temperature obtained until now (170 nanokelvin, by Anderson *et al.* in 1995 [2]) is still very far from the thermodynamic limit $T = 0$. Hence, we should regard the thermodynamic scale as extending logarithmically

TABLE I. Some meaningful values in the thermodynamic and logarithmic temperature scales.

T(K)	T_K^{L}(°L)	$10^{-4}\, T^{\mathrm{L}}(^\circ L)$ $T_0 = 273.15$ K	T^{L}(°L) $T_0 = 1$ K
$+\infty$	$+\infty$	$+\infty$	$+\infty$
10^8	4107	9559	896
10^4	1154	2686	448
273.15	0	0	273.16
1	−1798	−4186	0
10^{-2}	−3274	−7622	−224
10^{-4}	−4751	−11058	−448
10^{-8}	−7703	−17931	−896
0	$-\infty$	$-\infty$	$-\infty$

downward to absolute zero, giving an infinite region for exploration [3]. This fact certainly shows the effort needed to approach absolute zero on the Kelvin scale [9].

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