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On the evaluation of the capacitance of toroidal capacitors with a moon-shape meridian cross section

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ABSTRACT. The toroidal capacitors studied in this paper consist of electrodes with meridian cross sections that are circular arcs meeting at the axis, and separated from each other by two small insulating spheres at their meeting points. The description and analysis of such capacitors is carried out by using bispherical coordinates. The R-separability of the Laplace equation in these coordinates requires the use of the Green function technique, just like in the related problems of toroidal, spherical-cap-electrode, and bispherical capacitors [1–3]. An overall comparison of the solutions of the four problems is specially instructive.

RESUMEN. Los condensadores toroidales estudiados en este trabajo están formados por electrodos con secciones meridianas que son arcos de círculos que convergen en el eje, y separados entre sí por dos pequeñas esferas aislantes en los puntos de convergencia. La descripción y el análisis de tales condensadores se lleva a cabo usando coordenadas biesféricas. La R-separabilidad de la ecuación de Laplace en estas coordenadas requiere el uso de la técnica de la función de Green, como en los problemas afines de condensadores toroidales, con electrodos en forma de casquetes esféricos y biesféricos [1–3]. Una comparación global de las soluciones de los cuatro problemas es especialmente instructiva.

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1. INTRODUCTION

The writing of this paper and of its complementary companion [3] was done as a consequence of the Canadian-American-Mexican Physics Meeting in Cancún, 1994, in which we presented a teaching poster contribution on spherical-cap-electrode capacitors [2]. The figures in the poster attracted the attention of several colleagues. In particular, Dr. Luz J. Martínez-Miranda of Kent State University remarked "Those are like the electric fields that we observe in the liquid crystal samples in our laboratory"; the ensuing discussion allowed us to learn that her samples are actually based on circular-arc-cylinder electrodes, for which later on we were able to fax her the analytical description based on bipolar cylindrical coordinates. On the other hand, Dr. Jorge José of Northeastern University, who works on chaos in mesoscopic systems, asked us "Can you do the two-dimensional Pac-Man?", and also "Can you solve the case of two spheres?" For the latter we could say yes and constructed the solution based on bispherical coordinates [3]; for the former we have not found an answer and we are aware of its difficulties. Nevertheless, we can point out that the helmet-like capacitors of our poster are a good approximation to the three-dimensional Pac-Man, and that the cylindrical capacitors with a Pac-Man cross section can be approximated within the same description in bipolar cylindrical coordinates mentioned above. This introductory paragraph is intended to illustrate three points:

1. The current interest in different fields of physics in the electrical fields and/or capacitances of unorthodox electrode shapes.

2. The misinterpretation of a figure according to our preconceived notions or particular interests; of course, we have to be careful and find out, for example, if the figure is flat or if it is a transverse or a meridian cross section, and for the latter which is the axis of rotation.

3. The importance of distinguishing the dimensionalities of electrodes and fields. For instance, a flat 2-dimensional Pac-Man has an associated 3-dimensional field, while an infinite cylindrical capacitor with a Pac-Man cross section is 3-dimensional and its associated field is 2-dimensional.

In a wider didactic context, readers of *Revista Mexicana de Física* may be aware of our series of papers on electrostatics and magnetostatics in recent years, in which the common theme is the identification of harmonic components of sources and fields associated with specific geometries. The average physicist is familiar with the spherical harmonics, but without too much additional effort and using familiar methods of analysis he or she can become familiar with other ten types of harmonics in different geometries. There is no reason for teachers of physics to restrain themselves and their students to the solutions of Laplace's equation in cartesian, cylindrical and spherical coordinates only. Our aim in writing these papers is to help the interested teachers and students to learn about the diversity of harmonic functions, and also about the unity and systematics behind them, illustrating at the same time the current interest and need of them in the study of different physical phenomena.

This paper completes the sequel in the study of four types of capacitors having common geometrical roots, namely, toroidal [1], spherical-cap electrode [2], bispherical [3] and toroidal with a moon-shaped meridian cross-section capacitors. The common roots are the two-dimensional bipolar coordinates [4], formed by two sets of mutually orthogonal circles, with the circular arcs of one set meeting at two points ("the poles") and the circles in the other set nested around the poles. The toroidal coordinates [4] are generated by the rotation of the plane of bipolar coordinates around the straight line in the plane with points equidistant from the poles; thus the nested circles generate toroids [1], and the circular arcs meeting at the poles generate spherical caps [2]. The bispherical coordinates [4] are generated by the rotation of the plane of bipolar coordinates around the axis joining the poles; in this case, the nested circles generate nested spheres [3], and the circular arcs meeting at the poles generate toroids. Any pair of electrodes with the shape of such toroids and separated by small insulating spheres at the poles form a capacitor with a moon-shape meridian cross-section. Figure 1 illustrates some of these capacitors.

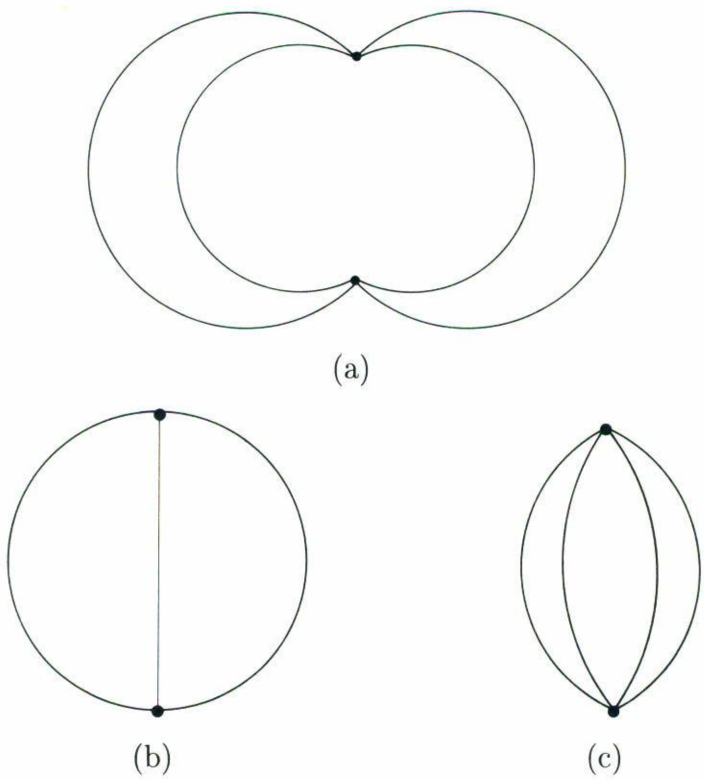

FIGURE 1. Toroidal capacitors with moon shape meridian cross sections: a) partially eclipsed moon $0 < \xi_1 < \pi/2$, $\xi_1 < \xi_2$, b) half moon $\xi_1 = \frac{\pi}{2}$, $\xi_2 = \pi$, c) American football $\pi/2 < \xi_1 < \xi_2 < \pi$.

The common geometrical roots translate into the R-separability [5] of the Laplace equation in both toroidal and bispherical coordinates, which makes the description of the electrostatic field of the four types of capacitors a nontrivial matter [1–3]. Section 2 contains the explicit construction of the electrostatic potential function for the moon-shape toroidal capacitors using the Green function technique [6], and the subsequent evaluation of the electric intensity field, the charge distributions and total charges on the electrodes, and the capacitance. Section 3 contains a discussion of the similarities and differences of the results of this study and those of Refs. 1–3. Appendix A introduces the bispherical coordinates, and Appendix B presents the construction of the Dirichlet Green function and some relevant integrals.

2. ELECTROSTATIC FIELD, CHARGES AND CAPACITANCE

In the notation of Appendix A for the bispherical coordinates ($0 \leq \xi \leq \pi$, $-\infty < \eta < \infty$, $0 < \varphi < 2\pi$), the electrostatic potential for the moon-shape toroidal capacitor is to be determined as a solution of the Laplace equation

$$\left[\frac{(\cosh\eta-\cos\xi)^3}{a^2\sin\xi}\left(\frac{\partial}{\partial\xi}\frac{\sin\xi}{\cosh\eta-\cos\xi}\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\frac{\sin\xi}{\cosh\eta-\cos\xi}\frac{\partial}{\partial\eta}\right)\right.$$
$$\left.+\frac{(\cosh\eta-\cos\xi)^2}{a^2\sin^2\xi}\frac{\partial^2}{\partial\varphi^2}\right]\phi(\xi,\eta,\varphi)=0, \quad (1)$$

satisfying the boundary conditions at the respective electrodes

$$\phi(\xi=\xi_1,\eta,\varphi)=V_1, \quad (2a)$$

$$\phi(\xi=\xi_2,\eta,\varphi)=V_2=0. \quad (2b)$$

The Laplace equation has R-separable solutions [5] of the general form

$$\phi(\xi,\eta,\varphi)=(\cosh\eta-\cos\xi)^{1/2}\sum_{m=0}^{\infty}\int_0^{\infty}dk\left[A_{km}P^m_{-\frac{1}{2}+ik}(\cos\xi)+B_{km}Q^m_{-\frac{1}{2}+ik}(\cos\xi)\right]$$
$$\times\left[C_k\sin k\eta+D_k\cos k\eta\right]\left[E_m\sin m\varphi+D_m\cos m\varphi\right], \quad (3)$$

where the Legendre functions having a complex-number order are of the conical type [7]. The presence of the square root of the binomial factor, reflecting the R-separability, makes the fulfillment of the boundary condition of Eq. (2a) a complicated problem. The solution is obtained via the Green function technique [6]:

$$\phi(\vec{r})=-\frac{1}{4\pi}\oint_S da'\phi(\vec{r}\,')\frac{\partial G_D(\vec{r},\vec{r}\,')}{\partial n'} \quad (4a)$$

where n' is the displacement perpendicular to the boundary S and the Dirichlet-Green function is constructed in the Appendix B.

By using Eq. (B8) for the normal derivative of the Green function at the electrode ξ_1 and the explicit forms of the scale factors, the harmonic expansion for the potential function is obtained:

$$\phi(\xi,\eta,\varphi)=-\frac{V_1}{4\pi}\int_{-\infty}^{\infty}\int_0^{2\pi}h_{\eta'}\,h_{\varphi'}\,d\eta'\,d\varphi'\left.\frac{\partial G_D(\vec{r},\vec{r}\,')}{\partial n'}\right|_{\xi'=\xi_1}$$
$$=V_1(\cosh\eta-\cos\xi)^{1/2}\sum_{m=0}^{\infty}\int_{-\infty}^{\infty}\frac{dk}{\sqrt{2\pi}}e^{ik\eta}$$

$$\int_0^{2\pi} d\varphi' \frac{\epsilon_m}{2\pi} \cos m(\varphi - \varphi') \int_{-\infty}^{\infty} \frac{d\eta' \, e^{-ik\eta'}}{\sqrt{2\pi}(\cos h \; \eta' - \cos\xi_1)^{1/2}}$$
$$\times \frac{P^m_{-\frac{1}{2}+ik}(\cos\xi_2)Q^m_{-\frac{1}{2}+ik}(\cos\xi) - Q^m_{-\frac{1}{2}+ik}(\cos\xi_2)P^m_{-\frac{1}{2}+ik}(\cos\xi)}{Q^m_{-\frac{1}{2}+ik}(\cos\xi_1)P^m_{-\frac{1}{2}+ik}(\cos\xi_2) - P^m_{-\frac{1}{2}+ik}(\cos\xi_1)Q^m_{-\frac{1}{2}+ik}(\cos\xi_2)}$$
$$= V_1(\cosh\eta - \cos\xi)^{1/2} \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} e^{ik\eta} C(k, \cos\xi_1)$$
$$\times \frac{P_{-\frac{1}{2}+ik}(\cos\xi_2)Q_{-\frac{1}{2}+ik}(\cos\xi) - Q_{-\frac{1}{2}+ik}(\cos\xi_2)P_{-\frac{1}{2}+ik}(\cos\xi)}{Q_{-\frac{1}{2}+ik}(\cos\xi_1)P_{-\frac{1}{2}+ik}(\cos\xi_2) - P_{-\frac{1}{2}+ik}(\cos\xi_1)Q_{-\frac{1}{2}+ik}(\cos\xi_2)}. \tag{4}$$

The integration over the azimuthal angle φ' eliminates all the terms with $m \neq 0$ from the summation, reflecting the invariance under rotation around the axis of the toroidal capacitor. The integral over η' is represented by $C(k, \cos\xi_1)$, Eq. (B9), and is identified as the Fourier integral transform of the inverse of the square root of the binomial appearing in the integral. The form of the potential in Eq. (4b) satisfies the boundary condition at the electrode $\xi = \xi_2$, Eq. (2b), in an obvious manner. The boundary condition at the electrode $\xi = \xi_1$, Eq. (2a), is also satisfied as shown next: the fraction involving the Legendre functions becomes one, the integral over k is the inverse Fourier transform of $C(k, \cos\xi_1)$, Eq. (B10), which is the inverse of the square root of the binomial, and therefore the potential is V_1.

The electric intensity field is evaluated as the negative gradient of the potential function, [Eq. (4b)],

$$\vec{E}(\xi, \eta, \varphi) = -\left[\frac{\hat{\xi}}{h_\xi}\frac{\partial}{\partial\xi} + \frac{\hat{\eta}}{h_\eta}\frac{\partial}{\partial\eta} + \frac{\hat{\varphi}}{h_\varphi}\frac{\partial}{\partial\varphi}\right]\phi(\xi, \eta, \varphi)$$
$$= -\frac{V_1}{a}\int_{-\infty}^{\infty}\frac{dk}{\sqrt{2\pi}}\frac{C(k, \cos\xi_1)e^{ik\eta}(\cosh\eta - \cos\xi)^{3/2}}{\left[Q_{-\frac{1}{2}+ik}(\cos\xi_1)P_{-\frac{1}{2}+ik}(\cos\xi_2) - P_{-\frac{1}{2}+ik}(\cos\xi_1)Q_{-\frac{1}{2}+ik}(\cos\xi_2)\right]}$$
$$\times\left\{\hat{\xi}\left[P_{-\frac{1}{2}+ik}(\cos\xi_2)\frac{d}{d\xi}Q_{-\frac{1}{2}+ik}(\cos\xi) - Q_{-\frac{1}{2}+ik}(\cos\xi_2)\frac{d}{d\xi}P_{-\frac{1}{2}+ik}(\cos\xi)\right]\right.$$
$$+\left[\hat{\xi}\frac{\sin\xi}{2(\cosh\eta - \cos\xi)} + \hat{\eta}\left(ik + \frac{\sinh\eta}{2(\cosh\eta - \cos\xi)}\right)\right]$$
$$\left.\times\left[P_{-\frac{1}{2}+ik}(\cos\xi_2)Q_{-\frac{1}{2}+ik}(\cos\xi) - Q_{-\frac{1}{2}+ik}(\cos\xi_2)P_{-\frac{1}{2}+ik}(\cos\xi)\right]\right\}. \tag{5}$$

The electric field lines at the electrodes $\xi = \xi_2$ are obviously in the direction $\hat{\xi}$ perpendicular to the toroidal surface, since the second term inside the curly brackets of Eq. (5) vanishes. The electric lines at the electrode $\xi = \xi_1$ are also in the $\hat{\xi}$ direction perpendicular to the corresponding toroidal surface; in fact, the integrals in the $\hat{\eta}$ direction in Eq. (5) when evaluated using Eq. (B10)

$$\int_{-\infty}^{\infty}\frac{dk}{\sqrt{2\pi}}C(k,\cos\xi_1)\,e^{ik\eta}ik=\frac{d}{d\eta}\frac{1}{(\cosh\eta-\cos\xi_1)^{1/2}}=-\frac{1}{2}\frac{\sinh\eta}{(\cosh\;\eta-\cos\xi_1)^{3/2}}$$

$$\frac{1}{2}\int_{-\infty}^{\infty}\frac{dk}{\sqrt{2\pi}}C(k,\cos\xi_1)\frac{e^{ik\eta}\sinh\eta}{(\cosh\eta-\cos\xi_1)}=\frac{\sinh\eta}{2(\cosh\eta-\cos\xi_1)^{3/2}}$$

are seen to cancel each other.

The charge distributions on the electrodes follow from Gauss' law:

$$\sigma(\xi=\xi_2,\eta,\varphi)=-\frac{\hat{\xi}\cdot\vec{E}(\xi_2,\eta,\varphi)}{4\pi}\bigg|_{\xi=\xi_2} \tag{6a}$$

$$=-\frac{V_1}{4\pi a}\int_{-\infty}^{\infty}\frac{dk}{\sqrt{2\pi}}\frac{C(k,\cos\xi_1)e^{ik\eta}(\cosh\eta-\cos\xi_2)^{3/2}}{\sin\xi_2\left[Q_{-\frac{1}{2}+ik}(\cos\xi_1)P_{-\frac{1}{2}+ik}(\cos\xi_2)-P_{-\frac{1}{2}+ik}(\cos\xi_1)Q_{-\frac{1}{2}+ik}(\cos\xi_2)\right]},$$

where the explicit form of the Wronskian of the Legendre functions has been used.

$$\sigma(\xi=\xi_1,\eta,\varphi)=\frac{\hat{\xi}\cdot\vec{E}(\xi,\eta,\varphi)}{4\pi}\bigg|_{\xi=\xi_1}$$

$$=-\frac{V_1}{4\pi a}\int_{-\infty}^{\infty}\frac{dk}{\sqrt{2\pi}}C(k,\cos\xi_1)e^{ik\eta}(\cosh\eta-\cos\xi_1)^{3/2}$$

$$\times\left\{\frac{P_{-\frac{1}{2}+ik}(\cos\xi_2)\frac{d}{d\xi_1}Q_{-\frac{1}{2}+ik}(\cos\xi_1)-Q_{-\frac{1}{2}+ik}(\cos\xi_2)\frac{d}{d\xi_1}P_{-\frac{1}{2}+ik}(\cos\xi_1)}{Q_{-\frac{1}{2}+ik}(\cos\xi_1)P_{-\frac{1}{2}+ik}(\cos\xi_2)-P_{-\frac{1}{2}+ik}(\cos\xi_1)Q_{-\frac{1}{2}+ik}(\cos\xi_2)}+\frac{\sin\xi_1}{2(\cosh\eta-\cos\xi_1)}\right\}. \tag{6b}$$

The total charges on the electrodes are evaluated by integrating Eqs. (6) over the respective surfaces:

$$Q_2=\int_{-\infty}^{\infty}h_\eta\,d\eta\int_0^{2\pi}h_\varphi d\varphi\sigma(\xi=\xi_2,\eta,\varphi)$$

$$=-\frac{V_1a^2}{4\pi a}\int_{-\infty}^{\infty}\frac{dk}{\sqrt{2\pi}}C(k,\cos\xi_1)\int_{-\infty}^{\infty}\frac{d\eta e^{ik\eta}}{(\cosh\eta-\cos\xi_2)^{3/2}}$$

$$\times\int_0^{2\pi}d\varphi\left[Q_{-\frac{1}{2}+ik}(\cos\xi_2)P_{-\frac{1}{2}+ik}(\cos\xi_1)-P_{-\frac{1}{2}+ik}(\cos\xi_2)Q_{-\frac{1}{2}+ik}(\cos\xi_2)\right]^{-1}$$

$$=-\frac{V_1a}{2}\int_{-\infty}^{\infty}\frac{dkC(k_1\cos\xi_1)C(k,\cos\xi_2)}{Q_{-\frac{1}{2}+ik}(\cos\xi_2)P_{-\frac{1}{2}+ik}(\cos\xi_1)-P_{-\frac{1}{2}+ik}(\cos\xi_2)Q_{-\frac{1}{2}+ik}(\cos\xi_2)}, \tag{7a}$$

where the integral over η is again Eq. (B9a) with ξ_2 replacing ξ_1.

$$Q_1 = \frac{V_1 a}{2} \int_{-\infty}^{\infty} \frac{dk\, C(k, \cos\xi_1)}{Q_{-\frac{1}{2}+ik}(\cos\xi_1) P_{-\frac{1}{2}+ik}(\cos\xi_2) - P_{-\frac{1}{2}+ik}(\cos\xi_1) Q_{-\frac{1}{2}+ik}(\cos\xi_2)}$$
$$\times \left\{ \int_{-\infty}^{\infty} \frac{d\eta\, e^{ik\eta}}{\sqrt{2\pi}(\cosh\eta - \cos\xi_1)^{1/2}} \left[P_{-\frac{1}{2}+ik}(\cos\xi_2) \frac{d}{d\xi_1} Q_{-\frac{1}{2}+ik}(\cos\xi_1) \right.\right.$$
$$\left. - Q_{-\frac{1}{2}+ik}(\cos\xi_2) \frac{d}{d\xi_1} P_{-\frac{1}{2}+ik}(\cos\xi_1) \right]$$
$$+ \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\eta\, e^{ik\eta} \sin\xi_1}{\sqrt{2\pi}(\cosh\eta - \cos\xi_1)^{3/2}} \left[Q_{-\frac{1}{2}+ik}(\cos\xi_2)\, P_{-\frac{1}{2}+ik}(\cos\xi_1) \right.$$
$$\left.\left. - P_{-\frac{1}{2}+ik}(\cos\xi_2)\, Q_{-\frac{1}{2}+ik}(\cos\xi_2) \right] \right\}. \tag{7b}$$

The integrals over η in the respective terms inside the curly brackets according to Eq. (B9a) are respectively, $C(k, \cos\xi_1)$ and $-dC(k, \cos\xi_1)/d\xi_1$. Furthermore, the expression inside the curly brackets can be reduced to $C(k, \cos\xi_2)$, showing that the charges in both electrodes have the same magnitude.

The capacitance is then obtained from either Eq. (7):

$$C = \frac{Q}{V} = \frac{a}{2} \int_{-\infty}^{\infty} \frac{dk\, C(k, \cos\xi_1) C(k, \cos\xi_2)}{Q_{-\frac{1}{2}+ik}(\cos\xi_1) P_{-\frac{1}{2}+ik}(\cos\xi_2) - P_{-\frac{1}{2}+ik}(\cos\xi_1) Q_{-\frac{1}{2}+ik}(\cos\xi_2)}. \tag{8}$$

3. DISCUSSION

The analysis of moon-shape toroidal capacitors has been carried out in bispherical coordinates obtaining the harmonic expansions for the electrostatic potential function, Eq. (4b), the electric intensity field, Eq. (5), the charge distributions on the electrodes, Eqs. (6), the total charges, Eq. (7), and the capacitance Eq. (8).

Since this article completes the sequel of the study of toroidal, spherical-cap-electrode, bispherical and moon-shape toroidal capacitors, the remaining discussion focuses on the relationships among them and peculiarities of each one. The capacitors studied in this paper and those of Ref. 1 share the toroidal shape, differing in their partial moon and full moon meridian cross sections associated with converging circular arcs *versus* nested circles, respectively; such a difference is reflected in the corresponding harmonic expansions involving Fourier integral transforms versus Fourier series. The comparison of the moon shape toroidal and spherical cap capacitors shows the need of the insulating elements to separate their converging electrodes; the result is that the electric properties of both types of capacitors are described by integral transforms, of the Fourier type and conical Legendre type, respectively. The moon shape toroidal and bispherical capacitors are complementary to each other, just like the ones of Refs. 1 and 2, being described by the same coordinates; the corresponding harmonic expansions as Fourier integral transforms and series in Legendre polynomials reflect that complementarity. Our closing remark is that the study of these capacitors revolves around four different representations of the inverse of the square-root of the binomial, of which the one of Ref. 1 is available in mathematical tables and the other three were explicitly constructed in Refs. 2, 3 and this paper.

4. Appendix A. Bispherical coordinates

In the notation of Ref. 3 the equations connecting the cartesian and bispherical coordinates are

$$x = \frac{a\sin\xi\cos\varphi}{\cosh\eta - \cos\xi} \qquad y = \frac{a\sin\xi\sin\varphi}{\cosh\eta - \cos\xi} \qquad z = \frac{a\sinh\eta}{\cosh\eta - \cos\xi}. \tag{A1}$$

Fixed values of ξ in the interval $(0, \pi)$ define toroidal surfaces with meridian cross-sections that are circular arcs meeting at the poles located at $(x = 0, y = 0, z = \pm a)$; the value $\xi = 0$ corresponds the external part of the z-axis $(x = 0, y = 0, |z| > a)$ and the value $\xi = \pi$ corresponds to the central part of the z-axis $(x = 0, y = 0, |z| < a)$. Fixed values of η in the intervals $(-\infty, 0)$ and $0, \infty)$ define spheres nested around the south $(\eta = -\infty)$ and north $(\eta = \infty)$ poles respectively, with $\eta = 0$ corresponding to the equatorial plame. Fixed values of φ in the interval $(0, 2\pi)$ define the familiar meridian half-planes meeting at the z-axis.

The expressions for the scale factors and unit vectors follow directly from Eqs. (A1):

$$d\vec{r} = \hat{\imath}\,dx + \hat{\jmath}\,dy + \hat{k}\,dz = \hat{\xi}h_\xi\,d\xi + \hat{\eta}h_\eta\,d\eta + \hat{\varphi}h_\varphi\,d\varphi, \tag{A2}$$

$$h_\xi = h_\eta = \frac{a}{\cosh\eta - \cos\xi}\,, \qquad h_\varphi = \frac{a\sin\xi}{\cosh\eta - \cos\xi} \tag{A3}$$

$$\begin{aligned} \hat{\xi} &= \frac{(\cosh\eta\cos\xi - 1)(\hat{\imath}\cos\varphi + \hat{\jmath}\sin\varphi) - \hat{k}\sinh\eta\sin\xi}{\cosh\eta - \cos\xi}, \\ \hat{\eta} &= \frac{-\sinh\eta\sin\xi(\hat{\imath}\cos\varphi + \hat{\jmath}\sin\varphi) - \hat{k}(\cosh\eta\cos\xi - 1)}{\cosh\eta - \cos\xi}, \\ \hat{\varphi} &= -\hat{\imath}\,\mathrm{sen}\,\varphi + \hat{\jmath}\cos\varphi. \end{aligned} \tag{A4}$$

5. Appendix B. Construction of the Green function

The Dirichlet-Green function is to be constructed as a solution of Poisson's equation with a unit point source,

$$\nabla^2 G_D(\vec{r}, \vec{r}\,') = -4\pi\delta(\vec{r} - \vec{r}\,') \tag{B1}$$

and satisfying the boundary conditions

$$G_D(\xi = \xi_1, \eta, \varphi; \xi', \eta', \varphi') = 0, \quad G_D(\xi = \xi_2, \eta, \varphi; \xi', \eta', \varphi') = 0. \tag{B2}$$

The Dirac-delta function is expressed as

$$\delta(\vec{r}-\vec{r}\,') = \frac{\delta(\xi-\xi')\delta(\eta-\eta')\delta(\varphi-\varphi')}{h_\xi h_\eta h_\varphi}$$
$$= \frac{\delta(\xi-\xi')}{h_\xi h_\eta h_\varphi}\int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{ik(\eta-\eta')}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2\pi}\cos m(\varphi-\varphi') \qquad \text{(B3)}$$

using the Fourier-integral and Fourier-series representations of the respective factors in the η and φ coordinates [4]. This suggests the corresponding harmonic expansion for the Green function,

$$G_D(\xi,\eta,\varphi;\xi',\eta',\varphi') = (\cosh\eta-\cos\xi)^{1/2}(\cosh\eta'-\cos\xi')^{1/2}$$
$$\times\sum_{m=0}^{\infty}\int_{-\infty}^{\infty}\frac{dk}{2\pi}g_{km}(\xi,\xi')e^{ik(\eta-\eta')}\frac{\varepsilon_m}{2\pi}\cos m(\varphi-\varphi'), \qquad \text{(B4a)}$$

where the symmetry under the exchange of the field and source points has been taken into account, and

$$g_{km}(\xi,\xi') = A_{km}\left[Q^m_{-\frac{1}{2}+ik}(\cos\xi_1)\,P^m_{-\frac{1}{2}+ik}(\cos\xi_<) - P^m_{-\frac{1}{2}+ik}(\cos\xi_1)\,Q^m_{-\frac{1}{2}+ik}(\cos\xi_<)\right]$$
$$\times\left[P^m_{-\frac{1}{2}+ik}(\cos\xi_2)\,Q^m_{-\frac{1}{2}+ik}(\cos\xi_>) - Q^m_{-\frac{1}{2}+ik}(\cos\xi_2)\,P^m_{-\frac{1}{2}+ik}(\cos\xi_>)\right], \qquad \text{(B4b)}$$

guarantees that Eq. (B4a) has the form of Eq. (3), is continuous at $\xi=\xi'$, and satisfies the boundary conditions of Eqs. (B2).

The coefficients A_{km} in Eq. (4b) are determined by the integration of Eq. (B1) in the following way. The expansions of Eqs. (B3) and (B4a) are substituted in Eq. (B1), and use is made of the linear independence of both Fourier bases as well as the R-separability of the Laplace equation to obtain

$$\left[\frac{1}{\sin\xi}\frac{d}{d\xi}\sin\xi\frac{d}{d\xi}-\frac{m^2}{\sin^2\xi}\right]g_{km}(\xi,\xi') = -\frac{4\pi}{a\sin\xi}\,\delta(\xi-\xi')\;. \qquad \text{(B5)}$$

The integration around the source point $\xi=\xi'$ is immediate and shows the discontinuity in the derivative,

$$\sin\xi\frac{d}{d\xi}g_{km}(\xi,\xi')\bigg|_{\xi=\xi'_+} - \sin\xi\frac{d}{d\xi}g_{km}(\xi,\xi')\bigg|_{\xi=\xi'_-} = -\frac{4\pi}{a} \qquad \text{(B6)}$$

when the explicit form of Eq. (B4b) is used in Eq. (eB6) the needed coefficients are found to be

$$A_{km} = \frac{4\pi}{a\sin^2\xi'}\left[Q^m_{-\frac{1}{2}+ik}(\cos\xi_1)\,P^m_{-\frac{1}{2}+ik}(\cos\xi_2) - P^m_{-\frac{1}{2}+ik}(\cos\xi_1)\,Q^m_{-\frac{1}{2}+ik}(\cos\xi_2)\right]^{-1}$$
$$\times\left[P^m_{-\frac{1}{2}+ik}(\cos\xi')\frac{dQ^m_{-\frac{1}{2}+ik}(\cos\xi')}{d(\cos\xi')} - Q^m_{-\frac{1}{2}+ik}(\cos\xi')\frac{dP^m_{-\frac{1}{2}+ik}(\cos\xi')}{d(\cos\xi')}\right]^{-1}. \qquad \text{(B7)}$$

The quantity inside the last pair of brackets is recognized as the Wronskian of the Legendre functions, which is proportional to $\csc^2\xi'$ [7].

The construction of the electrostatic potential function in Eq. (4) requires the derivative of the Green function at the electrode $\xi = \xi_1$:

$$\begin{aligned}
\left.\frac{\partial G_D}{h_{\xi'}\partial\xi'}\right|_{\xi'=\xi} &= \frac{(\cosh\eta - \cos\xi)^{1/2}}{h_{\xi'}} \sum_{m=0}^{\infty} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \cos k(\eta-\eta') \frac{\varepsilon_m}{2\pi} \cos m(\varphi-\varphi') \\
&\quad \times A_{km} \left[P^m_{-\frac{1}{2}+ik}(\cos\xi_2)\, Q^m_{-\frac{1}{2}+ik}(\cos\xi) - Q^m_{-\frac{1}{2}+ik}(\cos\xi_2)\, P^m_{-\frac{1}{2}+ik}(\cos\xi)\right] \\
&\quad \times \Bigg\{(\cosh\eta' - \cos\xi')^{1/2} \left[Q^m_{-\frac{1}{2}+ik}(\cos\xi_1) \frac{d}{d\xi'} P^m_{-\frac{1}{2}+ik}(\cos\xi')\right. \\
&\quad \left. - P^m_{-\frac{1}{2}+ik}(\cos\xi_1) \frac{d}{d\xi'} Q^m_{-\frac{1}{2}+ik}(\cos\xi')\right] \\
&\quad + \frac{1}{2}(\cosh\eta' - \cos\xi')^{-1/2} \sin\xi' \left[Q^m_{-\frac{1}{2}+ik}(\cos\xi_1)\, P^m_{-\frac{1}{2}+ik}(\cos\xi')\right. \\
&\quad \left.\left. - P^m_{-\frac{1}{2}+ik}(\cos\xi_1)\, Q^m_{-\frac{1}{2}+ik}(\cos\xi')\right]\Bigg\}\right|_{\xi'=\xi_1} \\
&= -\frac{4\pi}{a h_{\xi_1} \sin\xi_1} (\cosh\eta - \cos\xi)^{1/2} (\cosh\eta' - \cos\xi_1)^{1/2} \\
&\quad \times \sum_{m=0}^{\infty} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \cos k(\eta-\eta') \frac{\varepsilon_m}{2\pi} \cos m(\varphi-\varphi') \\
&\quad \times \frac{P^m_{-\frac{1}{2}+ik}(\cos\xi_2)\, Q^m_{-\frac{1}{2}+ik}(\cos\xi) - Q^m_{-\frac{1}{2}+ik}(\cos\xi_2)\, P^m_{-\frac{1}{2}+ik}(\cos\xi)}{Q^m_{-\frac{1}{2}+ik}(\cos\xi_1)\, P^m_{-\frac{1}{2}+ik}(\cos\xi_2) - P^m_{-\frac{1}{2}+ik}(\cos\xi_1)\, Q^m_{-\frac{1}{2}+ik}(\cos\xi_2)},
\end{aligned} \tag{B8}$$

where use has been made of Eq. (B7) to obtain the final form.

The integral over η' appearing in the electrostatic potential function, Eq. (4b), is the Fourier integral transform of the inverse of the square root of the binomial

$$\int_{-\infty}^{\infty} \frac{d\eta'\, e^{-ik\eta'}}{\sqrt{2\pi}(\cosh\eta' - \cos\xi_1)^{1/2}} = C(k,\cos\xi_1) = \sum_{t=0}^{\infty} \binom{-1/2}{t} (-1)^t (\cos\xi_1)^t S_t(k), \tag{B9a}$$

where

$$S_t(k) = \int_{-\infty}^{\infty} \frac{d\eta'}{\sqrt{2\pi}} e^{ik\eta'} (\operatorname{sech}\eta')^{\frac{1}{2}+t} = \sum_{r=0}^{\infty} \binom{-\frac{1}{2}-t}{r} \frac{2^{\frac{1}{2}+t}}{\sqrt{2\pi}} \frac{1+2t+4r}{(\frac{1}{2}+t+2r)^2 + k^2} \tag{B9b}$$

is the Fourier integral transform of the corresponding power of the hyperbolic secant.

In turn, the inverse of the square root of the binomial can be written in terms of its Fourier integral transform

$$\frac{1}{(\cosh\eta - \cos\xi_1)^{1/2}} = \int_{-\infty}^{\infty} \frac{d\eta}{\sqrt{2\pi}} e^{ik\eta} C(k,\cos\xi_1). \tag{B10}$$

REFERENCES

1. A. Góngora T. and E. Ley-Koo, *Rev. Mex. Fís.* **40** (1994) 814.
2. A. Góngora T. and E. Ley-Koo, *Supl. Bol. Soc. Mex. Fís.* **8** (1994) 152.
3. A. Góngora T. and E. Ley-Koo, *Rev. Mex. Fís.* **42** (1996) 663.
4. G. Arfken, *Mathematical Methods for Physicists*, Second Edition (Academic Press, New York, 1970), Chap. 2.
5. W. Miller Jr. *Symmetry and Separation of Variables*, (Addison-Wesley, Reading, Mass. 1977), Chap. 3.
6. J.D. Jackson, *Classical Electrodynamics*, Second Edition, (Wiley, 1975), Chap. 1.
7. M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions*, (Dover, New York, 1965), Chap. 6.