

Magnification and distortion in the modern schiefspiegler

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ABSTRACT. The modern schiefspiegler is an off-axis, reflecting optical system that consists of a pair of prolate spheroids coupled by superimposing two of their focal points. It has some unusual properties. The angle between the two spheroid axes may be set arbitrarily large *ad initio*. It has a pseudo axis; a special chief ray which acts as an axis of symmetry for all other chief rays. Conditions for image formation have been established in previous papers. With the pseudo axis, the modern schiefspiegler exhibits some of the properties of a rotational optical system. It also possesses bilateral symmetry with respect to a plane of symmetry on which lie the pseudo axis and the two spheroid axes. In particular, in the plane of symmetry all chief rays are symmetric with respect to the pseudo axis while chief rays skew to the plane of symmetry are symmetric with respect to that plane of symmetry. It will be shown that it is possible to define angular magnification and angular distortion on the plane of symmetry in a way analogous to that of an axially symmetric optical system.

RESUMEN. El schiefspiegler moderno es un sistema óptico reflector fuera de eje que está formado por dos esferoides prolatos que han sido acoplados al superponer dos de sus puntos focales resultando en algunas propiedades extraordinarias. El ángulo entre los dos ejes de los esferoides puede ser establecido arbitrariamente *ad initio*. Posee un pseudoeje; un rayo principal con características particulares que actúa como un eje de simetría para todos los rayos principales. Las condiciones para la formación de imágenes han sido establecidas en artículos anteriores. Con el pseudoeje, el schiefspiegler moderno exhibe algunas de las propiedades de un sistema óptico rotacional. También posee simetría bilateral con respecto al plano de simetría sobre el cual se encuentran el pseudoeje y los ejes de los dos esferoides. En particular, sobre el plano de simetría todos los rayos principales son simétricos con respecto al pseudoeje mientras que los rayos principales oblicuos al plano de simetría son simétricos con respecto a ese plano de simetría. Se mostrará que es posible definir una amplificación angular y una distorsión angular en el plano de simetría de manera análoga a la forma en que se hace para un sistema óptico axialmente simétrico.

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1. INTRODUCTION

Conventional ray tracing programs usually refer to the vertex plane of a surface; that plane tangent to the surface where it is intersected by its axis. It is tacitly assumed that the point of incidence of ray and surface is the one nearest this plane. In the case

of the schiefspiegler this arrangement is undesirable; the proper point of incidence need not be that nearest the vertex plane. Moreover, the distance between vertex plane and the point of incidence may be large providing, an opportunity for error. Presented here is a detailed analysis of the relationship between the pseudo axis and a chief ray in the modern schiefspiegler. The application is to more appropriate ray tracing routines. In addition to this are formulas for angular magnification of such a system as well as an expressions for angular distortion.

The prolate spheroid is generated by rotating an ellipse about its major axis. The surface so formed has two foci lying on the axis of rotation. Its optical property is that any ray passing through one focus, after reflection, must pass through the other. To fix ideas, we call the first spheroid focus encountered the *proximal focus*; the second, the *distal focus*.

The modern schiefspiegler consists of a pair of these spheroids arranged so that the distal focus of the first coincides with the proximal focus of the second. The axes of the two spheroids that intersect at the common focus subtend an angle, β , that is arbitrary. The remaining two foci, the proximal focus of the first spheroid and the distal focus of the second, serve as entrance pupil and exit pupil, respectively.

This system is bilaterally symmetric. The plane determined by the axes of the two spheroids divides the schiefspiegler into two identical halves and is therefore a plane of symmetry.

Any ray passing through one of the foci must necessarily pass through all the others. Such a ray, since it passes through the central point of the entrance pupil, is therefore a *chief* or *principal* ray.

We define the *pseudo axis* of this system as that unique chief ray, lying in the plane of symmetry, that has the property that it is a line of symmetry for all other chief rays. That is to say, any ray lying in the plane of symmetry that makes an angle ϕ with the pseudo axis must have a mirror image that makes an angle $-\phi$ with the pseudo axis and that this property persists as both rays propagate through the system. This property is exact for all chief rays; it does not apply to other rays.

The condition for the existence of the pseudo axis has been derived in an earlier paper [1]. (In subsequent papers [2, 3] conditions for image formation have also been derived but these are not particularly relevant here.) This condition is

$$\tan \alpha = \frac{\epsilon_2(1 - \epsilon_1^2) \sin \beta}{\epsilon_1(1 + \epsilon_2^2) - \epsilon_2(1 + \epsilon_1^2) \cos \beta}, \quad (1)$$

where α is the angle between the pseudo axis and the axis of the first spheroid; β , the angle between the two spheroid axes; and ϵ_1 and ϵ_2 , the eccentricities of the first and second spheroid, respectively. This can be written in a different way:

$$\begin{aligned} \sin \alpha &= \frac{\epsilon_2(1 - \epsilon_1^2) \sin \beta}{pq}, \\ \cos \alpha &= \frac{\epsilon_1(1 + \epsilon_2^2) - \epsilon_2(1 + \epsilon_1^2) \cos \beta}{pq}, \end{aligned} \quad (2)$$

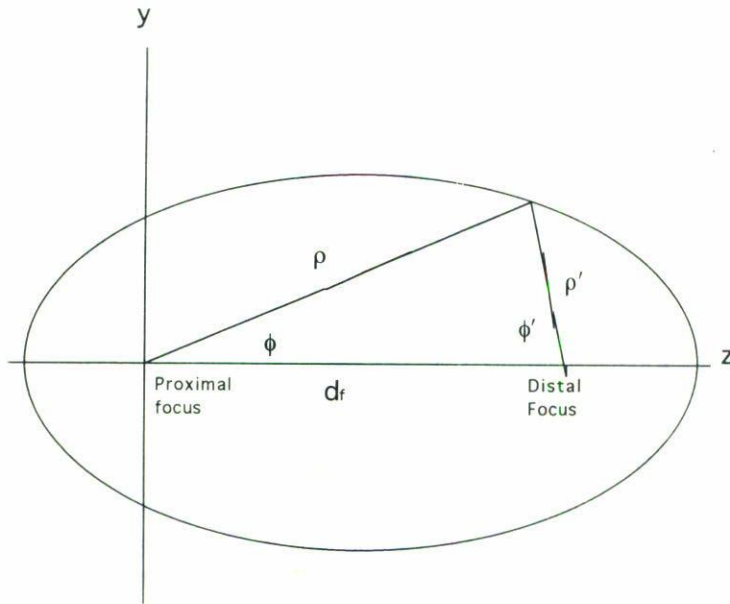


FIGURE 1. The ellipse. ϵ is the eccentricity; r , the semi latus rectum. Shown are the proximal and distal foci. d_f is the distance between the two foci. Relationships between ρ and ϕ , ρ' and ϕ' are given in Eqs. (5)–(9).

where

$$\begin{aligned} p^2 &= 1 + \epsilon_1^2 \epsilon_2^2 - 2\epsilon_1 \epsilon_2 \cos \beta, \\ q^2 &= \epsilon_1^2 + \epsilon_2^2 - 2\epsilon_1 \epsilon_2 \cos \beta. \end{aligned} \quad (3)$$

2. THE SINGLE SPHEROID

Shown in Fig. 1 is an ellipse with the various parameters that are used in its definition. Polar coordinates are used with the z -axis coinciding with the major axis. The y -axis passes through the proximal focus which is then the coordinate origin. The coordinates of a point are given by [4]

$$\begin{aligned} y &= \rho \sin \phi \\ z &= \rho \cos \phi, \end{aligned} \quad (4)$$

where

$$\rho = \frac{r}{1 - \epsilon \cos \phi}. \quad (5)$$

Here, r is the length of the semi latus rectum of the ellipse or, equivalently, its vertex radius of curvature.

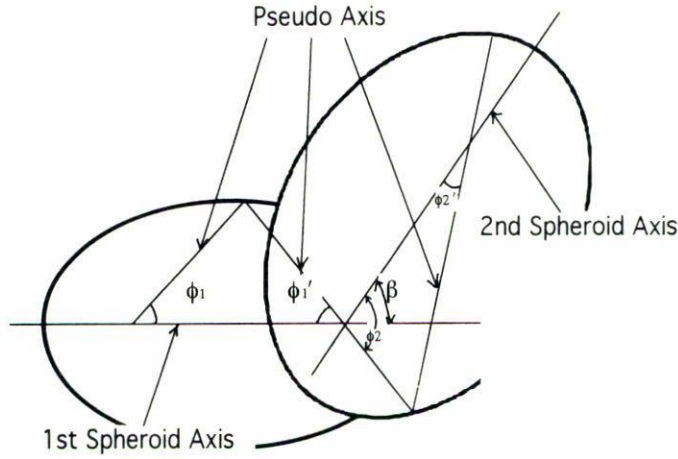


FIGURE 2. Two coupled spheroids. The distal focus of the 1st spheroid is made to coincide with the proximal focus of the 2nd. Note the pseudo axis ϕ_2 is shown as being negative to avoid confusion. The relationships between the various angle are found in Eqs. (10)–(12).

Referring now to the distal focus one can see that

$$\begin{aligned} y &= \rho' \sin \phi', \\ z &= d_f - \rho' \cos \phi', \end{aligned} \tag{6}$$

where

$$\rho' = \frac{r}{1 - \epsilon \cos \phi'}.$$

Here d_f is the distance between the two foci,

$$d_f = \frac{2\epsilon r}{1 - \epsilon^2}. \tag{7}$$

By using Eqs. (4)–(7) we can find that

$$\begin{aligned} \sin \phi' &= \frac{(1 - \epsilon^2) \sin \phi}{\kappa^2}, \\ \cos \phi' &= \frac{2\epsilon - (1 + \epsilon^2 \cos \phi)}{\kappa^2}, \end{aligned} \tag{8}$$

where

$$\kappa^2 = (1 + \epsilon^2) - 2\epsilon \cos \phi. \tag{9}$$

3. TWO COUPLED SPHEROIDS

Refer to Fig. 2. In what follows we shall neglect ρ , ρ' , and r . By applying the results of Eqs. (8) and (9) to each of the coupled spheroids we get

$$\begin{aligned}\kappa_1^2 &= (1 + \epsilon_1^2) - 2\epsilon_1 \cos \phi_1, \\ \kappa_1^2 \sin \phi_1' &= (1 - \epsilon_1^2) \sin \phi_1, \\ \kappa_1^2 \cos \phi_1' &= 2\epsilon_1 - (1 + \epsilon_1^2) \cos \phi_1,\end{aligned}\tag{10}$$

and

$$\begin{aligned}\kappa_2^2 &= (1 + \epsilon_2^2) - 2\epsilon_2 \cos \phi_2, \\ \kappa_2^2 \sin \phi_2' &= (1 - \epsilon_2^2) \sin \phi_2, \\ \kappa_2^2 \cos \phi_2' &= 2\epsilon_2 - (1 + \epsilon_2^2) \cos \phi_2,\end{aligned}\tag{11}$$

Now consider a chief ray that makes an angle θ with the pseudo axis, so that its angle with the axis of the first spheroid will be $\phi_1 = \alpha + \theta$. The relationship between the pseudo axis and the chief ray is shown in Fig. 3. Then Eq. (10) becomes

$$\begin{aligned}\kappa_1^2 &= (1 + \epsilon_1^2) - 2\epsilon_1(\cos \alpha \cos \theta - \sin \alpha \sin \theta), \\ \kappa_1^2 \sin \phi_1' &= (1 - \epsilon_1^2)(\sin \alpha \cos \theta + \cos \alpha \sin \theta), \\ \kappa_1^2 \cos \phi_1' &= 2\epsilon_1 - (1 + \epsilon_1^2)(\cos \alpha \cos \theta - \sin \alpha \sin \theta).\end{aligned}\tag{12}$$

The angle between the axis of the first spheroid and that of the second is β so that $\phi_2 = \phi_1' - \beta$. We return to Eq. (11). With this substitution and the results of the Eq. (12) we get

$$\kappa_2^2 = 1 + \epsilon_2^2 - 2\epsilon_2 \cos(\phi_1' - \beta).$$

Multiplying this by κ_1^2 yields

$$\begin{aligned}\kappa_1^2 \kappa_2^2 &= (1 + \epsilon_2^2)[(1 + \epsilon_1^2) - 2\epsilon_1(\cos \alpha \cos \theta - \sin \alpha \sin \theta)] \\ &\quad - 2\epsilon_2\{\cos \beta[2\epsilon_1 - (1 + \epsilon_1^2)(\cos \alpha \cos \theta - \sin \alpha \sin \theta)] \\ &\quad + \sin \beta[(1 - \epsilon_1^2)(\cos \alpha \cos \theta - \sin \alpha \sin \theta)]\}, \\ &= (1 + \epsilon_1^2)(1 + \epsilon_2^2) - 4\epsilon_1\epsilon_2 \cos \beta \\ &\quad + 2\{-[\epsilon_1(1 + \epsilon_2^2) - \epsilon_2(1 + \epsilon_1^2) \cos \beta] \cos \alpha - \epsilon_2(1 - \epsilon_1^2) \sin \beta \sin \alpha\} \cos \theta \\ &\quad + 2\{[\epsilon_1(1 + \epsilon_2^2) - \epsilon_2(1 + \epsilon_1^2) \cos \beta] \sin \alpha - \epsilon_2(1 - \epsilon_1^2) \sin \beta \cos \alpha\} \sin \theta.\end{aligned}$$

In what follows let

$$\mathcal{F}(\alpha + \theta) = \kappa_1^2 \kappa_2^2.\tag{13}$$

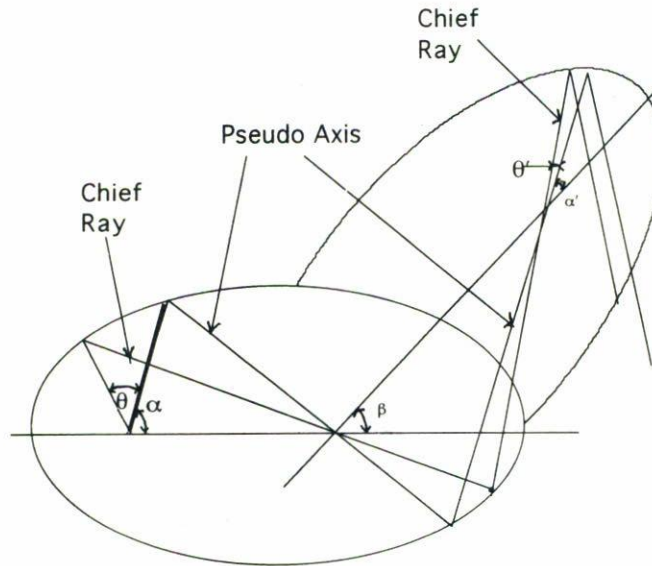


FIGURE 3. The modern schiefspiegler. Shown are the two coupled spheroids with the pseudo axis indicated. A second chief ray, making an angle θ with the pseudo axis is shown.

Then, using this and the results of Eq. (12) as well as Eqs. (2) and (3), we obtain the much simpler result

$$\mathcal{F}(\alpha + \theta) = p^2 + q^2 - 2pq \cos \theta. \quad (14)$$

The second equation of Eq. (11) now becomes

$$\begin{aligned} \mathcal{F}(\alpha + \theta) \sin \phi'_2 &= (1 - \epsilon_2^2) \{ \cos \beta [(1 - \epsilon_1^2)(\sin \alpha \cos \theta + \cos \alpha \sin \theta)] \\ &\quad - \sin \beta [2\epsilon_1 - (1 + \epsilon_1^2)(\cos \alpha \cos \theta - \sin \alpha \sin \theta)] \}, \\ &= (1 - \epsilon_2^2) \{ -2\epsilon_1 \sin \beta + [(1 - \epsilon_1^2) \cos \beta \sin \alpha + (1 + \epsilon_1^2) \sin \beta \cos \alpha] \cos \theta \\ &\quad + [(1 - \epsilon_1^2) \cos \beta \cos \alpha - (1 + \epsilon_1^2) \sin \beta \sin \alpha] \sin \theta \}. \end{aligned}$$

Again using Eqs. (12) and (14) as well as Eqs. (2) and (3), we find that this simplifies to

$$\begin{aligned} \mathcal{F}(\alpha + \theta) \sin \phi'_2 &= \frac{1}{pq} \{ \epsilon_1 (1 - \epsilon_2^2) \sin \beta [(p^2 + q^2) \cos \beta - 2pq] \\ &\quad - (p^2 - q^2) [\epsilon_2 (1 + \epsilon_1^2) - \epsilon_1 (1 + \epsilon_2^2) \cos \beta] \sin \beta \}. \quad (15) \end{aligned}$$

We do the same sort of thing for the third equation of Eq. (12), and get

$$\begin{aligned} \mathcal{F}(\alpha + \theta) \cos \phi'_2 &= 2[\epsilon_2 (1 + \epsilon_1^2) - \epsilon_1 (1 + \epsilon_2^2) \cos \beta] \\ &\quad + \{ [(1 + \epsilon_1^2)(1 + \epsilon_2^2) \cos \beta - 4\epsilon_1 \epsilon_2] \cos \alpha - (1 - \epsilon_1^2)(1 + \epsilon_2^2) \sin \beta \sin \alpha \} \cos \theta \\ &\quad - \{ [(1 + \epsilon_1^2)(1 + \epsilon_2^2) \cos \beta - 4\epsilon_1 \epsilon_2] \sin \alpha + (1 - \epsilon_1^2)(1 + \epsilon_2^2) \sin \beta \cos \alpha \} \sin \theta. \end{aligned}$$

Using Eqs. (12) and (14) together with Eqs. (2) and (3) one last time, we obtain

$$\mathcal{F}(\alpha + \theta) \cos \phi'_2 = \frac{1}{pq} \{ -[\epsilon_2(1 + \epsilon_1^2) - \epsilon_1(1 + \epsilon_2^2) \cos \beta][(p^2 + q^2) \cos \theta - 2pq] - \epsilon_1(1 - \epsilon_2^2)(p^2 - q^2) \sin \beta \sin \theta \}. \quad (16)$$

If θ is set equal to zero then ϕ'_2 becomes the angle between the pseudo axis and the axis of the second spheroid which we will designate as α' . Then, from Eq. (14)

$$\mathcal{F}(\alpha) = (p - q)^2, \quad (17)$$

from Eq. (15) we get

$$\sin \alpha' = \frac{1}{pq} \epsilon_1(1 - \epsilon_2^2) \sin \beta, \quad (18)$$

and from Eq. (16),

$$\cos \alpha' = -\frac{1}{pq} [\epsilon_2(1 + \epsilon_1^2) - \epsilon_1(1 + \epsilon_2^2) \cos \beta]. \quad (19)$$

Now going back to Eqs. (14) and (17) we find that

$$\mathcal{F}(\alpha + \theta) = \mathcal{F}(\alpha) + 2pq(1 - \cos \theta), \quad (20)$$

with Eqs. (15), (18), and (19), that

$$\mathcal{F}(\alpha + \theta) \sin \phi'_2 = [(p^2 + q^2) \cos \beta - 2pq] \sin \alpha' + (p^2 - q^2) \cos \alpha' \sin \theta, \quad (21)$$

and, from Eqs. (16), (18), and (19), that,

$$\mathcal{F}(\alpha + \theta) \cos \phi'_2 = [(p^2 + q^2) \cos \theta - 2pq] \cos \alpha' - (p^2 - q^2) \sin \alpha' \sin \theta. \quad (22)$$

Finally, let θ' be the angle that the ray makes with the pseudo axis in image space, so that $\theta' = \phi'_2 - \alpha'$. Then from Eqs. (20), (21), and (22) we obtain,

$$\begin{aligned} \sin \theta' &= \frac{(p^2 - q^2) \sin \theta}{p^2 + q^2 - 2pq \cos \theta}, \\ \cos \theta' &= \frac{-2pq + (p^2 + q^2) \cos \theta}{p^2 + q^2 - 2pq \cos \theta}. \end{aligned} \quad (23)$$

4. ANGULAR MAGNIFICATION AND DISTORTION

From Eq. (23) we may write

$$\tan \theta' = \frac{(p^2 - q^2) \sin \theta}{-2pq + (p^2 + q^2) \cos \theta}.$$

This is the same as

$$\tan \theta' = M(\theta) \tan \theta, \quad (24)$$

where

$$M(\theta) = \frac{(p^2 - q^2) \cos \theta}{-2pq + (p^2 + q^2) \cos \theta}.$$

It is clear that

$$M(0) = \frac{p + q}{p - q} \quad (25)$$

represents the angular magnification of the system and that

$$\mathcal{D}(\theta) = M(\theta) - M(0) = \frac{p + q}{p - q} \frac{2pq(1 - \cos \theta)}{-2pq + (p^2 + q^2) \cos \theta}, \quad (26)$$

is an expression for distortion. Fractional distortion or percent distortion is then the much simpler

$$\tilde{\mathcal{D}}(\theta) = \frac{M(\theta) - M(0)}{M(0)} = \frac{2pq(1 - \cos \theta)}{-2pq + (p^2 + q^2) \cos \theta}. \quad (27)$$

5. CONCLUSIONS

This completes the task. Relationships between a chief ray, as it progresses through the system, and the pseudo axis have been found. This provides part of the requirements for a more efficient ray tracing system. Also obtained are formulas for angular magnification and angular distortion for the modern schiefspiegler. They depend only upon the angle between the two prolate axes and the eccentricities of the prolate spheroids. The condition for the existence of the pseudo axis determines a third angle, the angle between the pseudo axis and the axis of the prolate spheroid.

But this applies only to rays on the plane of symmetry. Ray off this plane are not symmetric with respect to the pseudo axis but are with respect to the plane of symmetry itself. Therefore angular magnification and angular distortion need to be determined for those rays lying in a plane perpendicular to the plane of symmetry. In general magnification in this plane is not equal to the magnification calculated here. This leads to the fact that the modern schiefspiegler is anamorphic.

Future work will lead to the determination of the degree of this anamorphotism and a means for correcting it.

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