Recursive relations for ray-tracing through three-dimensional reflective confocal prolate spheroids

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RESUMEN. Modern, large telescopes incorporate segmented primary mirrors, with off-axis spheroidal shapes. Each off-axis mirror and the secondary mirror represent an off-axis optical system without an axis of symmetry and for which the paraxial optics does not apply. Simple, recursive ray-trace equations are presented that describe exactly a ray propagating through a series of coupled, confocal, prolate spheroids. Vector formulation using the ray direction cosines is employed for characterizing the ray propagation in different planes through the spheroid axis of symmetry.

ABSTRACT. Los grandes telescopios modernos incorporan espejos primarios segmentados de formas esferoides fuera de eje. Cada segmento y el espejo secundario representan un sistema óptico fuera de eje sin simetría rotacional para los cuales la óptica paraxial no se aplica. Se presentan ecuaciones recursivas sencillas que describen exactamente un rayo propagándose a través de una serie de esferoides prolatos de focos coincidentes. Debido a la presencia de varios segmentos en diferentes planos, para caracterizar la propagación del rayo se emplea una formulación vectorial haciendo uso de los cosenos directores.

1. INTRODUCTION

The benefits of incorporating the prolate spheroids have been realized in the operational optical systems that were designed to meet high performance requirements. The segmented and the dilute aperture telescopes are probably the most notable example of the trend to design optical systems with coupled, confocal prolate spheroids. [1, 2]

We present ray-trace equations for a ray propagating through two coupled, reflective confocal spheroids in three-dimensions. The angles of reflection from the second surface are related to the angles of incidence on the first surface using the direction cosines and the optical path distance along the ray. [3] Both of these formulations are required for the development of a wavefront aberration function for an optical system without an axis of symmetry. [4] A dilute aperture array is one example of an optical system which does not possess an axis of symmetry.

Due to its potential of increasing the effective aperture size using small aperture optical sub-systems [5, 6] it remains of great current interest. Its performance cannot be

described adequately with the currently-available theories. [7, 8] The inherent limitations of the multiple aperture phased arrays may be better understood once their theoretical performance has been predicted and the engineering control issues have been decoupled from the optical system design and optimization. Each of the individual apertures may be coupled with the beam-combining mirror using one pseudo axis. [9] However, the *complete* optical system does not posses an axis of symmetry. The results presented here are an extension to three-dimensions of the theoretical development for a ray propagating in a single plane of symmetry through reflective confocal prolate spheroids. [10]

In the following Section we define the coordinate systems and summarize the characteristics of the confocal spheroids. In Sect. 3, the ray reflection from the point on the spheroid is presented in terms of the ray direction cosines. The direction cosines are evaluated using the optical path difference along the ray propagating between two spheroid foci. In Sect. 4, the direction cosines of the ray reflected from the second spheroid are related to the direction cosines of the ray incident on the first spheroid, in terms of eccentricities only.

2. GEOMETRICAL DESCRIPTION OF A PROLATE SPHEROID

A prolate spheroid is a three-dimensional, geometrical surface, generated when a circle, ellipse, parabola, or a hyperbola is rotated about its geometrical axis of symmetry, resulting in a sphere, an ellipsoid, a paraboloid, or a hyperboloid, all of revolution. [11] The same mathematical representation may be used for all these geometrical surfaces, with different eccentricity value, ε. The value of $\varepsilon = 0$ results in a sphere; the value of ε between 0 and 1 results in an ellipsoid; the choice of $\varepsilon = \pm 1$ gives a paraboloid ($+1$ paraboloid opens to the right); and any value of ε greater than 1, and smaller than -1 generates a hyperboloid.

2.1. COORDINATE SYSTEMS

We use an ellipsoid to illustrate the representative surface for the purpose of this discussion. We adopt a modified spherical coordinate system, shown in Fig. 1. The x-, y-, z-Cartesian coordinate system has the origin at O, with the positive z-axis along the axis of symmetry of the geometrical object. (Suitably-chosen) center of the off-axis segment has coordinates $(x_c, 0, z_c)$, *i.e.*, it defines the direction of the x-axis. The ellipsoid is drawn so that its proximal focus coincides with the coordinate origin at O. The R-axis is defined so that a point $P(x, y, z)$, on the off-axis spheroid segment, lie s in the R-z plane. The x- and y-axes are normal to each other. The R-axis makes an angle Phi with the x-axis. The projection of the ellipsoid on the R-z plane is an ellipse. The projection of the ellipsoid on the x-y plane is a circle.

The azimuthal angle θ is measured from the positive z-axis. Usually, its value is restricted to the range between 0 and π. In the modified spherical coordinate system that we are using here, its range of values is extended to include values from 0 to 2π. This is done so that the results evaluated in the polar R-θ coordinate system may be used for comparison with previous work. Earlier results were presented in two dimensions, on
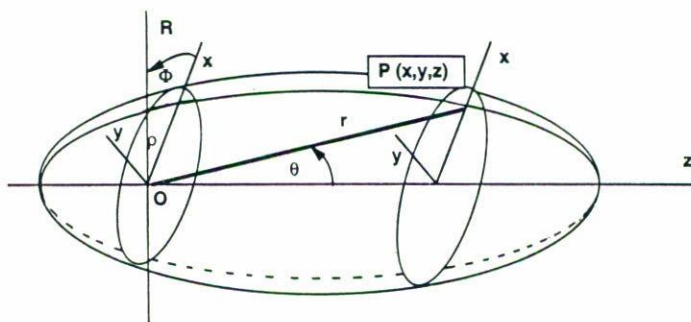
FIGURA 1. An ellipsoid with z axis along the axis of symmetry, in a polar and a cartesian coordinate system.

the basis of symmetry considerations. The angle Φ, measured in x, y plane is the second coordinate, and the radius-vector r is the third coordinate. The coordinates of a general point $P(x, y, z)$ in this coordinate system are given by

$$x = r \sin\theta \cos\Phi, \tag{1}$$

$$y = r \sin\theta \sin\Phi, \tag{2}$$

$$z = r \cos\theta. \tag{3}$$

We also note that

$$R = r \sin\theta. \tag{4}$$

Then we get

$$x = R \cos\Phi, \tag{5}$$

$$y = R \sin\Phi. \tag{6}$$

When the point R in Eqs. (4)–(6) is on the spheroid, R becomes ρ.

2.2. SPHEROID REPRESENTATION

The description of an ellipsoid is very simple in the modified spherical coordinate system. Figure 2 shows the spheroid parameters, in any R-z plane, due to the ellipsoid cylindrical symmetry. The radius-vector r, the distance of an arbitrary point $P(x, y, z)$ from the proximal focus may be given in terms of only one angle:

$$r = \frac{\rho}{1 - \varepsilon \cos\theta}. \tag{7}$$

Here, ρ denotes the semilatus rectum or the vertex radius of curvature and ε is the eccentricity. If the origin of the modified spherical coordinate system were chosen at the second, the distal, focus, the negative sign in the denominator of Eq. (7) would change
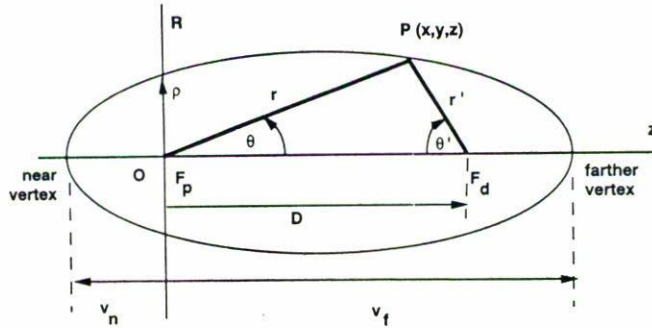
FIGURA 2. Ellipsoid parameters.

and become positive. (This change may be accomplished formally by replacing θ by $\theta + \pi$). Same range of r-values is traced by the radius-vector. This radius-vector is also the distance of an arbitrary point $P(x, y, z)$ from the distal focus:

$$r = \frac{\rho}{1 + \varepsilon\theta}. \tag{8}$$

Considering the formal nature of the change of coordinates, it is clear that θ is measured counterclockwise from the positive z-axis.

We wish to develop recursive relationships for tracing a ray through two confocal prolate spheroids. Next, we consider the ray propagation within a single i-th spheroid.

3. Ray reflection at an off-axis point of the i-th prolate spheroid

Here we present the ray reflection both in terms of the radius-vector and in terms of the ray direction cosines. The direction cosines are evaluated upon the consideration of the total optical path length for a ray propagating between two foci.

3.1. Ray reflection in terms of the radius-vectors

A ray segment is represented as a distance along the radius-vector. The optical path distance may therefore be obtained by adding two appropriate radius-vectors. Now we express the radius-vector of the reflected ray in terms of the angles of incidence Φ_i and θ_i, and the spheroid parameters ρ_i and ε_i. With the interpretations presented in the previous section, we may use a single equation, Eq. (7), to describe a point on a prolate spheroid, or a ray segment from the proximal focus to the point $P(x_i, y_i, z_i)$ on a prolate spheroid. As illustrates in Fig. 3, a general ray is assumed incident through a near focus in the direction specified by angles θ_i and Φ_i. (The subscript i and the spheroid itself have been omitted in Fig. 3 for clarity)

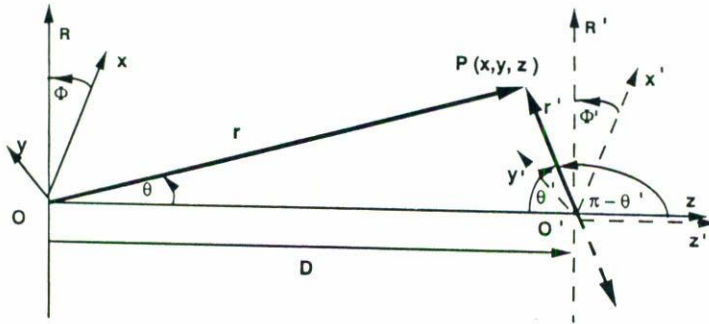$$r_i = \frac{\rho_i}{1 - \varepsilon_i \cos \theta_i}. \tag{7a}$$

FIGURA 3. The (ray) optical path distance between two foci is equal to the addition of two radii-vectors, each originating at a focus and both pointing to the same point.

The cartesian coordinates of a point $P(x_i, y_i, z_i)$ on the surface of the prolate spheroid, x_i, y_i, and z_i may be given in the spherical coordinate system of r_i, Φ_i, and θ_i

$$x_i = \frac{r_i \sin \theta_i \cos \Phi_i}{1 - \varepsilon_i \cos \theta_i}, \tag{9}$$

$$y_i = \frac{r_i \sin \theta_i \sin \Phi_i}{1 - \varepsilon_i \cos \theta_i}, \tag{10}$$

$$z_i = \frac{r_i \cos \theta_i}{1 - \varepsilon_i \cos \theta_i}. \tag{11}$$

The ray is reflected from the point $P(x_i, y_i, z_i)$ on the spheroid in such a way that it passes through the distal focus, by the descriptive, functional, operational definition of the ellipsoid. The distance between the distal focus and the point $P(x_i, y_i, z_i)$, r'_i, is denoted by r' in Fig. 3.

By replacing θ_i with $\pi - \theta'_i$ in Eq. (8) and denoting the resulting distance with r'_i, the following relationship is obtained:

$$r'_i = \frac{r_i}{1 - \varepsilon_i \cos \theta'_i}. \tag{12}$$

We note that ρ_i and ε_i are the same as in Eq. (7a), since we are describing the same ellipsoid. However, the angle θ'_i is measured from the negative z-axis in the clockwise direction.

The cartesian coordinates of a point $P(x_i, y_i, z_i)$ on the surface of the prolate spheroid, x_i, y_i, and z_i may also be expressed in the primed coordinate system. This coordinate system, also illustrated in Fig. 3, is defined as follows:

$$x' = x, \tag{13}$$

$$z' = D - z, \tag{14}$$

$$y' = y. \tag{15}$$

The coordinates of the point $P(x_i, y_i, z_i)$ are given in the primed spherical coordinate system, r_i', Φ_i', and θ_i', as follows:

$$x_i = \frac{r_i' \sin \theta_i' \cos \Phi_i'}{1 - \varepsilon_i \cos \theta_i'}, \tag{16}$$

$$y_i = \frac{r_i' \sin \theta_i' \sin \Phi_i'}{1 - \varepsilon_i \cos \theta_i'}, \tag{17}$$

$$z_i = D_i - \frac{r_i' \cos \theta_i'}{1 - \varepsilon_i \cos \theta_i'}. \tag{18}$$

Two sets of equations, Eqs. (9)–(11) and Eqs. (16)–(18) describe the same point on the spheroid. Upon setting the corresponding coordinates equal, we get the following equalities:

$$x_i = \frac{r_i \sin \theta_i \cos \Phi_i}{1 - \varepsilon_i \cos \theta_i} = \frac{r_i' \sin \theta_i' \cos \Phi_i'}{1 - \varepsilon_i \cos \theta_i'}, \tag{19}$$

$$y_i = \frac{r_i \sin \theta_i \sin \Phi_i}{1 - \varepsilon_i \cos \theta_i} = \frac{r_i' \sin \theta_i' \sin \Phi_i'}{1 - \varepsilon_i \cos \theta_i'}, \tag{20}$$

$$z_i = \frac{r_i \cos \theta_i}{1 - \varepsilon_i \cos \theta_i} = D_i - \frac{r_i' \cos \theta_i'}{1 - \varepsilon_i \cos \theta_i'}. \tag{21}$$

Next we evaluate the coordinates x_i and y_i and Eqs. (5) and (6) for the points on the spheroid. We obtain the following relationship from Eq. (19) or (20):

$$r_i = \frac{r_i \sin \theta_i}{1 - \varepsilon_i \cos \theta_i} = \frac{r_i' \sin \theta_i'}{1 - \varepsilon_i \cos \theta_i'} \tag{22}$$

We denote the quantities in Eq. (5) with a subscript i. We divide the left and right side of Eq. (22) with the corresponding sides of Eq. (21). Then we obtain a relationship between the angle of reflection θ_i' and the angle of incidence θ_i and eccentricity ε_i:

$$\tan \theta_i' = \frac{(1 - \varepsilon_i^2) \sin \theta_i}{2\varepsilon_i - (1 + \varepsilon_i^2) \cos \theta_i}. \tag{23}$$

The expressions for $\sin \theta_i'$ and $\cos \theta_i'$ are derived by a applying the trigonometric relationships defined in a right triangle:

$$\sin \theta_i' = \frac{(1 - \varepsilon_i^2) \sin \theta_i}{K_i}, \tag{24}$$

$$\cos \theta_i' = 2 \frac{\varepsilon_i - (1 + \varepsilon_i^2) \cos \theta_i}{K_i}. \tag{25}$$

Here, K_i denotes the common denominator:

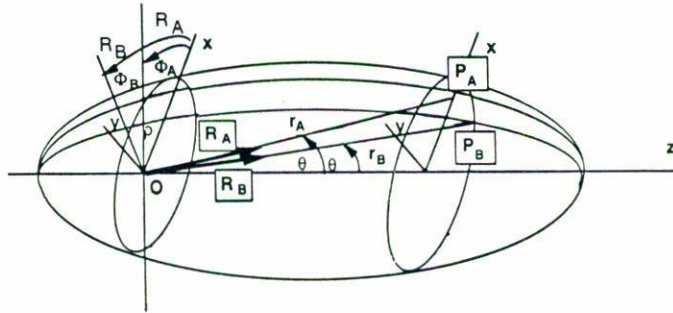$$K_i = (1 + \varepsilon_i^2) - 2\varepsilon_i \cos \theta_i \tag{26}$$

FIGURA 4. Two rays with different direction cosines and the same azimuthal angle θ.

The second part of the angle of incidence, Φ_i, may also be evaluated from Eqs. (19) and (20). Dividing the corresponding sides of these equations, we get

$$\tan \Phi_i' = \tan \Phi_i. \tag{27}$$

This (predictable) result says that the reflected ray and the incident ray lie in the same plane, defined as the R-z plane in Fig. 1. Thus we obtain from Eq. (27) that the angles are also equal:

$$\Phi_i' = \Phi_i. \tag{28}$$

While an individual ray indeed remains in one plane, there will be rays in infinitely many different planes. In particular, for the case of the multiple-apertura primary in a multiple mirror telescope, the center of each aperture and the secondary mirror lie on a separate line.

We are interested in understanding the image-forming properties of a system consisting of a series of confocal prolate spheroids. An off-axis spheroid segment is expected to display different amounts of aberration in the x-y plane and x-z plane. In order to obtain a ray trajectory in three-dimensions, we develop next a direction of a reflected ray in terms of its direction cosines.

3.2. REFLECTION IN TERMS OF RAY DIRECTION COSINES

The rays incident on the spheroid will be in different R-z planes as illustrated in Fig. 4. Two rays A and B, incident from the proximal focus, subtend the same angle θ and both satisfy Eq. (7). However, they are incident with a different angle Φ. To uniquely characterize rays in a non-rotationally symmetric system, I am additionally interested in expressing the direction of propagating ray in terms of its direction cosines. The direction cosines of the incident ray are denoted by l_i, m_i, n_i:

$$l_i = \sin \theta_i \cos \Phi_i, \tag{29}$$

$$m_i = \sin \theta_i \sin \Phi_i, \tag{30}$$

$$n_i = \cos \theta_i. \tag{31}$$

The direction cosines of the reflected ray l_i', m_i', n_i' are parallel to r_i', but are directed in the opposite direction to the direction of the radius-vector r_i'.

$$l_i' = \sin(-\theta_i') \cos(-\Phi_i'), \tag{32}$$

$$m_i' = \sin(-\theta_i') \sin(-\Phi_i'), \tag{33}$$

$$n_i' = \cos(-\theta_i'). \tag{34}$$

With the trigonometric relationships for negative angles, we get

$$l_i' = -\sin\theta_i' \cos\Phi_i', \tag{35}$$

$$m_i' = \sin\theta_i' \sin\Phi_i', \tag{36}$$

$$n_i' = \cos\theta_i'. \tag{37}$$

Using equations developed in the previous section for the azimuthal angle of the reflected ray, Eqs. (24)–(26) and (28), we can evaluate the direction cosines of the reflected ray:

$$l_i' = -\frac{(1-\varepsilon_i^2) \cos\Phi_i \sin\theta_i}{K_i}, \tag{38}$$

$$m_i' = \frac{(1-\varepsilon_i^2) \sin\Phi_i \sin\theta_i}{K_i}, \tag{39}$$

$$n_i' = \frac{2\varepsilon_i - (1+\varepsilon_i^2) \cos\theta_i}{K_i}. \tag{40}$$

Here the common denominator depends on the angle of incidence and the eccentricity:

$$K_i = (1 + \varepsilon_i^2) - 2\varepsilon_i \cos\theta_i. \tag{41}$$

The direction cosines of the ray reflected of the point on the spheroid depend on both angles of incidence Φ_i and θ_i of the incident ray.

The expressions developed in this section for the geometrical relationships between the radius-vectors connecting foci and the points on the prolate spheroids may be used to derive ray transfer equations within a single prolate spheroid characterized by only three parameters, θ_i, ρ_i, and ε_i.

Next, we apply the ray transfer relationships among the spheroid quantities developed here for a single spheroid to two consecutive spheroids. The two consecutive spheroids are related in that the ray that leaves the first spheroid is the very same ray as the ray that is incident on the second spheroid.

4. Exact recursive ray trace equations for two (off-axis) confocal prolate spheroids

we wish to evaluate the direction of propagation of the ray reflected from the second spheroid in terms of the direction cosines. We first evaluate the ray transfer equations for the ray in an arbitrary plane R–z.

4.1. RAY TRANSFER IN TERMS OF RADIUS-VECTOR

A general ray is incident in any azimuthal plane, characterized by any angle θ_{i+1}. This ray reflects according to the laws of reflection developed earlier. Equations (23)–(26) are evaluated for $(i + 1)$-st prolate spheroid, characterized by ε_{i+1} and ρ_{i+1}:

$$\tan \theta'_{i+1} = \frac{(1 - \varepsilon^2_{i+1}) \sin \theta_{i+1}}{2\varepsilon_{i+1} - (1 + \varepsilon^2_{i+1}) \cos \theta_{i+1}}, \tag{42}$$

$$\sin \theta'_{i+1} = \frac{(1 - \varepsilon^2_{i+1}) \sin \theta_{i+1}}{K_{i+1}}, \tag{43}$$

$$\cos \theta'_{i+1} = \frac{3\varepsilon_{i+1} - (1 + \varepsilon^2_{i+1}) \cos \theta_{i+1}}{K_{i+1}}. \tag{44}$$

Here, K_{i+1} denotes the common denominator

$$K_{i+1} = (1 + \varepsilon^2_{i+1}) - 2\varepsilon_{i+1} \cos \theta_{i+1}. \tag{45}$$

Φ_{i+1} is the angle of incidence for the ray incident on the $(i + 1)$-st spheroid. Also, from Eq. (27), we know that

$$\tan \Phi'_{i+1} = \tan \Phi_{i+1}. \tag{46}$$

The radius-vector of the ray leaving the i-th spheroid is collinear with the radius-vector of the ray incident on the $(i + 1)$-st spheroid. Thus, we note the following relationships from Fig. 3:

$$\theta_{i+1} = \theta'_i, \tag{47}$$

$$\Phi_{i+1} = \Phi'_i. \tag{48}$$

Using Eq. (35), we get

$$\Phi_{i+1} = \Phi_i + \pi. \tag{49}$$

Using Eq. (47), we may eliminate θ_{i+1} from Eq. (23) (with the subscript i replaced by $i + 1$)

$$\tan \theta'_{i+1} = -\frac{(1 - \varepsilon^2_{i+1}) \sin \theta'_i}{2\varepsilon_{i+1}(1 + \varepsilon^2_{i+1}) \cos \theta'_i}. \tag{50}$$

After the substitution of Eqs. (24) and (25) for $\sin \theta'_i$ and $\cos \theta'_i$, respectively, we get an expression for the angle θ'_{i+1}:

$$\tan \theta'_{i+1} = -\frac{(1 - \varepsilon^2_{i+1})(1 - \varepsilon^2_i) \sin \theta_i}{2\varepsilon_{i+1}[(1 + \varepsilon^2_i) - 2\varepsilon_i \cos \theta_i] - (1 + \varepsilon^2_{i+1})[2\varepsilon_i - (1 + \varepsilon^2_i) \cos \theta_i]}. \tag{51}$$

We regroup to show explicit dependence on $\sin\theta_i$ and $\cos\theta_i$ terms. Also, we introduce constants A_{i+1}, B_{i+1}, and C_{i+1}, that depend only on eccentricities:

$$\tan\theta'_{i+1} = -\frac{A_{i+1}\sin\theta_i}{B_{i+1} + C_{i+1}\cos\theta_i}. \tag{52}$$

The constants associated with each spheroid are defined as follows:

$$A_{i+1} = (1 - \varepsilon_{i+1}^2)(1 - \varepsilon_i^2), \tag{53}$$

$$B_{i+1} = 2(\varepsilon_{i+1} - \varepsilon_i)(1 - \varepsilon_{i+1}\varepsilon_i), \tag{54}$$

$$C_{i+1} = (1 - \varepsilon_{i+1}\varepsilon_i)^2 + (\varepsilon_{i+1} - \varepsilon_i)^2. \tag{55}$$

In Eqs. (52) through (55), the angle of reflection from the second spheroid has been obtained in terms of the angle of incidence on the first ellipsoid, and the spheroid eccentricities. Due to the repeated application of the law of reflection, a ray remains in a single plane of incidence after multiple reflections. However, when considering rays in different planes of incidence, the direction cosines of the reflected ray needs to be evaluated in terms of the spheroid parameters and in terms of the angles of incidence.

4.2. DIRECTION COSINES

The direction cosines of rays incident on the $(i + 1)$-st spheroid are the same as the direction cosines of the ray leaving the i-th spheroid:

$$l_{i+1} = l'_i, \tag{56}$$

$$m_{i+1} = m'_i, \tag{57}$$

$$n_{i+1} = n'_i. \tag{58}$$

The direction cosines of the ray leaving the i-th spheroid, l'_i, m'_i, n'_i, are parallel to r'_i, but directed in the opposite direction:

$$\theta_{i+1} = -\theta'_i, \tag{59}$$

$$\Phi_{i+1} = -\Phi'_i. \tag{60}$$

The direction cosines of the ray leaving the $(i + 1)$-st spheroid are expressed in terms of the quantities defined for the i-th spheroid, using Eqs. (38)–(39):

$$l_{i+1} = -\frac{(1 - \varepsilon_i^2)\cos\Phi_i\sin\theta_i}{K_i}, \tag{61}$$

$$m_{i+1} = \frac{(1 - \varepsilon_i^2)\sin\Phi_i\sin\theta_i}{K_i}, \tag{62}$$

$$n_{i+1} = \frac{2\varepsilon_i - (1 + \varepsilon_i^2)\cos\theta_i}{K_i}. \tag{63}$$

Equations (24)–(26) give $\sin\theta_i'$, $\cos\theta_i'$ and K_i. We also know that the direction cosines of the reflected ray $(l_{i+1}', m_{i+1}', n_{i+1}')$ are parallel to r_{i+1}', but directed in the opposite direction.

If we replace i with $(i+1)$ Eqs. (38)–(39), we get the following relationships:

$$l_{i+1}' = -\frac{(1-\varepsilon_{i+1}^2)\cos\Phi_{i+1}\sin\theta_{i+1}]}{Ki+1},\tag{64}$$

$$m_{i+1}' = \frac{(1-\varepsilon_{i+1}^2)\sin\Phi_{i+1}\sin\theta_{i+1}}{Ki+1},\tag{65}$$

$$n_{i+1}' = \frac{2\varepsilon_{i+1}-(1+\varepsilon_{i+1}^2)\cos\theta_{i+1}}{K_{i+1}}.\tag{66}$$

In Eqs. (69)–(71), the denominator K_{i+1} is given as

$$K_{i+1} = (1+\varepsilon_{i+1}^2)-2\varepsilon_{i+1}\cos\theta_{i+1}.\tag{67}$$

The angles have been found in the section dealing with the ray transfer in terms of the radius-vector. From Eq. (48), we can eliminate Φ_{i+1}:

$$l_{i+1}' = -\frac{(1-\varepsilon_{i+1}^2)\cos\Phi_i\sin\theta_{i+1}}{K_{i+1}},\tag{68}$$

$$m_{i+1}' = -\frac{(1-\varepsilon_{i+1}^2)\sin\Phi_i\sin\theta_{i+1}}{K_{i+1}},\tag{69}$$

$$n_{i+1}' = \frac{2\varepsilon_{i+1}-(1+\varepsilon_{i+1}^2)\cos\theta_{i+1}}{K_{i+1}}.\tag{70}$$

Upon replacing i with $(i+1)$ Eqs. (24), (25) and (59), we obtain the following identities:

$$\sin\theta_{i+1} = -\frac{(1-\varepsilon_i^2)\sin\theta_i}{K_i},\tag{71}$$

$$\cos\theta_{i+1} = \frac{2\varepsilon_i-(1+\varepsilon_i^2)\cos\theta_i}{K_i}.\tag{72}$$

And, K_i is given in Eq. (26). Upon the substitution of Eqs. (71) and (72) into Eqs. (68)–(70), we get the direction cosines of the ray leaving the $(i+1)$-st spheroid:

$$l_{i+1}' = \frac{A_{i+1}\cos\Phi_i\sin\theta_i}{B_{i+1}+C_{i+1}\cos\theta_i},\tag{73}$$

$$m_{i+1}' = \frac{A_{i+1}\sin\Phi_i\sin\theta_i}{B_{i+1}+C_{i+1}\cos\theta_i},\tag{74}$$

$$n_{i+1}' = \frac{B_{i+1}+C_{i+1}\cos\theta_i}{C_{i+1}+B_{i+1}\cos\theta_i}.\tag{75}$$

The constants A_{i+1}, B_{i+1}, C_{i+1} that depend only on the spheroid eccentricities have been defined in Eqs. (53)–(55).

5. Summary

We developed ray trace equation for the light propagating in three dimensions, through a series of confocal prolate spheroids. This development is different from the traditional ray trace approach as no real ray is assumed to satisfy the conditions of paraxial optics.

We employed the optical path distance traditionally employed in the rotationally-symmetric optical systems for the image assessment. Additionally, we found it necessary to express the propagation of the ray in terms of the ray vector in a three-dimensional space, because the image forming rays for the distinct off-axis segment are not restricted to a single plane. The set of spheroid mirror segments does not constitute a system with rotationall symmetry.

In the standard ray trace analysis, two equations are used to trace a ray through an optical system: the refraction at the surface and the ray transfer between surfaces. Eq. (35) gives the reflection at the spheroid mirror with an eccentricity ε in terms of the angle measured from the geometrical axis of the parent. Ray transfer distance along the geometrical axis of the parent spheroid is presented in Eq. (15). Ray transfer distance along the ray itself is given in Eq. (30).

The ray transfer equations obtained for individual spheroids have been evaluated to develop recursive relations for ray propagation through a two confocal prolate spheroids. Due to the non-rotationally symmetric nature of this problem, a vector formulation is used to describe the ray reflections. The optical path difference equations are used to evaluate the angles.

Acknowledgment

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