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General relativistic hydrodynamics: a new approach

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We provide a new general relativistic version of the Euler equation for perfect fluids and its corresponding equation of conservation of matter-energy. Such equations are written in a component language based on the 3+1 and tetrad formalisms of general relativity.

Keywords: General relativity: 3+1 formalism, tetrads; Hydrodynamics: Euler equation

En este artículo se presenta una nueva versión de las ecuaciones de Euler de los fluidos perfectos y de su correspondiente ecuación de conservación de materia-energía para el caso de relatividad general. Las ecuaciones han sido escritas en un lenguaje de componentes con base en los formalismos 3+1 y de tétradas de la relatividad general.

Descriptores: Relatividad general: formalismo 3+1, tétradas; Hidrodinámica: ecuación de Euler

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1. Introduction

According to the 3+1 formulation of general relativity (see Refs. 1–6 for details) the spacetime $(M^4, \boldsymbol{g})$ is foliated by a family of spacelike hypersurfaces Σ_t, labeled by a real parameter t called *the coordinate time*. The hypersurfaces Σ_t are thus a family of Riemannian subspaces $(M^3, \boldsymbol{h})$, where $\boldsymbol{h}$ is the *induced metric* (the 3-metric) on M^3. Moreover, on every hypersurface one can introduce a spatial coordinate system $(x^i) = (x^1, x^2, x^3)$.

The *induced metric* (first fundamental form) of the hypersurfaces Σ_t is given by

$$\boldsymbol{h} := \boldsymbol{n} \otimes \boldsymbol{n} + \boldsymbol{g}, \tag{1}$$

where $\boldsymbol{n}$ stands for the timelike unit fourvector normal to Σ_t. This vector defines the fourvelocity of the *Eulerian* observer $\mathcal{O}_\circ$ which is orthogonal to Σ_t [7]. Its components with respect to the coordinates (t, x^1, x^2, x^3) are

$$n_\alpha = (-N, 0, 0, 0), \quad n^\alpha = \left(\frac{1}{N}, \frac{N^1}{N}, \frac{N^2}{N}, \frac{N^3}{N}\right); \tag{2}$$

where $\boldsymbol{N} = N^i \boldsymbol{e}_i$ is a 4-vector tangent to Σ_t, called the *shift* vector, N is the *lapse* function and $\{\boldsymbol{e}_i\}$ represents a coordinate basis of Σ_t. Therefore, $\{\boldsymbol{n}\, , \, \boldsymbol{e}_i\}$ constitutes a basis of the tangent space to $(M^4, \boldsymbol{g})$.

We can decompose tensors with respect to this basis and obtain components parallel and orthogonal to Σ_t [8]. In particular, a 2-rank symmetric tensor $\boldsymbol{T}$, like the stress energy-momentum tensor, is decomposed as follows [7]:

$$S^{\beta\mu} := T^{\alpha\nu}\, h^\beta_\alpha\, h^\mu_\nu\, , \tag{3}$$

$$\tilde{J}^\alpha := -T^{\alpha\nu}\, n_\nu\, , \tag{4}$$

$$J^\beta := \tilde{J}^\alpha\, h^\beta_\alpha\, , \tag{5}$$

$$E := T^{\alpha\nu}\, n_\alpha\, n_\nu\, , \tag{6}$$

where $\boldsymbol{S}$ is called the *tensor of constraints*, $\boldsymbol{J}$ is the *momentum density vector*, and E is to be interpreted as the *total* mass-energy density measured by the Eulerian observer $\mathcal{O}_\circ$. Then, $\boldsymbol{T}$ reads,

$$T^{\beta\mu} = S^{\beta\mu} + J^\beta n^\mu + n^\beta J^\mu + E n^\beta n^\mu\, . \tag{7}$$

It it clear that Eq. (7) can be also written in terms of the components associated to a tetrad constructed from the basis $\{\boldsymbol{n}, \boldsymbol{e}_i\}$ (see the Appendix B and Refs. 9–12 and 24 for a review of the tetrad formalism).

1.1. The 3+1 Equation of conservation of energy

The general relativistic equations of motion for matter are obtained by applying the covariant divergence to (7):

$$\nabla \cdot \mathbf{T} = -\boldsymbol{\mathcal{F}}\, , \tag{8}$$

where $\boldsymbol{\mathcal{F}}$ stands for some external fields.

Applying the operator $\nabla \cdot$ to each term of the tetrad components of $\boldsymbol{T}$ we obtain for the time component,

$$S^{(i)(j)}\, \mathcal{O}^{(0)}_{(i)(j)} + J^{(i)}\, \mathcal{O}^{(0)}_{(0)(i)} + E\, \nabla \cdot \mathbf{n}$$
$$+ \partial_{(i)} J^{(i)} + J^{(i)}\, \mathcal{O}^{(\beta)}_{(\beta)(i)} + \partial_{(0)} E = -\mathcal{F}^{(0)}\, , \tag{9}$$

where the indices in parenthesis refer to the components with respect to the tetrad (hereafter referred to as physical components), and we have used the fact that the physical temporal components of vectors and tensors tangent to Σ_t vanish identically (see the Appendix C), and also that $n^{(\mu)} = (1,0,0,0)$. The quantities (see the Appendix B)

$$\mathcal{O}^{(\alpha)}_{(\beta)(\gamma)} := e^{(\alpha)}_{\mu}\, q^{\mu}_{(\gamma);\nu}\, q^{\nu}_{(\beta)}\,, \tag{10}$$

are called the Ricci rotation coefficients (hereafter referred to as RRC). We remind that, unlike the Christoffel symbols, the RRC are not symmetric with respect to the covariant indices, and therefore they satisfy some remarkable properties (see Refs. 11 and 12 for a review). In particular, they are related with the 3+1 variables in a very particular way (see Appendix C). Using such relations in Eq. (9) we find,

$$-S^{(i)(j)}K_{(i)(j)} + 2J^{(i)}a_{(i)} - EK + \partial_{(i)}J^{(i)} + J^{(i)}\,{}^3\mathcal{O}^{(j)}_{(j)(i)} + \partial_{(0)}E = -\mathcal{F}^{(0)}\,, \tag{11}$$

where ${}^3\mathcal{O}^{(j)}_{(j)(i)}$ are the 3-RRC (*i.e.*, the RRC compatible with the triad associated to the 3-metric $\boldsymbol{h}$); $K_{(i)(j)}$ are the physical components of the second fundamental form of the hypersurfaces Σ_t (also known as the *extrinsic curvature*), which is defined in terms of the Lie derivative by [7]

$$\mathbf{K} := -\frac{1}{2}\mathcal{L}_n \boldsymbol{h}\,, \tag{12}$$

with the property that its trace is simply

$$\mathrm{K} := \mathrm{Tr}\,\mathbf{K} = -\nabla\cdot\boldsymbol{n}\,. \tag{13}$$

The quantities $a_{(i)}$ which contract with $J^{(i)}$ in Eq. (11) are the physical components of the 4-*acceleration* of the Eulerian observer $\mathcal{O}_\circ$. This is defined by

$$\mathbf{a} := \nabla_n \boldsymbol{n} = \boldsymbol{h}\cdot\nabla(\ln N)\,, \tag{14}$$

which suggests the interpretation of the lapse function as the *acceleration potential* for $\mathcal{O}_\circ$ (*cf.* [7]). From the normalization condition $\boldsymbol{n}\cdot\boldsymbol{n} = -1$, it is clear that $\boldsymbol{a}$ and $\boldsymbol{n}$ are mutually orthogonal: $\boldsymbol{n}\cdot\boldsymbol{a} = 0$.

Finally, we remark that the operator $\partial_{(0)}$ is the time component of the so-called *directional* or *Pfaffian* derivative [11–13], which in terms of the usual derivatives, and the tetrad fields $q^{\mu}_{(\alpha)}$ reads (see Appendix B)

$$\partial_{(\alpha)} := q^{\mu}_{(\alpha)}\partial_\mu\,. \tag{15}$$

Equation (11) can be arranged as to obtain

$$\partial_{(0)}E + {}^3\nabla\cdot\mathrm{J} - EK + 2J^{(i)}a_{(i)} - S^{(i)(j)}K_{(i)(j)} = -\mathcal{F}^{(0)}\,. \tag{16}$$

We can identify this as the triad version of the equation of conservation of matter-energy of Bardeen & Piran [14].†

1.2. The 3+1 equation of conservation of momentum

In the same way, the spatial components of (8) lead to

$$\partial_{(j)}S^{(i)(j)} + S^{(i)(j)}a_{(j)} + S^{(i)(j)}\,{}^3\mathcal{O}^{(l)}_{(l)(j)} + S^{(j)(l)}\,{}^3\mathcal{O}^{(i)}_{(l)(j)} + \partial_{(0)}J^{(i)} - J^{(l)}\left(K^{(i)}_{(l)} - \mathcal{O}^{(i)}_{(0)(l)}\right) - J^{(i)}K + Ea^{(i)} = -\mathcal{F}^{(i)}\,. \tag{17}$$

This can be written as

$$\partial_{(0)}J^{(i)} + S^{(i)(j)}_{\;|(j)} + S^{(i)(j)}a_{(j)} - J^{(i)}K - J^{(l)}\left(K^{(i)}_{(l)} - \mathcal{O}^{(i)}_{(0)(l)}\right) + Ea^{(i)} = -\mathcal{F}^{(i)}\,, \tag{18}$$

where $S^{(i)(j)}_{\;|(j)} := \partial_{(j)}S^{(i)(j)} + S^{(j)(l)}\,{}^3\mathcal{O}^{(i)}_{(l)(j)}$, and the identities of Appendix C were used. The term in parenthesis can take different forms whether we use (C.32), (C.33) or (C.34). Let us take for instance (C.32). Then,

$$\partial_{(0)}J^{(i)} + S^{(i)(j)}_{\;|(j)} + S^{(i)(j)}a_{(j)} - J^{(i)}K + J^{(l)}\mathcal{C}^{(i)}_{(l)} + Ea^{(i)} = -\mathcal{F}^{(i)}\,. \tag{19}$$

This can be identified as the *triad* version of the equation of conservation of momentum of Bardeen & Piran [14].

2. General relativistic hydrodynamics

Let us consider the tetrad components of the energy-momentum tensor for perfect fluids:

$$T^{(\alpha)(\beta)} = (e+p)\,u^{(\alpha)}u^{(\beta)} + \eta^{(\alpha)(\beta)}p\,. \tag{20}$$

The physical components of the fourvelocity of the fluid $u^{(\alpha)}$ contains indeed the kinetic relativistic effects (Lorentz factors) which are a landmark of the relative motion of observers (namely, the *Eulerian* observer $\mathcal{O}_\circ$ and the observer comoving to the fluid $\mathcal{O}_f$). It will be convenient to remove such effects by introducing the more meaningful quantities:

$$U^{(\alpha)} := \frac{u^{(\alpha)}}{u^{(0)}} = \frac{u^{(\alpha)}}{\Gamma}, \tag{21}$$

where $\Gamma = -n_\mu u^\mu = u^{(0)}$ is the Lorentz factor between the hereabove quoted observers:

$$\Gamma = \left(-U^{(\alpha)}U^{(\beta)}\eta_{(\alpha)(\beta)}\right)^{-1/2} = \left(1 - {U^{(i)}}^2\right)^{-1/2} \tag{22}$$

Therefore, from the expression of the fourvelocity $u^\mu = \Gamma/N(1, dx^i/dt)$, and Eqs. (21), (B.5) and (C.19), we obtain

$${}^3U^{(i)} = \frac{1}{N}\left(V^{(l)} - N^{(l)}\right), \tag{23}$$

where $V^{(i)}$ and $N^{(i)}$ are the triad components of the coordinate 3-velocity $V^i = dx^i/dt$, and the shift vector N^i respectively. Moreover, we have added an upperscript '3' to stress that ${}^3U^{(i)}$ are in fact projections on Σ_t [*cf.* Eq. (C.22)].

The quantities ${}^3U^{(i)}$ are the general relativistic generalizations of the Newtonian fluid velocity components as "measured" by the Eulerian observer.

As we mentioned, the meaning of the term $\left(V^{(l)} - N^{(l)}\right)/N$ is clear, it gives the physical components of the fluid-velocity relative to the Eulerian observer $\mathcal{O}_\circ$, and therefore we notice that such an observer is forced to move with a physical 3-velocity $N^{(i)}$ with respect to the coordinates x^i. This is the well known phenomenon of *dragging of inertial frames* (*cf.* [2]). Furthermore, the *lapse* factor N in (23), indicates that the time intervals measured by $\mathcal{O}_\circ$ undergo a contraction due to the gravitational field.

Moreover, the relative velocity $V^{(l)} - N^{(l)} = e^{(i)}_l\left(V^l - N^l\right)$ contains in addition some other general relativistic effects like those associated to the coordinate factors and metric potentials given by the triad fields $e^{(i)}_l$, which in fact provide the physical units (units of *length/time*) to this new variables. For instance, in the case of a fluid in relativistic rigid axisymmetric motion, the angular component of ${}^3U^{(i)}$ in the direction of the angle of symmetry (*i.e.*, ϕ-component) reads in the so-called *maximal slicing quasi-isotropic coordinates* (MSQI) as follows [15, 16]:

$${}^3U^{(\phi)} = \frac{A^2 Br\sin\theta}{N}\left(\Omega - N^\phi\right). \tag{24}$$

Here $A(r,\theta)$, $B(r,\theta)$ are the metric potentials which give the actual length of the corresponding curved spacetime; $\Omega := V^\phi = d\phi/dt$ is the coordinate angular velocity of the fluid, and $r\sin\theta$ is a coordinate factor which give the units of length/time to this velocity component. In other words, the Eulerian observer measures *with his rods* the infinitesimal fluid displacements $A^2Br\sin\theta d\phi$ in the ϕ-direction. However, since he is also moving with coordinate angular displacements $N^\phi dt = d\phi_\circ$, he has to make a subtraction in order to find the actual infinitesimal fluid distances measured with respect to his reference frame (his triad). These actual distances are then given by $A^2Br\sin\theta(d\phi - N^\phi dt)$. On the other hand, *his clock* measures infinitesimal time intervals $N\,dt$. When dividing the above local displacements by these time intervals, he finally obtains the fluid velocity with respect to his triad frame [*i.e.*, Eq. (24)].

It is to note, that the entire factor $A^2Br\sin\theta$ is the *circumferential* radius of a circle centered around the axis of symmetry in the MSQI coordinates, *i.e.*, the radius obtained from the quotient between the circumference of such a circle and the factor 2π. This suggest the following quasi-Newtonian expression:

$${}^3U^{(\phi)} = R_\circ\Omega_\circ, \tag{25}$$

with $R_\circ := A^2Br\sin\theta$, and $\Omega_\circ := \left(\Omega - N^\phi\right)/N$, which in the Newtonian limit reduces to the well known tangential velocity $\Omega r\sin\theta$.

Substitution of (21) into (20) leads to

$$T^{(\alpha)(\beta)} = (E+p)\,U^{(\alpha)}U^{(\beta)} + \eta^{(\alpha)(\beta)}p, \tag{26}$$

where, $E := (e+p)\,\Gamma^2 - p$ is, as mentioned, the total mass-energy density measured by $\mathcal{O}_\circ$ [*cf.* Eq. (6)].

The remaining 3+1-energy-momentum entities for the perfect fluid, *i.e.*, the tensor of constraints and the momentum density vector given by Eqs. (3) and (5), lead together with (26) and (23) to the 3+1-triad components

$$S^{(i)(j)} = (E+p)\;{}^3U^{(i)}\;{}^3U^{(j)} + \delta^{(i)(j)}p, \tag{27}$$

$$J^{(i)} = (E+p)\;{}^3U^{(i)}. \tag{28}$$

2.1. Equation of conservation of energy for perfect fluids

Using Eqs. (16), (27) and (28) we obtain:

$$\partial_{(0)}E + \left[(E+p)\;{}^3U^{(l)}\right]_{||l} + (E+p)\left[2\;{}^3U^{(j)}a_{(j)} - {}^3U^{(l)}\;{}^3U^{(j)}K_{(l)(j)} - K\right] = -\mathcal{F}^{(0)}. \tag{29}$$

We are now in position to interpret the different terms appearing in this equation. The operator '$\partial_{(0)}$' for example, is identified according to (B.6) and Eqs. (C.13) and (C.15) with the directional derivative '$n^\mu\partial_\mu$'. Such a term represents the time variation (as measured by $\mathcal{O}_\circ$) of the energy density E. The general relativity corrections are contained in n^μ via the shift and the lapse. In the flat limit, '$\partial_{(0)}$' reduces to the ordinary operator $\partial/\partial t$. The second term gives the flow of the momentum density relative to $\mathcal{O}_\circ$. The general relativity corrections to the flat limit appear in this term not only from the covariant divergence, but also from the lapse and the *dragging of inertial frame* terms included in $J^i = N^{-1}(E+p)[V^i - N^i]$.

The term such as $2\,{}^3U^{(j)}(E+p)a_{(j)} \equiv 2J^{(j)}a_{(j)}$, is to be interpreted as a power (by unit of volume) developed by the system due to the coupling between the acceleration of $\mathcal{O}_\circ$ and the fluid current $J^{(j)}$; the term $(E+p)\,{}^3U^{(l)}\,{}^3U^{(j)}K_{(l)(j)}$ gives a kind of energy stored by the coupling of the velocity field with the extrinsic curvature tensor. Finally, we can appreciate the trace of $\boldsymbol{K}$ (K), which is in fact related with the *slicing condition*. For instance, the *maximal* slicing condition corresponds to coordinates with vanishing K [7].

2.2. The Euler equation in general relativity

The 3+1 equation of conservation of momentum (19) applied to perfect fluids leads to

$$(E+p)\left[\partial_{(0)}\,{}^3U^{(i)} + {}^3U^{(j)}\,{}^3U^{(i)}_{|(j)}\right] + {}^3U^{(i)}\partial_{(0)}p + {}^3U^{(i)}\left[E_{,(0)} + J^{(l)}_{|(l)} - (E+p)K\right] + p\left({}^3\mathcal{O}^{(l)}_{(l)(i)} + {}^3\mathcal{O}^{(i)}_{(l)(l)}\right)$$
$$+ {}^3\partial^{(i)}p + (E+p)\,{}^3U^{(i)}\,{}^3U^{(l)}a_{(l)} + (E+p)\,{}^3U^{(l)}\mathcal{C}^{(i)}_{(l)} + (E+p)a^{(i)} = -\mathcal{F}^{(i)}. \quad (30)$$

Using the antisymmetry property of the 3-RRC, and the equation of conservation of energy we obtain after some simplifications [15]

$$\partial_{(0)}\,{}^3U^{(i)} + {}^3U^{(j)}\,{}^3U^{(i)}_{|(j)} = -\frac{1}{E+p}\left[{}^3\partial^{(i)}p + {}^3U^{(i)}\partial_{(0)}p\right] - a^{(i)}$$
$$+ {}^3U^{(i)}\,{}^3U^{(l)}\left(a_{(l)} - {}^3U^{(j)}K_{(l)(j)}\right) - {}^3U^{(l)}\mathcal{C}^{(i)}_{(l)} + \frac{1}{E+p}\left({}^3U^{(i)}\mathcal{F}^{(0)} - \mathcal{F}^{(i)}\right). \quad (31)$$

We begin the interpretation of (31) from the left to the right hand side. We can first split up the term ${}^3U^{(j)}\,{}^3U^{(i)}_{|(j)}$ into ${}^3U^{(j)}\partial^3_{(j)}U^{(i)} + {}^3U^{(l)}\,{}^3U^{(j)}\mathcal{O}^{(i)}_{(l)(j)}$. The expression $\partial_{(0)}\,{}^3U^{(i)} + {}^3U^{(j)}\partial^3_{(j)}U^{(i)}$ is the convective or material derivative of the velocity field. This generalizes the Newtonian expression by the operator '$\partial_{(0)}$', and also by the fact that '$\partial_{(j)} = q^l_{(j)}\,{}^3\partial_l$' includes the metric potentials $q^l_{(j)}$, which provide the physical *lengths* in the directions of the coordinates x^i.

The remaining term, *i. e.*, ${}^3U^{(l)}\,{}^3U^{(j)}\mathcal{O}^{(i)}_{(l)(j)}$, generalizes the Newtonian *inertial accelerations* acting on the fluid, such as the *centrifugal* ones. Note, that it is quadratic on the velocity field (see Ref. 17).

In the right-hand side, we can recognize the factor $-1/(E+p)$ which is well known from the special-relativistic limit [*cf.* Eq. (2.10.16) of Ref. 18]. Inside the square bracket we appreciate also a special-relativistic term which couples the velocity field with the "physical" time derivative of the pressure [*cf.* Eq. (2.10.16) of Ref. 18]. Such a term vanishes in the Newtonian limit. The remaining term inside the brackets gives the physical pressure gradients.

Next to the bracket we see the acceleration of the Eulerian observer (the term $a^{(i)}$), which generalizes the Newtonian acceleration potential $-\nabla\Phi$ [*cf.* Eq.(14)].

The second line of (31), shows the term ${}^3U^{(i)}\,{}^3U^{(l)}a_{(l)}$ which is also quadratic in the fluid-velocity field, and which couples with the acceleration of the Eulerian observer. Such a term represents a kind of *flow of power* (velocity×power)—per unit of mass—. Moreover, this term is the equivalent of the relativistic correction of the inertial acceleration which is present in the formula that governs the motion of a particle in the *proper reference frame of an accelerated observer* [2]. It is to be stressed, that in such a formula usually a factor of '2' appears multiplying the equivalent term to ${}^3U^{(i)}\,{}^3U^{(l)}a_{(l)}$. In our case, that factor has been split up in two parts since the use of physical components. The "hidden" one is in fact contained in the convective term ${}^3U^{(j)}\partial^3_{(j)}U^{(i)}$. So, the well known factor of '2' is recovered when returning to coordinate components $V^i = dx^i/dt$ via Eq. (23).

As far as we know, the remaining term in parenthesis (cubic! on the velocity field) as well as the linear one which multiplies some structure constants, do not have an equivalent in the Newtonian or flat limits, so they do not allow an interpretation as easy as the previous terms. Nevertheless, in some cases (*i.e.*, spherical symmetry), it is possible to use the 3+1-Einstein equations (see [14]) for replacing them (namely, the cubic term on the velocity field), in terms of metric potentials and its derivatives, such as to recover after some manipulations some Newtonian effects [19, 20].

Finally, we find the external forces $\mathcal{F}$ (other than the inertial and gravitational ones, *e.g.*, the electromagnetic Lorentz force) acting on the fluid.

It is to be emphasized that such a quasi-Newtonian Euler equation, up to our knowledge, has not been considered before, except in spherical symmetry with the so-called *radial gauge polar slicing* coordinates (see [19, 20]). We also point out that equations for relativistic spinning fluids has been already studied in the past by [21–23].

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Appendix A: Tangent vectors to Σ_t

Let ${}^3\boldsymbol{w}$ be a 4-vector tangent to Σ_t given by

$$ {}^3\boldsymbol{w} := \boldsymbol{h} \cdot \boldsymbol{w}\,, \tag{A.1}$$

where $\boldsymbol{w}$ is a fourvector of $\mathcal{T}_p$ [the tangent space of $(M^4, \boldsymbol{g})$]. It is easy to check that the following holds,

$$ {}^3w^t = 0\,, \tag{A.2}$$

$$ {}^3w^i = w^i - w^t N^i\,. \tag{A.3}$$

In the same way we find

$$ {}^3w_t = -N^j w_j\,, \tag{A.4}$$

$$ {}^3w_i = w_i\,. \tag{A.5}$$

On the other hand we have,

$$ {}^3w_\mu = g_{\mu\nu}\,{}^3w^\nu = h_{\mu\nu}\,{}^3w^\nu\,. \tag{A.6}$$

The last equality comes from (1) and the fact that ${}^3\boldsymbol{w}$ is orthogonal to $\boldsymbol{n}$. It yields then

$$ {}^3w_t = -N_j\,{}^3w^j\,, \tag{A.7}$$

$$ {}^3w_i = h_{ij}\,{}^3w^j\,. \tag{A.8}$$

We conclude upon equation (A.6), that for lowering or rising indices of tangent vectors to Σ_t we can employ both the 4-metric or the 3-metric. In particular, Eq. (A.8) shows that we can use the 3-metric to lower or rise the spatial indices of tangent vectors by considering these as 3-vectors on Σ_t.

Appendix B: The tetrad formalism

According to the tetrad formalism (see Refs. 10–12 and 24 for a review), a linear transformation $e_\mu^{(\alpha)}$ between a coordinate basis $\boldsymbol{e}_\mu$, and an orthogonal one $\boldsymbol{e}_{(\mu)}$ allows to write

$$ g_{\mu\nu} = e_\mu^{(\alpha)}\, e_\nu^{(\beta)}\, \eta_{(\alpha)(\beta)}\,. \tag{B.1}$$

In the same way we find

$$ g^{\mu\nu} = q^\mu_{(\alpha)}\, q^\nu_{(\beta)}\, \eta^{(\alpha)(\beta)}\,, \tag{B.2}$$

where the tetrad fields $e_\mu^{(\alpha)}$ and $q^\mu_{(\alpha)}$ are simply related by

$$ e_\mu^{(\alpha)} q^\mu_{(\beta)} = \delta^{(\alpha)}_{(\beta)}\,, \tag{B.3}$$

$$ e_\nu^{(\alpha)} q^\mu_{(\alpha)} = \delta^\mu_\nu\,. \tag{B.4}$$

The $q^\mu_{(\alpha)}$ coefficients are to be seen as the components of the tetrad vectors with respect the coordinate basis. Here, the (*internal*) index 'μ' is an index of general coordinate transformations (*i.e.*, a 4-vector index).

A consequence of these results is that the coordinate and orthonormal components of 4-tensors are related by

$$ P^{(\mu_1)(\mu_2)\cdots(\mu_n)}_{(\nu_1)(\nu_2)\cdots(\nu_m)} = e^{(\mu_1)}_{\sigma_1} e^{(\mu_2)}_{\sigma_2} \cdots e^{(\mu_n)}_{\sigma_n} \times q^{\pi_1}_{(\nu_1)}\, q^{\pi_2}_{(\nu_2)} \cdots q^{\pi_m}_{(\nu_m)} P^{\sigma_1\sigma_2\cdots\sigma_n}_{\pi_1\pi_2\cdots\pi_m}\,. \tag{B.5}$$

The inverse relationships follows immediately.

A particular case of (B.5) is the *directional* or *Pfaffian* derivative [11–13] given by

$$ \partial_{(\alpha)} := q^\mu_{(\alpha)} \partial_\mu\,. \tag{B.6}$$

Moreover, the indices of tetrad components (*physical* components) are lowered and raised by the Lorentz metric $\eta_{(\alpha)(\beta)}$ and $\eta^{(\alpha)(\beta)}$, respectively.

It will useful to recall that covariant derivatives of arbitrary rank 4-tensors when projected on a tetrad take the following form:

$$ \begin{aligned} P^{(\mu_1)(\mu_2)\cdots(\mu_n)}_{(\nu_1)(\nu_2)\cdots(\nu_m);(\rho)} ={}& \partial_{(\rho)} P^{(\mu_1)(\mu_2)\cdots(\mu_n)}_{(\nu_1)(\nu_2)\cdots(\nu_m)} + P^{(\sigma)(\mu_2)\cdots(\mu_n)}_{(\nu_1)(\nu_2)\cdots(\nu_m)}\, \mathcal{O}^{(\mu_1)}_{(\rho)(\sigma)} + P^{(\mu_1)(\sigma)\cdots(\mu_n)}_{(\nu_1)(\nu_2)\cdots(\nu_m)}\, \mathcal{O}^{(\mu_2)}_{(\rho)(\sigma)} + \cdots + P^{(\mu_1)(\mu_2)\cdots(\sigma)}_{(\nu_1)(\nu_2)\cdots(\nu_m)}\, \mathcal{O}^{(\mu_n)}_{(\rho)(\sigma)} \\ &- P^{(\mu_1)(\mu_2)\cdots(\mu_n)}_{(\sigma)(\nu_2)\cdots(\nu_m)}\, \mathcal{O}^{(\sigma)}_{(\rho)(\nu_1)} - P^{(\mu_1)(\mu_2)\cdots(\mu_n)}_{(\nu_1)(\sigma)\cdots(\nu_m)}\, \mathcal{O}^{(\sigma)}_{(\rho)(\nu_2)} - \cdots - P^{(\mu_1)(\mu_2)\cdots(\mu_n)}_{(\nu_1)(\nu_2)\cdots(\sigma)}\, \mathcal{O}^{(\sigma)}_{(\rho)(\nu_m)}, \end{aligned} \tag{B.7}$$

where, as mentioned before, the quantities

$$ \mathcal{O}^{(\alpha)}_{(\beta)(\gamma)} := e^{(\alpha)}_\mu q^\mu_{(\gamma);\nu} q^\nu_{(\beta)}\,, \tag{B.8}$$

are called the Ricci rotation coefficients (RRC).

Appendix C: The 3+1 decomposition of a tetrad

Returning to Eq. (B.1), and then comparing this with each term of $\boldsymbol{g}$ given by (1), we find (see Refs. 8, 25–27 for details about this appendix)

$$ -N^2 + N_k N^k = g_{tt} = -(e_t^{(0)})^2 + e_t^{(i)} e_t^{(j)} \eta_{(i)(j)}\,, \tag{C.1}$$

$$ -N_i = g_{ti} = e_t^{(0)} e_i^{(0)} + e_t^{(j)} e_i^{(k)} \eta_{(j)(k)}\,, \tag{C.2}$$

$$ h_{ij} = g_{ij} = e_i^{(0)} e_j^{(0)} + e_i^{(k)} e_j^{(l)} \eta_{(k)(l)}\,. \tag{C.3}$$

Since the matrix $e_\beta^{(\alpha)}$ is not symmetric, then its sixteen entries seem to be linearly independent. However, Eqs. (C.1)–(C.3) form a system of ten linearly independent algebraic equations. So, there are six coefficients $e_\mu^{(a)}$ which are linearly dependent (actually those related to the six parameters which define a general Lorentz transformation between two observers). In fact, the 3+1 decomposition fixes a natural choice for the coefficients $e_\mu^{(a)}$. Such a choice is closely related to the way a tetrad carried by the local observer $\mathcal{O}_\circ$ is constructed with the normal vector $\boldsymbol{n}$ and a *triad* lying on Σ_t.

C.1. Gauge choice

In order to determine the matrix of coefficients $e_\mu^{(a)}$ in terms of the metric-tensor components we impose the following condition:

$$e_t^{(i)} := -e_j^{(i)} N^j\,, \tag{C.4}$$

$$e_i^{(0)} := 0\,. \tag{C.5}$$

Then Eq. (C.3) reduces to

$$\begin{aligned} h_{ij} &= e_i^{(k)} e_j^{(l)} \eta_{(k)(l)} \\ &= e_i^{(l)} e_j^{(l)}\,. \end{aligned} \tag{C.6}$$

Indeed, Eqs. (C.4)–(C.6), allow us to recover via Eq. (C.2) the rule for the lowering of spatial indices of tangent vectors to Σ_t [*cf.* (A.8)]:

$$-N_i = -e_k^{(j)} e_i^{(l)} N^k \eta_{(j)(l)} = -h_{ik} N^k\,. \tag{C.7}$$

Furthermore, Eq. (C.1), and the use of (C.4)–(C.6) give the condition

$$N = e_t^{(0)}\,. \tag{C.8}$$

To summarize, we have

$$e_t^{(i)} := -e_j^{(i)} N^j\,, \tag{C.9}$$

$$e_i^{(0)} := 0\,, \tag{C.10}$$

$$e_t^{(0)} = N\,, \tag{C.11}$$

$$\begin{aligned} h_{ij} &= e_i^{(k)} e_j^{(l)} \eta_{(k)(l)} \\ &= e_i^{(l)} e_j^{(l)}\,. \end{aligned} \tag{C.12}$$

Again, we stress that the coefficient matrix $e_i^{(k)}$ is not symmetric, however three coefficients are linearly dependent; the other six are to be obtained in terms of the 3-metric components from Eq. (C.12). This means that modulo a rotation SO(3), the triad is 'single' defined (the three rotation parameters are those related with the three linearly dependent coefficients). In the Appendix D we show that if h_{ij} has a certain form, then $e_i^{(k)}$ may be solved in terms of h_{ij} as a triangular-matrix (with its lowering part equal to zero).

We can follow the same procedure to determine the coefficients $q_{(a)}^\mu$ by means of Eqs. (1) and (B.2). This leads to

$$q_{(0)}^i = \frac{N^i}{N}\,, \tag{C.13}$$

$$q_{(i)}^t = 0\,, \tag{C.14}$$

$$q_{(0)}^t = N^{-1}\,, \tag{C.15}$$

$$\begin{aligned} h^{ij} &= q_{(k)}^i q_{(l)}^j \eta^{(k)(l)} \\ &= q_{(l)}^i q_{(l)}^j\,. \end{aligned} \tag{C.16}$$

C.2. Physical components of tangent vectors to Σ_t

Let ${}^3\boldsymbol{w}$ be a 4-vector tangent to Σ_t.

Then, according to Eq. (B.5) we find

$${}^3w^{(\alpha)} = e_\mu^{(\alpha)}\, {}^3w^\mu\,. \tag{C.17}$$

From the gauge condition (C.5), and the fact that ${}^3w^t = 0$ [*cf* Eq. (A.2)] we have

$${}^3w^{(0)} = 0\,, \tag{C.18}$$

$${}^3w^{(i)} = e_l^{(i)}\, {}^3w^l\,. \tag{C.19}$$

By lowering the indices we obtain also

$${}^3w_{(0)} = 0\,, \tag{C.20}$$

$${}^3w_{(i)} = {}^3w^{(i)}\,. \tag{C.21}$$

We appreciate that the temporal physical (covariant or contravariant) components of tangent vectors to Σ_t vanish identically, and that the covariant and contravariant spatial physical components coincide with each other. In addition, Eqs. (1), (A.1) and (C.17), simply lead to

$${}^3w^{(i)} = w^{(i)}\,. \tag{C.22}$$

Previous results can be generalized to higher rank tangent tensors to Σ_t. For instance, the extrinsic curvature verifies:

$$K^{(0)(\alpha)} = K_{(\alpha)}^{(0)} = K_{(0)}^{(\alpha)} = K_{(0)(\alpha)} = 0\,, \tag{C.23}$$

$$K^{(i)(j)} = K^{lm} e_l^{(i)} e_m^{(j)}\,, \tag{C.24}$$

$$K_{(j)}^{(i)} = K_{(i)}^{(j)} = K_{(i)(j)}\,. \tag{C.25}$$

C.3. Link between the 4-Ricci rotation coefficients and the 3+1 variables

In the following we present a series of identities which link the 4-Ricci rotation coefficients with some of the 3+1 variables introduced in previous Sections. As we saw in Sec. I, these identities allows us to incorporate easily the curvature effects (as "forces") acting on the Eulerian observer $\mathcal{O}_\circ$.

These identities are the following [8]:

$$i) \qquad \mathcal{O}^{(0)}_{(0)(0)} = 0. \tag{C.26}$$

$$ii) \qquad \mathcal{O}^{(i)}_{(0)(0)} = a^{(i)}. \tag{C.27}$$

$$iii) \qquad \mathcal{O}^{(0)}_{(i)(0)} = 0. \tag{C.28}$$

$$iv) \qquad \mathcal{O}^{(0)}_{(i)(j)} = -K_{(i)(j)}. \tag{C.29}$$

$$v) \qquad \mathcal{O}^{(0)}_{(0)(i)} = a_{(i)}. \tag{C.30}$$

$$vi) \qquad \mathcal{O}^{(i)}_{(j)(0)} = -K^{(i)}_{(j)}. \tag{C.31}$$

$$vii) \qquad \mathcal{O}^{(i)}_{(0)(j)} = \mathcal{O}_{(0)(j)(i)} = K_{(j)(i)} + {}^3C_{(j)(i)}. \tag{C.32}$$

Or

$$\mathcal{O}_{(0)(j)(i)} = -K_{(j)(i)} - {}^3C_{(i)(j)}. \tag{C.33}$$

Or even more

$$\mathcal{O}_{(0)(j)(i)} = \frac{1}{2}\left[{}^3C_{(j)(i)} - {}^3C_{(i)(j)}\right]. \tag{C.34}$$

where

$$\begin{aligned}{}^3C_{(i)(j)} := -\frac{\partial_{(i)} N^{(j)}}{N} + q^l_{(m)} \frac{N^{(m)}}{N} \partial_{(i)} e^{(j)}_l \\ + q^l_{(i)} \partial_{(0)} e^{(j)}_l.\end{aligned} \tag{C.35}$$

$$viii) \qquad \mathcal{O}^{(i)}_{(j)(k)} = {}^3\mathcal{O}^{(i)}_{(j)(k)}. \tag{C.36}$$

Appendix D: Potential representation of the 3-metric

In the Appendix C we showed that the triad representation of the 3-metric obeys

$$h_{ij} = e^{(l)}_i e^{(l)}_j. \tag{D.1}$$

The question that arises is the following: *given a 3-metric, Is it always possible to solve the above non-linear algebraic equations for $e^{(l)}_i$ in terms of h_{ij} ?*

For our purposes, it suffices to answer this question in the positive for the special case in which h_{ij} is given in terms of *potentials*. Then, it is possible to solve (D.1) for $e^{(l)}_i$. From the practical point of view, the potential representation of the 3-metric is well adapted to treat a broad set of problems on gravitational collapse as well as for constructing static and stationary equilibrium configurations of relativistic objects.

So let us assume

$$(h_{ij}) = \begin{pmatrix} {\alpha_1}^2 & \alpha_1\alpha_2 & \alpha_1\alpha_3 \\ \alpha_1\alpha_2 & {\alpha_2}^2 + {\beta_2}^2 & \alpha_2\alpha_3 + \beta_2\beta_3 \\ \alpha_1\alpha_3 & \alpha_2\alpha_3 + \beta_2\beta_3 & {\alpha_3}^2 + {\beta_3}^2 + {\gamma_3}^2 \end{pmatrix} \tag{D.2}$$

where α_i, β_i and γ_i are scalar functions of t, r, θ, ϕ.

Then, it is easy to show that

$$\left(e^{(i)}_j\right) = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & \beta_2 & \beta_3 \\ 0 & 0 & \gamma_3 \end{pmatrix}, \tag{D.3}$$

solves equation (D.1). Notice the triangular form of $e^{(l)}_i$.

By choosing conveniently α_i, β_i and γ_i, in terms of new metric potentials we can recover isotropic or radial gauges. For instance, by imposing $\alpha_1 := A^2CB^{-1}$, $\alpha_2 := A^2\alpha B^{-1}$, $\alpha_3 := A^2B\gamma$, $\beta_2 := A^2B^{-1}$, $\beta_3 := A^2B\beta$, and $\gamma_3 := A^2\, B$, *i.e.*,

$$\left(e^{(i)}_j\right) = A^2 \begin{pmatrix} CB^{-1} & \alpha B^{-1} & B\gamma \\ 0 & B^{-1} & B\beta \\ 0 & 0 & B \end{pmatrix}, \tag{D.4}$$

one generalizes the maximal slicing quasi-isotropic coordinates (MSQI coordinates) [16], employed for constructing stationary relativistic axisymmetric rotating bodies. For recovering the MSQI coordinates we set $\alpha = \beta = \gamma = 0$ and $C = 1$.

†. Indeed their equation has a wrong sign in the divergence of $\boldsymbol{J}$.

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