Exact fields of electrostatic and magnetostatic multipoles

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We show that the n-th order terms in the multipole expansions of the scalar potential, electrostatic field, vector potential, and magnetostatic field depend on time independent electric charge and current densities, $\rho(\mathbf{r})$ and $\mathbf{j}(\mathbf{r})$, only through suitably constructed electric and magnetic multipole moment vectors, $\boldsymbol{\rho}^{(n)}(\hat{\mathbf{r}})$ and $\mathbf{M}^{(n)}(\hat{\mathbf{r}})$, according to

$$\varphi^{(n)} = \frac{1}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}\cdot\boldsymbol{\rho}^{(n)}}{r^{n+1}}, \qquad \mathbf{E}^{(n)} = \frac{1}{4\pi\epsilon_0}\frac{(2n+1)\hat{\mathbf{r}}\left[\hat{\mathbf{r}}\cdot\boldsymbol{\rho}^{(n)}\right] - n\boldsymbol{\rho}^{(n)}}{r^{n+2}},$$

$$\mathbf{A}^{(n)} = \frac{\mu_0}{4\pi}\frac{\mathbf{M}^{(n)}\times\hat{\mathbf{r}}}{r^{n+1}}, \qquad \mathbf{B}^{(n)} = \frac{\mu_0}{4\pi}\frac{(2n+1)\hat{\mathbf{r}}\left[\hat{\mathbf{r}}\cdot\mathbf{M}^{(n)}\right] - n\mathbf{M}^{(n)}}{r^{n+2}}.$$

We calculate the explicit forms of $\mathbf{M}^{(n)}$, $\mathbf{A}^{(n)}$, and $\mathbf{B}^{(n)}$ for the example of a circular loop carrying a current I.

Keywords: Spherical tensors, multipole expansion, magnetostatic field, electrostatic field

Mostramos que los términos de orden n en las expansiones multipolares del potencial escalar, campo electrostático, potencial vectorial y campo magnetostático dependen de las densidades de carga eléctrica y corriente, independientes del tiempo, $\rho(\mathbf{r})$ and $\mathbf{j}(\mathbf{r})$, sólo a través de los vectores de los momentos multipolares eléctrico y magnético apropiados $\rho^{(n)}(\hat{\mathbf{r}})$ y $\mathbf{M}^{(n)}(\hat{\mathbf{r}})$ de acuerdo a

$$\varphi^{(n)} = \frac{1}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}\cdot\boldsymbol{\rho}^{(n)}}{r^{n+1}}, \qquad \mathbf{E}^{(n)} = \frac{1}{4\pi\epsilon_0}\frac{(2n+1)\hat{\mathbf{r}}\left[\hat{\mathbf{r}}\cdot\boldsymbol{\rho}^{(n)}\right] - n\boldsymbol{\rho}^{(n)}}{r^{n+2}},$$

$$\mathbf{A}^{(n)} = \frac{\mu_0}{4\pi}\frac{\mathbf{M}^{(n)}\times\hat{\mathbf{r}}}{r^{n+1}}, \qquad \mathbf{B}^{(n)} = \frac{\mu_0}{4\pi}\frac{(2n+1)\hat{\mathbf{r}}\left[\hat{\mathbf{r}}\cdot\mathbf{M}^{(n)}\right] - n\mathbf{M}^{(n)}}{r^{n+2}}.$$

Calculamos como ejemplo las formas explícitas de $\mathbf{M}^{(n)}$, $\mathbf{A}^{(n)}$ y $\mathbf{B}^{(n)}$ para una corriente I en un rizo circular.

Descriptores: Tensores esféricos, expansión multipolar, campo magnetostático, campo electrostático

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1. Introduction

In classical electrodynamics, the electrostatic and magnetostatic fields due to time independent electric charge and current densities, $\rho(\mathbf{r})$ and $\mathbf{j}(\mathbf{r})$, are frequently expanded in inverse powers of the distance between the observation point and the origin of the sources [1]:

$$\mathbf{E}(\mathbf{r}) = \sum_{n=0}^{\infty}\mathbf{E}^{(n)}(\mathbf{r}), \qquad \mathbf{E}^{(n)}(\mathbf{r}) \sim \frac{1}{r^{n+2}}, \tag{1}$$

$$\mathbf{B}(\mathbf{r}) = \sum_{n=0}^{\infty}\mathbf{B}^{(n)}(\mathbf{r}), \qquad \mathbf{B}^{(n)}(\mathbf{r}) \sim \frac{1}{r^{n+2}}. \tag{2}$$

Most textbooks give explicit formulas for the electric multipole fields $\mathbf{E}^{(n)}(\mathbf{r})$ and their corresponding electric multipole scalar potentials $\varphi^{(n)}(\mathbf{r})$ and electric multipole moments to all orders n and for the magnetic multipole fields $\mathbf{B}^{(n)}(\mathbf{r})$ and their corresponding magnetic multipole vector potentials $\mathbf{A}^{(n)}(\mathbf{r})$ and magnetic multipole moments only to the lowest non-zero order $n = 1$ (magnetic dipole). In this paper we make use of tensor methods to obtain formulas for the magnetic multipole fields and their corresponding magnetic multipole vector potentials, magnetic multipole moment vectors, and magnetic multipole moment tensors to all orders n. We apply the resulting formulas to the example of a circular loop carrying a current I.

Section 2 introduces tensor methods, including basic definitions and theorems [2]. Section 3 contains properties of spherical tensors. Section 4 contains a discussion for the electrostatic field. Section 5 contains the discussion for the magnetostatic field. A discussion of the consequences of gauge invariance on the final form of the vector potential is contained in the Appendix. Section 6 applies the results of Sect. 5 to the example of a circular loop carrying a current I. Section 7 contains conclusions.

2. Tensor methods

This Section presents properties of tensors that will be used in subsequent sections. We consider only the cartesian coordinates $x_1 = x$, $x_2 = y$, $x_3 = z$ of tensors in a flat, 3-dimensional space [3]. Throughout this paper we will use the summation convention which states that if an index is repeated twice in a formula, then a summation from 1 to 3 is being performed over this index.

An object $T_{i_1\ldots i_n}$ that transforms under coordinate transformations as the product of the coordinates $x_{i_1}\cdots x_{i_n}$ is called a *(contravariant) tensor of rank* n. A *trace* of a tensor $T_{i_1\ldots i_n}$ of rank $n \geq 2$ is defined by the sum (contraction) over two of its indices; *e. g.*, $T_{iii_3\ldots i_n}$, and is a tensor of rank $n-2$[‡]. A tensor of rank n has in general $\frac{1}{2}n(n-1)$ traces.

Tensors that satisfy the properties

$$T^{\mathrm{s}}_{i_1\ldots i_n} = T^{\mathrm{s}}_{(i_1\ldots i_n)} \qquad \text{(symmetry)}, \tag{3}$$

$$T^{\mathrm{p}}_{i_1 i_2\ldots i_n} = T^{\mathrm{p}}_{i_1(i_2\ldots i_n)} \qquad \text{(partial symmetry)}, \tag{4}$$

$$T^{0}_{iii_3\ldots i_n} = 0 \qquad \text{(vanishing trace)}, \tag{5}$$

are called *symmetric tensors, partially symmetric tensors*, and *traceless tensors*, respectively. Indices within brackets $(i_1\ldots i_n)$ denote indices with respect to which a tensor is symmetric; tensor components related by any permutation of such indices have the same value. The partially symmetric tensor $T^{\mathrm{p}}_{i_1 i_2\ldots i_n}$ is symmetric with respect to the indices $(i_2\ldots i_n)$ with no symmetry assumed for the index i_1. A symmetric tensor has $\frac{1}{2}(n+1)(n+2)$ independent components and only one trace, which is also a symmetric tensor. A traceless symmetric tensor $T^{\mathrm{s}0}_{i_1\ldots i_n}$ is a symmetric tensor whose trace is zero. A traceless symmetric tensor has $2n+1$ independent components. A traceless partially symmetric tensor $T^{\mathrm{p}0}_{i_1 i_2\ldots i_n}$ has $(n+1)^2-1$ independent components.

Tensors that have the same form in all coordinate systems are called *invariant tensors*. Two important invariant tensors are the Kronecker delta tensor

$$\delta_{ij} = \begin{cases} 1, & \text{if } i=j, \\ 0, & \text{if } i\neq j, \end{cases} \tag{6}$$

and the totally antisymmetric Levi-Civita tensor

$$\varepsilon_{ijk} = \begin{cases} 1, & \text{if } ijk \text{ is an even permutation of 1,2,3}, \\ -1, & \text{if } ijk \text{ is an odd permutation of 1,2,3}, \\ 0, & \text{otherwise}. \end{cases} \tag{7}$$

These two tensors fulfill the determinant relation

$$\varepsilon_{ijk}\varepsilon_{i'j'k'} = \begin{vmatrix} \delta_{ii'} & \delta_{ij'} & \delta_{ik'} \\ \delta_{ji'} & \delta_{jj'} & \delta_{jk'} \\ \delta_{ki'} & \delta_{kj'} & \delta_{kk'} \end{vmatrix} \tag{8}$$

and, summing over one pair of indices,

$$\varepsilon_{ijk}\varepsilon_{i'j'k} = \begin{vmatrix} \delta_{ii'} & \delta_{ij'} \\ \delta_{ji'} & \delta_{jj'} \end{vmatrix} = \delta_{ii'}\delta_{jj'} - \delta_{ij'}\delta_{ji'} \tag{9}$$

and, summing over two pairs of indices,

$$\varepsilon_{ijk}\,\varepsilon_{i'jk} = 2\delta_{ii'}. \tag{10}$$

One can show that when a tensor is contracted with a traceless and/or symmetric tensor only its traceless and/or symmetric part contributes. Similarly, when a tensor is contracted with an antisymmetric tensor only its antisymmetric part contributes.

Given a rank n symmetric tensor $T^{\mathrm{s}}_{i_1\ldots i_n}$ one can obtain the rank $n-2$ symmetric tensors $T^{\mathrm{s}}_{mmi_1\ldots i_{k-1}i_{k+1}\ldots i_{l-1}i_{l+1}\ldots i_n}$, *i. e.*, $T^{\mathrm{s}}_{mmi_1\ldots i_{n-2}}$, and the rank n traceless symmetric tensor

$$T^{\mathrm{s}0}_{i_1\ldots i_n} = T^{\mathrm{s}}_{i_1\ldots i_n} - \frac{1}{2n-1}\sum_{k=1}^{n-1}\sum_{l=k+1}^{n}\delta_{i_k i_l}T^{\mathrm{s}}_{mmi_1\ldots i_{k-1}i_{k+1}\ldots i_{l-1}i_{l+1}\ldots i_n} + \frac{1}{(2n-1)(2n-3)}\sum_{\substack{k>l,\,p>q,\\(k,l)\neq(p,q)}}\delta_{i_k i_l}\delta_{i_p i_q}T^{\mathrm{s}}_{mmjji_1\ldots i_{k-1}i_{k+1}\ldots i_{l-1}i_{l+1}\ldots i_{p-1}i_{p+1}\ldots i_{q-1}i_{q+1}\ldots i_n} + \cdots. \tag{11}$$

The rank n symmetric tensor $T^{\mathrm{s}}_{i_1\ldots i_n}$ may then be expressed as

$$T^{\mathrm{s}}_{i_1\ldots i_n} = T^{\mathrm{s}0}_{i_1\ldots i_n} + \frac{1}{2n-1}\sum_{k=1}^{n-1}\sum_{l=k+1}^{n}\delta_{i_k i_l}T^{\mathrm{s}}_{mmi_1\ldots i_{k-1}i_{k+1}\ldots i_{l-1}i_{l+1}\ldots i_n} - \frac{1}{(2n-1)(2n-3)}\sum_{\substack{k>l,\,p>q,\\(k,l)\neq(p,q)}}\delta_{i_k i_l}\delta_{i_p i_q}T^{\mathrm{s}}_{mmjji_1\ldots i_{k-1}i_{k+1}\ldots i_{l-1}i_{l+1}\ldots i_{p-1}i_{p+1}\ldots i_{q-1}i_{q+1}\ldots i_n} + \cdots. \tag{12}$$

Furthermore, the following theorem follows using mathematical induction:

Theorem 1 *A symmetric tensor $T^{\rm s}_{i_1 i_2 \ldots i_n}$ of even (odd) rank n can be expressed in terms of traceless symmetric tensors of even (odd) rank smaller or equal to n, a scalar (vector), and the invariant tensor δ_{ij}.*

A rank 2 symmetric tensor $T^{\rm s}_{ij}$ can be expressed in terms of the scalar $T = T^{\rm s}_{kk}$ and the rank 2 traceless symmetric tensor $T^{\rm s0}_{ij} = T^{\rm s}_{ij} - \frac{1}{3}\delta_{ij} T^{\rm s}_{kk}$ according to

$$T^{\rm s}_{ij} = T^{\rm s0}_{ij} + \frac{1}{3}\delta_{ij} T. \tag{13}$$

A rank 3 symmetric tensor $T^{\rm s}_{ijk}$ can be expressed in terms of the vector $T_i = T^{\rm s}_{ikk}$ and the rank 3 traceless symmetric tensor $T^{\rm s0}_{ijk} = T^{\rm s}_{ijk} - \frac{1}{5}(\delta_{ij} T^{\rm s}_{kll} + \delta_{ik} T^{\rm s}_{jll} + \delta_{jk} T^{\rm s}_{ill})$ according to

$$T^{\rm s}_{ijk} = T^{\rm s0}_{ijk} + \frac{1}{5}\Big(\delta_{ij} T_k + \delta_{ik} T_j + \delta_{jk} T_i\Big). \tag{14}$$

Given a rank n traceless partially symmetric tensor $T^{\rm p0}_{i_1 \ldots i_n}$ one can obtain the rank n symmetric tensor

$$T^{\rm s}_{i_1 \ldots i_n} = \frac{1}{n}\sum_{k=1}^{n} T^{\rm p0}_{i_k i_1 \ldots i_{k-1} i_{k+1} \ldots i_n} \tag{15}$$

and the rank $n-1$ traceless partially symmetric tensor

$$T^{\rm p0}_{l i_2 \ldots i_{k-1} i_{k+1} \ldots i_n} = \frac{1}{2n}\varepsilon_{i_1 i_k l}\Big(T^{\rm p0}_{i_1 \ldots i_n} - T^{\rm p0}_{i_k i_1 \ldots i_{k-1} i_{k+1} \ldots i_n}\Big) \tag{16}$$

and express $T^{\rm p0}_{i_1 \ldots i_n}$ according to

$$T^{\rm p0}_{i_1 \ldots i_n} = T^{\rm s}_{i_1 \ldots i_n} + \sum_{k=2}^{n} \varepsilon_{i_1 i_k l} T^{\rm p0}_{l i_2 \ldots i_{k-1} i_{k+1} \ldots i_n}. \tag{17}$$

Furthermore, since theorem 1 applies to $T^{\rm s}_{i_1 \ldots i_n}$, the following theorem follows using mathematical induction:

Theorem 2 *A traceless partially symmetric tensor $T^{\rm p0}_{i_1 i_2 \ldots i_n}$ can be expressed in terms of traceless symmetric tensors of rank n or smaller, a vector, and the invariant tensors δ_{ij} and ε_{ijk}.*

A rank 2 traceless partially symmetric tensor $T^{\rm p0}_{ij}$ can be expressed in terms of the vector $T_k = \frac{1}{2}\varepsilon_{kij} T^{\rm p0}_{ij}$ and the rank 2 traceless symmetric tensor $T^{\rm s0}_{ij} = \frac{1}{2}(T^{\rm p0}_{ij} + T^{\rm p0}_{ji})$ according to

$$T^{\rm p0}_{ij} = \tfrac{1}{2}\Big(T^{\rm p0}_{ij} + T^{\rm p0}_{ji}\Big) + \tfrac{1}{2}\Big(T^{\rm p0}_{ij} - T^{\rm p0}_{ji}\Big) = T^{\rm s0}_{ij} + \varepsilon_{ijk} T_k. \tag{18}$$

A rank 3 traceless partially symmetric tensor $T^{\rm p0}_{ijk}$ can be expressed in terms of the vector $T_k = T^{\rm p0}_{kii}$ and the traceless symmetric tensors $T^{\rm s0}_{ijk} = \frac{1}{3}(T^{\rm p0}_{ijk} + T^{\rm p0}_{jik} + T^{\rm p0}_{kij}) - \frac{1}{15}(\delta_{ij} T^{\rm p0}_{kll} + \delta_{ik} T^{\rm p0}_{jll} + \delta_{jk} T^{\rm p0}_{ill})$ and $T^{\rm s0}_{lk} = \varepsilon_{ijl} T^{\rm p0}_{ijk} + \frac{1}{2}\varepsilon_{lki} T^{\rm p0}_{imm}$ according to

$$\begin{aligned} T^{\rm p0}_{ijk} &= \left[\frac{1}{3}\Big(T^{\rm p0}_{ijk} + T^{\rm p0}_{jik} + T^{\rm p0}_{kij}\Big) - \frac{1}{15}\Big(\delta_{ij} T^{\rm p0}_{kll} + \delta_{ik} T^{\rm p0}_{jll} + \delta_{jk} T^{\rm p0}_{ill}\Big)\right] \\ &\quad + \frac{1}{3}\Big(T^{\rm p0}_{ijk} - T^{\rm p0}_{jik}\Big) + \frac{1}{3}\Big(T^{\rm p0}_{ijk} - T^{\rm p0}_{kij}\Big) + \frac{1}{15}\Big(\delta_{ij} T^{\rm p0}_{kll} + \delta_{ik} T^{\rm p0}_{jll} + \delta_{jk} T^{\rm p0}_{ill}\Big) \\ &= T^{\rm s0}_{ijk} + \frac{1}{3}\varepsilon_{ijl} T^{\rm s0}_{lk} + \frac{1}{3}\varepsilon_{ikl} T^{\rm s0}_{lj} - \frac{1}{10}\delta_{ij} T_k - \frac{1}{10}\delta_{ik} T_j + + \frac{2}{5}\delta_{jk} T_i. \end{aligned} \tag{19}$$

3. Spherical tensors and expansions of $\dfrac{1}{|\mathbf{r} - \boldsymbol{\xi}|}$

The cartesian coordinates of a spherical tensor of rank n are defined by

$$Q^{(n)}_{i_1 \ldots i_n}(\hat{\mathbf{r}}) \equiv r^{n+1}\frac{(-1)^n}{n!}\frac{\partial^n}{\partial x_{i_1} \ldots \partial x_{i_n}}\frac{1}{r}. \tag{20}$$

Spherical tensors depend on only the direction $\hat{\mathbf{r}} \equiv \mathbf{r}/r$ and, for $n \geq 2$, are traceless symmetric tensors. That $Q^{(n)}_{i_1 \ldots i_n}(\hat{\mathbf{r}})$ is a symmetric tensor follows from symmetry under interchange of the order of the partial derivatives. That $Q^{(n)}_{i_1 \ldots i_n}(\hat{\mathbf{r}})$ is traceless follows from the fact that the trace involves the Laplacian of $1/r$:

$$\begin{aligned} Q^{(n)}_{jji_1 \ldots i_{n-2}}(\hat{\mathbf{r}}) &= r^{n+1}\frac{(-1)^n}{n!}\frac{\partial^n}{\partial x_{i_1} \ldots \partial x_{i_{n-2}} \partial x_j \partial x_j}\frac{1}{r} \\ &= r^{n+1}\frac{(-1)^n}{n!}\frac{\partial^{n-2}}{\partial x_{i_1} \ldots \partial x_{i_{n-2}}}\Delta\frac{1}{r} \\ &= 0, \qquad \text{for } r \neq 0. \end{aligned} \tag{21}$$

The spherical tensor $Q^{(n)}_{i_1 \ldots i_n}(\hat{\mathbf{r}})$ has $2n+1$ independent components which, for $n = l$, are linear combinations of the spherical harmonics $Y_{lm}(\theta, \phi)$, $m \in \{l, l-1, \ldots, -l\}$.

Spherical tensors satisfy the recurrence relations

$$Q^{(n)}_{i_1 \ldots i_{n-1} j}(\hat{\mathbf{r}}) = -\frac{r^{n+1}}{n}\frac{\partial}{\partial x_j}\left[\frac{Q^{(n-1)}_{i_1 \ldots i_{n-1}}(\hat{\mathbf{r}})}{r^n}\right] \tag{22}$$

and

$$\hat{x}_{i_n} Q^{(n)}_{i_1 \ldots i_n}(\hat{\mathbf{r}}) = Q^{(n-1)}_{i_1 \ldots i_{n-1}}(\hat{\mathbf{r}}). \tag{23}$$

Explicit formulas for the lowest rank spherical tensors are given by

$$Q^{(0)}(\hat{\mathbf{r}}) = 1,$$

$$Q^{(1)}_i(\hat{\mathbf{r}}) = -r^2 \frac{\partial}{\partial x_i}\left[\frac{Q^{(0)}(\hat{\mathbf{r}})}{r}\right] = \frac{x_i}{r} = \hat{x}_i,$$

$$Q^{(2)}_{i_1 i_2}(\hat{\mathbf{r}}) = -\frac{r^3}{2}\frac{\partial}{\partial x_{i_2}}\left[\frac{Q^{(1)}_{i_1}(\hat{\mathbf{r}})}{r^2}\right] = \frac{3}{2}\hat{x}_{i_1}\hat{x}_{i_2} - \frac{1}{2}\delta_{i_1 i_2},$$

$$Q^{(3)}_{i_1 i_2 i_3}(\hat{\mathbf{r}}) = -\frac{r^4}{3}\frac{\partial}{\partial x_{i_3}}\left[\frac{Q^{(2)}_{i_1 i_2}(\hat{\mathbf{r}})}{r^3}\right]$$
$$= \frac{5}{2}\hat{x}_{i_1}\hat{x}_{i_2}\hat{x}_{i_3} - \frac{1}{2}\left(\hat{x}_{i_1}\delta_{i_2 i_3} + \hat{x}_{i_2}\delta_{i_1 i_3} + \hat{x}_{i_3}\delta_{i_1 i_2}\right).$$

Multipole expansions for which the observation point $\mathbf{r}$ is farther from the origin than the positions $\boldsymbol{\xi}$ of all sources are based on the following expansion of $1/|\mathbf{r}-\boldsymbol{\xi}\,|$ valid for $r > \xi$:

$$\frac{1}{|\mathbf{r}-\boldsymbol{\xi}\,|} = \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \xi_{i_1} \ldots \xi_{i_n} Q^{(n)}_{i_1 \ldots i_n}(\hat{\mathbf{r}})$$
$$= \sum_{n=0}^{\infty} \frac{\xi^n}{r^{2n+1}} x_{i_1} \ldots x_{i_n} Q^{(n)}_{i_1 \ldots i_n}(\hat{\boldsymbol{\xi}}\,) \tag{24}$$
$$= \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{\xi}{r}\right)^n \frac{2^n (n!)^2}{(2n)!} Q^{(n)}_{i_1 \ldots i_n}(\hat{\mathbf{r}}) Q^{(n)}_{i_1 \ldots i_n}(\hat{\boldsymbol{\xi}}\,).$$

Multipole expansions for which the observation point $\mathbf{r}$ is closer to the origin than the positions $\boldsymbol{\xi}$ of all sources are based on the following expansion of $1/|\mathbf{r}-\boldsymbol{\xi}\,|$ valid for $r < \xi$:

$$\frac{1}{|\mathbf{r}-\boldsymbol{\xi}\,|} = \sum_{n=0}^{\infty} \frac{r^n}{\xi^{2n+1}} \xi_{i_1} \ldots \xi_{i_n} Q^{(n)}_{i_1 \ldots i_n}(\hat{\mathbf{r}})$$
$$= \sum_{n=0}^{\infty} \frac{1}{\xi^{n+1}} x_{i_1} \ldots x_{i_n} Q^{(n)}_{i_1 \ldots i_n}(\hat{\boldsymbol{\xi}}\,) \tag{25}$$
$$= \frac{1}{\xi}\sum_{n=0}^{\infty}\left(\frac{r}{\xi}\right)^n \frac{2^n (n!)^2}{(2n)!} Q^{(n)}_{i_1 \ldots i_n}(\hat{\mathbf{r}}) Q^{(n)}_{i_1 \ldots i_n}(\hat{\boldsymbol{\xi}}\,).$$

Multipole expansions for which the observation point $\mathbf{r}$ is farther from the origin than some sources and closer to the origin than other sources are based on a combination of the expansions valid for $r > \xi$ and valid for $r < \xi$.

Legendre polynomials $P_l(\cos\theta)$ are given in terms of spherical tensors by

$$P_n(\cos\theta) = P_n(\hat{\boldsymbol{\xi}}\cdot\hat{\mathbf{r}}) = Q^{(n)}_{i_1 \ldots i_n}(\hat{\boldsymbol{\xi}}\,)\hat{x}_{i_1}\cdots\hat{x}_{i_n}$$
$$= Q^{(n)}_{i_1 \ldots i_n}(\hat{\mathbf{r}})\hat{\xi}_{i_1}\cdots\hat{\xi}_{i_n} \tag{26}$$

where $\hat{\boldsymbol{\xi}}$ is the direction of a vector $\boldsymbol{\xi}$ and θ is the angle between the vectors $\boldsymbol{\xi}$ and $\mathbf{r}$. Equation (26) may be used in Eqs. (24) and (25) to obtain

$$\frac{1}{|\mathbf{r}-\boldsymbol{\xi}\,|} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{\xi}{r}\right)^n P_n(\hat{\boldsymbol{\xi}}\cdot\hat{\mathbf{r}})$$
$$= \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{\xi}{r}\right)^n P_n(\cos\theta), \qquad \text{for} \quad r > \xi, \tag{27}$$

and

$$\frac{1}{|\mathbf{r}-\boldsymbol{\xi}\,|} = \frac{1}{\xi}\sum_{n=0}^{\infty}\left(\frac{r}{\xi}\right)^n P_n(\hat{\boldsymbol{\xi}}\cdot\hat{\mathbf{r}})$$
$$= \frac{1}{\xi}\sum_{n=0}^{\infty}\left(\frac{r}{\xi}\right)^n P_n(\cos\theta), \qquad \text{for} \quad r < \xi. \tag{28}$$

4. The electrostatic field

In this Section we assume that the electric charge density $\rho(\mathbf{r})$ is zero beyond a distance R from the origin. For $r > R$, we obtain exact, closed formulas that express how the electric scalar potential $\varphi^{(n)}$ and electric field $\mathbf{E}^{(n)}$ of the n-th order electric multipole depend on the charge density ρ only through an electric multipole moment vector $\boldsymbol{\rho}^{(n)}$.

The scalar potential for the electrostatic field of a time independent charge density $\rho(\mathbf{r})$ is given by

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int d^3\xi\, \frac{\rho(\boldsymbol{\xi}\,)}{|\mathbf{r}-\boldsymbol{\xi}\,|}. \tag{29}$$

Using Eqs. (24), we immediately obtain

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{x_{i_1}\ldots x_{i_n}}{r^{2n+1}}\int d^3\xi\, \rho(\boldsymbol{\xi}\,)\xi^n Q^{(n)}_{i_1 \ldots i_n}(\hat{\boldsymbol{\xi}}\,)$$
$$= \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{n!}\frac{x_{i_1}\ldots x_{i_n}}{r^{2n+1}}\rho^{s0}_{i_1 \ldots i_n} \tag{30}$$
$$= \sum_{n=0}^{\infty}\varphi^{(n)}(\mathbf{r})$$

where

$$\rho^{s0}_{i_1 \ldots i_n} \equiv n!\int d^3\xi\, \rho(\boldsymbol{\xi}\,)\xi^n Q^{(n)}_{i_1 \ldots i_n}(\hat{\boldsymbol{\xi}}\,) \tag{31}$$

is the tensor for the n-th order electric multipole moments of the charge density and where

$$\varphi^{(n)}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{x_{i_1}\ldots x_{i_n}}{r^{2n+1}}\frac{1}{n!}\rho^{s0}_{i_1 \ldots i_n} \tag{32}$$

is the contribution of the of the n-th order electric multipole to the scalar potential. The tensor $\rho^{s0}_{i_1 \ldots i_n}$ is traceless and symmetric and therefore has $2n+1$ independent components. The electric field of the n-th order electric multipole is given as the negative gradient of $\varphi^{(n)}(\mathbf{r})$:

$$\mathbf{E}^{(n)}(\mathbf{r}) = -\nabla\varphi^{(n)}(\mathbf{r}) = -\hat{\mathbf{e}}_j \frac{\partial}{\partial x_j}\varphi^{(n)}(\mathbf{r}) = -\hat{\mathbf{e}}_j \frac{\partial}{\partial x_j}\frac{1}{4\pi\epsilon_0}\frac{x_{i_1}\dots x_{i_n}}{r^{2n+1}}\frac{1}{n!}\rho^{s0}_{i_1\dots i_n}$$
$$= \frac{1}{4\pi\epsilon_0}\frac{1}{r^{n+2}}\hat{\mathbf{e}}_j\left[(2n+1)\frac{1}{n!}\rho^{s0}_{i_1\dots i_n}\hat{x}_j\hat{x}_{i_1}\dots\hat{x}_{i_n} - \frac{1}{n!}\rho^{s0}_{i_1\dots i_n}\sum_{k=1}^{n}\hat{x}_{i_1}\dots\hat{x}_{i_{k-1}}\delta_{i_k j}\hat{x}_{i_{k+1}}\dots\hat{x}_{i_n}\right] \tag{33}$$
$$= \frac{1}{4\pi\epsilon_0}\frac{1}{r^{n+2}}\hat{\mathbf{e}}_j\left[(2n+1)\frac{1}{n!}\rho^{s0}_{i_1\dots i_n}\hat{x}_j\hat{x}_{i_1}\dots\hat{x}_{i_n} - n\frac{1}{n!}\rho^{s0}_{j i_1\dots i_{n-1}}\hat{x}_{i_1}\dots\hat{x}_{i_{n-1}}\right]$$

where the last equality relies on the fact that the tensor $\rho^{s0}_{i_1\dots i_n}$ is symmetric. For $n=0$, the second term does not appear and the first term reduces to

$$\mathbf{E}^{(0)}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\hat{\mathbf{e}}_j\hat{x}_j\rho^{s0}$$
$$= \frac{1}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{r^2}\int d^3\xi\ \rho(\boldsymbol{\xi}) = \frac{1}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}\rho^{(0)}}{r^2} \tag{34}$$

where $\rho^{(0)} = \int d^3\xi\ \rho(\boldsymbol{\xi})$ is just the total charge. For $n \geq 1$, the second term is proportional to the vector field

$$\boldsymbol{\rho}^{(n)} = \hat{\mathbf{e}}_j\rho_j^{(n)} \equiv \hat{\mathbf{e}}_j\frac{1}{n!}\rho^{s0}_{j i_1\dots i_{n-1}}\hat{x}_{i_1}\cdots\hat{x}_{n-1} \tag{35}$$

and $\varphi^{(n)}(\mathbf{r})$ and $\mathbf{E}^{(n)}(\mathbf{r})$ may be expressed as

$$\varphi^{(n)}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}\cdot\boldsymbol{\rho}^{(n)}(\hat{\mathbf{r}})}{r^{n+1}}, \tag{36}$$

$$\mathbf{E}^{(n)}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{(2n+1)\hat{\mathbf{r}}\left[\hat{\mathbf{r}}\cdot\boldsymbol{\rho}^{(n)}(\hat{\mathbf{r}})\right] - n\boldsymbol{\rho}^{(n)}(\hat{\mathbf{r}})}{r^{n+2}}. \tag{37}$$

The vector $\boldsymbol{\rho}^{(n)}(\hat{\mathbf{r}})$ is the n-th order electric multipole moment vector. The n-th order electric scalar potential $\varphi^{(n)}(\mathbf{r})$ and electric field $\mathbf{E}^{(n)}(\mathbf{r})$ depend on the electric charge density $\rho(\mathbf{r})$ only through the vector $\boldsymbol{\rho}^{(n)}(\hat{\mathbf{r}})$ or only through the $\mathbf{r}$-independent tensor $\rho^{s0}_{j i_1\dots i_{n-1}}$.

Using Eqs. (24), $\varphi^{(n)}(\mathbf{r})$ may also be written as

$$\varphi^{(n)}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^{n+1}}Q^{(n)}_{i_1\dots i_n}(\hat{\mathbf{r}})\rho_{i_1\dots i_n}$$
$$= \frac{1}{4\pi\epsilon_0}\frac{1}{r^{n+1}}Q^{(n)}_{i_1\dots i_n}(\hat{\mathbf{r}})\frac{2^n n!}{(2n)!}\rho^{s0}_{i_1\dots i_n} \tag{38}$$

where

$$\rho_{i_1\dots i_n} \equiv \int d^3\xi\ \rho(\boldsymbol{\xi})\xi_{i_1}\dots\xi_{i_n} \tag{39}$$

is the tensor for the n-th order moments of the charge density. The tensor $\rho_{i_1\dots i_n}$ is symmetric and the tensor $\rho^{s0}_{i_1\dots i_n}$ is the traceless symmetric part of $\frac{(2n)!}{2^n n!}\rho_{i_1\dots i_n}$.

The electric multipole moment tensor, electric multipole moment vector, electric scalar potential, and electric field of an n-th order electric multipole may be calculated using the formulas

$$\rho_{i_1\dots i_n} = \int d^3\xi\ \rho(\boldsymbol{\xi})\xi_{i_1}\cdots\xi_{i_n}, \tag{40}$$

$$\rho^{s0}_{i_1\dots i_n} = \text{traceless symmetric part of } \frac{(2n)!}{2^n n!}\rho_{i_1\dots i_n} \tag{41}$$

$$= n!\int d^3\xi\ \rho(\boldsymbol{\xi})\xi^n Q^{(n)}_{i_1\dots i_n}(\hat{\boldsymbol{\xi}}), \tag{42}$$

$$\rho_j^{(n)} = \frac{1}{n!}\rho^{s0}_{j i_1\dots i_{n-1}}\hat{x}_{i_1}\cdots\hat{x}_{i_{n-1}}, \tag{43}$$

$$\varphi^{(n)}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}\cdot\boldsymbol{\rho}^{(n)}(\hat{\mathbf{r}})}{r^{n+1}}, \tag{44}$$

$$\mathbf{E}^{(n)}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{(2n+1)\hat{\mathbf{r}}\left[\hat{\mathbf{r}}\cdot\boldsymbol{\rho}^{(n)}(\hat{\mathbf{r}})\right] - n\boldsymbol{\rho}^{(n)}(\hat{\mathbf{r}})}{r^{n+2}}, \tag{45}$$

where $\rho^{s0}_{i_1\dots i_n}$ may be calculated using either Eqs. (40) and (41) or Eq. (42). For an electric dipole these formulas reduce to

$$\rho_i^{(1)} = \rho_i^{s0} = \int d^3\xi\ \rho(\boldsymbol{\xi})\xi_i, \tag{46}$$

$$\varphi^{(1)}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}\cdot\boldsymbol{\rho}^{(1)}(\mathbf{r})}{r^2}, \tag{47}$$

$$\mathbf{E}^{(1)}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{3\hat{\mathbf{r}}\left[\hat{\mathbf{r}}\cdot\boldsymbol{\rho}^{(1)}(\mathbf{r})\right] - \boldsymbol{\rho}^{(1)}(\mathbf{r})}{r^3} \tag{48}$$

and for an electric quadrupole these formulas reduce to

$$\rho^{s0}_{ij} = \int d^3\xi\ \rho(\boldsymbol{\xi})(3\xi_i\xi_j - \xi^2\delta_{ij}), \tag{49}$$

$$\rho_i^{(2)} = \frac{1}{2}\rho^{s0}_{ij}\frac{x_j}{r}, \tag{50}$$

$$\varphi^{(2)}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}\cdot\boldsymbol{\rho}^{(2)}(\mathbf{r})}{r^3}, \tag{51}$$

$$\mathbf{E}^{(2)}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{5\hat{\mathbf{r}}\left[\hat{\mathbf{r}}\cdot\boldsymbol{\rho}^{(2)}(\mathbf{r})\right] - 2\boldsymbol{\rho}^{(2)}(\mathbf{r})}{r^4}. \tag{52}$$

For the case that all of the charges are located outside a sphere of radius R, one can use Eq. (25) in Eq. (29) to obtain the following formulas for the multipole expansion inside the sphere:

$$\rho^{s0}_{i_1\ldots i_n} = n! \int d^3\xi \; \rho(\boldsymbol{\xi}) \frac{Q^{(n)}_{i_1\ldots i_n}(\hat{\boldsymbol{\xi}})}{\xi^{n+1}}, \tag{53}$$

$$\rho^{(n)}_j = \frac{1}{n!}\rho^{s0}_{ji_1\ldots i_{n-1}}\hat{x}_{i_1}\cdots\hat{x}_{i_{n-1}}, \tag{54}$$

$$\varphi^{(n)}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} r^n \hat{\mathbf{r}}\cdot\boldsymbol{\rho}^{(n)}(\hat{\mathbf{r}}), \tag{55}$$

$$\mathbf{E}^{(n)}(\mathbf{r}) = -\frac{1}{4\pi\epsilon_0} r^{n-1} n\boldsymbol{\rho}^{(n)}(\hat{\mathbf{r}}). \tag{56}$$

The only term that contributes to the electric field at the origin is the dipole term:

$$E_i(\mathbf{r}=\mathbf{0}) = E^{(1)}_i(\mathbf{r}=\mathbf{0}) = -\frac{1}{4\pi\epsilon_0}\rho^{s0}_i. \tag{57}$$

The tensor $\rho^{s0}_{i_1\ldots i_n}$ of Eq. (53) is the n-th order tensor of the *internal* electric multipole moments. Like the corresponding tensor for the *external* electric multipole expansion given in Eq. (31), it is traceless and symmetric and therefore has $2n+1$ independent components. The tensors of Eq. (53) and Eq. (31) are not equal and have different dimensions.

External and internal multipole expansions have been discussed in Ref. 4. Reference 4 uses spherical harmonics to simplify part of the discussion but the derivation of the field is complicated because it requires detailed knowledge of the spherical harmonics. Moreover, unlike Eqs. (37) and (56), the expressions for the external and internal electric fields involve products of spherical polar basis vectors with spherical harmonics and their derivatives. Thanks to our use of cartesian tensors, Eqs. (37) and (56) give the explicit forms of the n-th multipole fields without any special functions or their derivatives.

5. The magnetostatic field

In this section we assume that the electric current density $\mathbf{j}(\mathbf{r})$ is zero beyond a distance R from the origin. For $r > R$, we obtain exact, closed formulas that express how the magnetic vector potential $\mathbf{A}^{(n)}$ and magnetic field $\mathbf{B}^{(n)}$ of the n-th order magnetic multipole depend on the current density $\mathbf{j}$ only through a magnetic multipole moment vector $\mathbf{M}^{(n)}$.

The vector potential for the magnetostatic field of a time independent current density $\mathbf{j}(\mathbf{r})$ is given by

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3\xi \; \frac{\mathbf{j}(\boldsymbol{\xi})}{|\mathbf{r}-\boldsymbol{\xi}|}. \tag{58}$$

For each component of $\mathbf{j}$ and $\mathbf{A}$, this relation is similar to the relation between ρ and φ. Using the same procedure as that used for φ, we may write

$$\mathbf{A}(\mathbf{r}) = \sum_{n=0}^{\infty}\mathbf{A}^{(n)}(\mathbf{r}) = \hat{\mathbf{e}}_i\sum_{n=0}^{\infty}A^{(n)}_i(\mathbf{r}) \tag{59}$$

where

$$\begin{aligned} A^{(n)}_i(\mathbf{r}) &= \frac{\mu_0}{4\pi}\frac{1}{r^{n+1}}Q^{(n)}_{i_1\ldots i_n}(\hat{\mathbf{r}})M_{ii_1\ldots i_n} \\ &= \frac{\mu_0}{4\pi}\frac{1}{n!}\frac{x_{i_1}\ldots x_{i_n}}{r^{2n+1}}\widetilde{M}_{ii_1\ldots i_n} \\ &= \frac{\mu_0}{4\pi}\frac{1}{r^{n+1}}Q^{(n)}_{i_1\ldots i_n}(\hat{\mathbf{r}})\frac{2^n n!}{(2n)!}\widetilde{M}_{ii_1\ldots i_n} \end{aligned} \tag{60}$$

is the contribution of the n-th order magnetic multipole to the i-th coordinate of the vector potential, where

$$M_{ii_1\ldots i_n} \equiv \int d^3\xi \; j_i(\boldsymbol{\xi})\xi_{i_1}\ldots\xi_{i_n} \tag{61}$$

is the tensor for the n-th order moments of the i-th component of the current density, and where

$$\widetilde{M}_{ii_1\ldots i_n} \equiv n!\int d^3\xi \; j_i(\boldsymbol{\xi})\xi^n Q^{(n)}_{i_1\ldots i_n}(\hat{\boldsymbol{\xi}}). \tag{62}$$

Both of the tensors $M_{ii_1\ldots i_n}$ and $\widetilde{M}_{ii_1\ldots i_n}$ are partially symmetric. With respect to the indices $(i_1\ldots i_n)$, but not the index i, the tensor $M_{ii_1\ldots i_n}$ is symmetric and the tensor $\widetilde{M}_{ii_1\ldots i_n}$ is the traceless symmetric part of $\frac{(2n)!}{2^n n!}M_{ii_1\ldots i_n}$.

The components of the tensor $M_{ii_1\ldots i_n}$, or $\widetilde{M}_{ii_1\ldots i_n}$, are not independent but fulfill additional constraints that follow from electric current conservation [5]. For any surface over which the current density is zero, $\mathbf{j}(\mathbf{r}) = 0$, the following surface integral is zero:

$$0 = \int d\sigma \; \mathbf{n}\cdot[\mathbf{j}(\boldsymbol{\xi})\xi_i\xi_{i_1}\ldots\xi_{i_n}] \tag{63}$$

where $\mathbf{n}$ is a unit normal to the surface and $d\sigma$ is the area element. The constraint equations are obtained by considering a closed surface that contains all of the currents. For such a closed surface, Eq. (63), together with Gauss' theorem and charge conservation, leads to the following identity involving the tensors $M_{ii_1\ldots i_n}$ and $\rho_{ii_1\ldots i_n}$:

$$\begin{aligned} 0 &= \oint d\sigma \; \mathbf{n}\cdot[\mathbf{j}(\boldsymbol{\xi})\xi_i\xi_{i_1}\ldots\xi_{i_n}] \\ &= \int d^3\xi \; \nabla\cdot[\mathbf{j}(\boldsymbol{\xi})\xi_i\xi_{i_1}\ldots\xi_{i_n}] \\ &= \int d^3\xi \, \mathbf{j}(\boldsymbol{\xi})\cdot\nabla(\xi_i\xi_{i_1}\ldots\xi_{i_n}) + \int d^3\xi \; \xi_i\xi_{i_1}\ldots\xi_{i_n}\nabla\cdot\mathbf{j}(\boldsymbol{\xi}) \\ &= \int d^3\xi \; j_k(\boldsymbol{\xi})\frac{\partial}{\partial\xi_k}(\xi_i\xi_{i_1}\ldots\xi_{i_n}) - \int d^3\xi \; \xi_i\xi_{i_1}\ldots\xi_{i_n}\frac{\partial\rho}{\partial t} \\ &= M_{ii_1\ldots i_n} + M_{i_1i\ldots i_n} + \cdots \\ &\qquad + M_{i_nii_1\ldots i_{n-1}} - \frac{\partial}{\partial t}\int d^3\xi \; \rho\xi_i\xi_{i_1}\ldots\xi_{i_n} \\ &= M_{ii_1\ldots i_n} + \sum_{k=1}^{n} M_{i_ki_1\ldots i_{k-1}i_{k+1}\ldots i_ni} - \frac{\partial\rho_{ii_1\ldots i_n}}{\partial t}. \end{aligned} \tag{64}$$

For $n \geq 1$, we define from the rank $n+1$ tensor $M_{ii_1\ldots i_n}$ a rank n tensor

$$M^{\mathrm{p}0}_{ji_1\ldots i_{n-1}} \equiv \frac{(2n)!}{2^n n!}\frac{n}{n+1}\varepsilon_{iji_n}M_{ii_1\ldots i_n}. \qquad (65)$$

After a straightforward calculation using Eqs. (65) and (9), we obtain

$$\sum_{k=1}^{n}\varepsilon_{iji_k}M^{\mathrm{p}0}_{ji_1\ldots i_{k-1}i_{k+1}\ldots i_n} = \frac{(2n)!}{2^n n!}\frac{n}{n+1} \times\left(nM_{ii_1\ldots i_n} - \sum_{k=1}^{n}M_{i_k i_1\cdots i_{k-1}i_{k+1}\cdots i_n i}\right) \qquad (66)$$

and then, using Eq. (64),

$$M_{ii_1\ldots i_n} = \frac{1}{n+1}\frac{\partial\rho_{ii_1\ldots i_n}}{\partial t} + \frac{2^n n!}{(2n)!}\frac{1}{n}\sum_{k=1}^{n}\varepsilon_{iji_k}M^{\mathrm{p}0}_{ji_1\ldots i_{k-1}i_{k+1}\ldots i_n}. \qquad (67)$$

For $n = 0$, the sums over k in Eqs. (64) and (67) do not appear and M_i is given simply by

$$M_i = \frac{\partial\rho_i}{\partial t}. \qquad (68)$$

The identity (67) is valid for time dependent charge and current densities.

For the case that the charge density is time independent, Eq. (67) reduces to

$$M_{ii_1\ldots i_n} = \frac{2^n n!}{(2n)!}\frac{1}{n}\sum_{k=1}^{n}\varepsilon_{iji_k}M^{\mathrm{p}0}_{ji_1\ldots i_{k-1}i_{k+1}\ldots i_n}, \qquad (69)$$

with $M_i = 0$ for $n = 0$. Using Eq. (69) in Eq. (60) gives

$$\begin{aligned} A_i^{(n)}(\mathbf{r}) =& \frac{\mu_0}{4\pi}\frac{1}{r^{n+1}}Q^{(n)}_{i_1\ldots i_n}(\hat{\mathbf{r}}) \times \frac{2^n n!}{(2n)!}\frac{1}{n}\sum_{k=1}^{n}\varepsilon_{iji_k}M^{\mathrm{p}0}_{ji_1\ldots i_{k-1}i_{k+1}\ldots i_n} \qquad (70)\\ =& \frac{\mu_0}{4\pi}\frac{1}{r^{n+1}}\frac{2^n n!}{(2n)!}Q^{(n)}_{i_1\ldots i_n}(\hat{\mathbf{r}})\varepsilon_{iji_n}M^{\mathrm{p}0}_{ji_1\ldots i_{n-1}}, \end{aligned}$$

with $A_i^{(0)}(\mathbf{r}) = 0$ for $n = 0$. The magnetic field of the n-th order magnetic multipole is given as the curl of $\mathbf{A}_i^{(n)}(\mathbf{r})$:

$$\begin{aligned} B_l^{(n)}(\mathbf{r}) &= \varepsilon_{lki}\frac{\partial A_i^{(n)}}{\partial x_k} \\ &= \frac{\mu_0}{4\pi}\frac{2^n n!}{(2n)!}\varepsilon_{lki}\varepsilon_{iji_n}M^{\mathrm{p}0}_{ji_1\ldots i_{n-1}}\frac{\partial}{\partial x_k}\left[\frac{Q^{(n)}_{i_1\ldots i_n}(\hat{\mathbf{r}})}{r^{n+1}}\right] \\ &= -\frac{\mu_0}{4\pi}\frac{2^n n!}{(2n)!}\varepsilon_{lki}\varepsilon_{iji_n}M^{\mathrm{p}0}_{ji_1\ldots i_{n-1}}(n+1)\frac{Q^{(n+1)}_{ki_1\ldots i_n}(\hat{\mathbf{r}})}{r^{n+2}} \\ &= \frac{\mu_0}{4\pi}\frac{1}{r^{n+2}}\frac{2^n n!}{(2n)!}(n+1)\left[M^{\mathrm{p}0}_{ji_1\ldots i_{n-1}}Q^{(n+1)}_{lji_1\ldots i_{n-1}}(\hat{\mathbf{r}}) - M^{\mathrm{p}0}_{li_1\ldots i_{n-1}}Q^{(n+1)}_{i_n i_n i_1\ldots i_{n-1}}(\hat{\mathbf{r}})\right] \\ &= \frac{\mu_0}{4\pi}\frac{1}{r^{n+2}}\frac{2^n n!}{(2n)!}(n+1)M^{\mathrm{p}0}_{ji_1\ldots i_{n-1}}Q^{(n+1)}_{lji_1\ldots i_{n-1}}(\hat{\mathbf{r}}). \end{aligned} \qquad (71)$$

Now, since $Q^{(n+1)}_{lji_1\ldots i_{n-1}}(\hat{\mathbf{r}})$ is a traceless symmetric tensor, the only part of the tensor $M^{\mathrm{p}0}_{ji_1\ldots i_{n-1}}$ that contributes to the magnetic field is its traceless symmetric part $M^{\mathrm{s}0}_{ji_1\ldots i_{n-1}}$ [6]:

$$B_l^{(n)}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^{n+2}}\frac{2^n n!}{(2n)!}(n+1)M^{\mathrm{s}0}_{ji_1\ldots i_{n-1}}Q^{(n+1)}_{lji_1\ldots i_{n-1}}(\hat{\mathbf{r}}), \qquad (72)$$

where $M^{\mathrm{s}0}_{ji_1\ldots i_{n-1}}$ is constructed from $M^{\mathrm{p}0}_{ji_1\ldots i_{n-1}}$ according to Eqs. (15) and (11).

Since $M^{\mathrm{s}0}_{ji_1\ldots i_{n-1}}$ is the only part of $M^{\mathrm{p}0}_{ji_2\ldots i_n}$ that contributes to the magnetic field, it is also the only part that needs to be included in the vector potential. The vector potential may therefore be changed to

$$\begin{aligned} A_i^{(n)}(\mathbf{r}) &= \frac{\mu_0}{4\pi}\frac{1}{r^{n+1}}\frac{2^n n!}{(2n)!}Q^{(n)}_{i_1\ldots i_n}(\hat{\mathbf{r}})\varepsilon_{iji_n}M^{\mathrm{s}0}_{ji_1\ldots i_{n-1}} \\ &= \frac{\mu_0}{4\pi}\frac{1}{r^{n+1}}\frac{1}{n!}\varepsilon_{iji_n}M^{\mathrm{s}0}_{ji_1\ldots i_{n-1}}\hat{x}_{i_1}\ldots\hat{x}_{i_n} \\ &= \frac{\mu_0}{4\pi}\frac{\left[\mathbf{M}^{(n)}(\hat{\mathbf{r}})\times\hat{\mathbf{r}}\right]_i}{r^{n+1}} \end{aligned} \qquad (73)$$

where the components of the vector $\mathbf{M}^{(n)}(\hat{\mathbf{r}})$ are given by

$$\begin{aligned} M_j^{(n)}(\hat{\mathbf{r}}) &= \frac{1}{n!}M^{\mathrm{s}0}_{ji_1\ldots i_{n-1}}\hat{x}_{i_1}\cdots\hat{x}_{i_{n-1}} \\ &= \frac{1}{n!}M^{\mathrm{s}0}_{ji_1\ldots i_{n-1}}\frac{x_{i_1}\ldots x_{i_{n-1}}}{r^{n-1}}. \end{aligned} \qquad (74)$$

The magnetic vector potential of the n-th order magnetic multipole is then given by

$$\mathbf{A}^{(n)}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{M}^{(n)}(\hat{\mathbf{r}}) \times \hat{\mathbf{r}}}{r^{n+1}} \tag{75}$$

and, taking the curl of this, the magnetic field of the n-th order magnetic multipole may be expressed as

$$\mathbf{B}^{(n)}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{(2n+1)\hat{\mathbf{r}}\left[\hat{\mathbf{r}}\cdot\mathbf{M}^{(n)}(\hat{\mathbf{r}})\right] - n\mathbf{M}^{(n)}(\hat{\mathbf{r}})}{r^{n+2}}. \tag{76}$$

The traceless symmetric tensor $M^{\mathrm{s}0}_{ji_1\ldots i_{n-1}}$ has $2n+1$ independent components and is the n-th order magnetic multipole moment tensor. The vector $\mathbf{M}^{(n)}(\hat{\mathbf{r}})$ is the n-th order magnetic multipole moment vector. The n-th order magnetic vector potential $\mathbf{A}^{(n)}(\mathbf{r})$ and magnetic field $\mathbf{B}^{(n)}(\mathbf{r})$ depend on the electric current density $\mathbf{j}(\mathbf{r})$ only through the vector $\mathbf{M}^{(n)}(\hat{\mathbf{r}})$ or only through the $\mathbf{r}$ independent tensor $M^{\mathrm{s}0}_{ji_1\ldots i_{n-1}}$.

The magnetic multipole moment tensor, magnetic multipole moment vector, magnetic vector potential, and magnetic field of an n-th order magnetic multipole may be calculated using the formulas

$$M_{ii_1\ldots i_n} = \int d^3\xi\, j_i(\boldsymbol{\xi})\xi_{i_1}\cdots\xi_{i_n}, \tag{77}$$

$$\begin{aligned} M^{\mathrm{p}0}_{ji_1\ldots i_{n-1}} &= \frac{(2n)!}{2^n n!}\frac{n}{n+1}\varepsilon_{iji_n}M_{ii_1\ldots i_n} \\ &= \frac{(2n)!}{2^n n!}\frac{n}{n+1}\int d^3\xi \\ &\qquad \times\left[\boldsymbol{\xi}\times\mathbf{j}(\boldsymbol{\xi})\right]_j\xi_{i_1}\cdots\xi_{i_{n-1}}, \end{aligned} \tag{78}$$

$$M^{\mathrm{s}0}_{ji_1\ldots i_{n-1}} = \text{traceless symmetric part of } M^{\mathrm{p}0}_{ji_1\ldots i_{n-1}}, \tag{79}$$

$$M^{(n)}_j = \frac{1}{n!}M^{\mathrm{s}0}_{ji_1\ldots i_{n-1}}\hat{x}_{i_1}\cdots\hat{x}_{i_{n-1}}, \tag{80}$$

$$\mathbf{A}^{(n)}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{M}^{(n)}(\hat{\mathbf{r}})\times\hat{\mathbf{r}}}{r^{n+1}}, \tag{81}$$

$$\mathbf{B}^{(n)}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{(2n+1)\hat{\mathbf{r}}\left[\hat{\mathbf{r}}\cdot\mathbf{M}^{(n)}(\hat{\mathbf{r}})\right] - n\mathbf{M}^{(n)}(\hat{\mathbf{r}})}{r^{n+2}}. \tag{82}$$

For a magnetic dipole these formulas reduce to

$$M_{ii_1} = \int d^3\xi\, j_i(\boldsymbol{\xi})\xi_{i_1}, \tag{83}$$

$$M^{\mathrm{p}0}_j = \frac{1}{2}\varepsilon_{iji_1}M_{ii_1} = \frac{1}{2}\int d^3\xi\,\left[\boldsymbol{\xi}\times\mathbf{j}(\boldsymbol{\xi})\right]_j, \tag{84}$$

$$M^{(1)}_j = M^{\mathrm{s}0}_j = \frac{1}{2}\int d^3\xi\,\left[\boldsymbol{\xi}\times\mathbf{j}(\boldsymbol{\xi})\right]_j, \tag{85}$$

$$\mathbf{A}^{(1)}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{M}^{(1)}(\hat{\mathbf{r}})\times\hat{\mathbf{r}}}{r^2}, \tag{86}$$

$$\mathbf{B}^{(1)}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{3\hat{\mathbf{r}}\left[\hat{\mathbf{r}}\cdot\mathbf{M}^{(1)}(\hat{\mathbf{r}})\right] - \mathbf{M}^{(1)}(\hat{\mathbf{r}})}{r^3} \tag{87}$$

and for a magnetic quadrupole these formulas reduce to

$$M_{ii_1i_2} = \int d^3\xi\, j_i(\boldsymbol{\xi})\xi_{i_1}\xi_{i_2}, \tag{88}$$

$$M^{\mathrm{p}0}_{ji_1} = 2\varepsilon_{iji_2}M_{ii_1i_2} = 2\int d^3\xi\left[\boldsymbol{\xi}\times\mathbf{j}(\boldsymbol{\xi})\right]_j\xi_{i_1}, \tag{89}$$

$$M^{\mathrm{s}0}_{ji_1} = \int d^3\xi\left\{\left[\boldsymbol{\xi}\times\mathbf{j}(\boldsymbol{\xi})\right]_j\xi_{i_1} + \left[\boldsymbol{\xi}\times\mathbf{j}(\boldsymbol{\xi})\right]_{i_1}\xi_j\right\}, \tag{90}$$

$$\begin{aligned} M^{(2)}_j &= \frac{1}{2}M^{\mathrm{s}0}_{ji_1}\frac{x_{i_1}}{r} = \frac{1}{2r}\int d^3\xi \\ &\qquad\times\left\{\left[\boldsymbol{\xi}\times\mathbf{j}(\boldsymbol{\xi})\right]_j\boldsymbol{\xi}\cdot\mathbf{r} + \left[\boldsymbol{\xi}\times\mathbf{j}(\boldsymbol{\xi})\right]\cdot\mathbf{r}\xi_j\right\}, \end{aligned} \tag{91}$$

$$\mathbf{A}^{(2)}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{M}^{(2)}(\hat{\mathbf{r}})\times\hat{\mathbf{r}}}{r^3}, \tag{92}$$

$$\mathbf{B}^{(2)}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{5\hat{\mathbf{r}}\left[\hat{\mathbf{r}}\cdot\mathbf{M}^{(2)}(\hat{\mathbf{r}})\right] - 2\mathbf{M}^{(2)}(\hat{\mathbf{r}})}{r^4}. \tag{93}$$

6. Circular loop carrying current *I*

In this section we consider the circular loop carrying a constant current I as an example and obtain explicit formulas for the n-th order magnetic multipole moment vector $\mathbf{M}^{(n)}(\hat{\mathbf{r}})$ and its corresponding vector potential $\mathbf{A}^{(n)}(\mathbf{r})$ and magnetic field $\mathbf{B}^{(n)}(\mathbf{r})$ in terms of the Legendre polynomials $P_{n-1}(\hat{\mathbf{n}}\cdot\hat{\mathbf{r}})$, $P_n(\hat{\mathbf{n}}\cdot\hat{\mathbf{r}})$, and $P_{n+1}(\hat{\mathbf{n}}\cdot\hat{\mathbf{r}})$.

We assume that the current loop C has radius R, is centered at the origin, and lies in a plane perpendicular to the direction $\hat{\mathbf{n}}$. The current density of the current loop is given by

$$\mathbf{j}(\mathbf{r}) = I\oint_C d\mathbf{l}\,\delta^3(\mathbf{r}-\mathbf{l}) \tag{94}$$

where the points along the current loop C are parameterized according to

$$\mathbf{l}(\alpha) = \begin{pmatrix}\hat{\mathbf{t}}_1 & \hat{\mathbf{t}}_2 & \hat{\mathbf{n}}\end{pmatrix}\begin{pmatrix}R\cos\alpha \\ R\sin\alpha \\ 0\end{pmatrix}, \qquad 0\le\alpha\le 2\pi \tag{95}$$

where $\hat{\mathbf{t}}_1, \hat{\mathbf{t}}_2, \hat{\mathbf{n}}$ constitute a right-handed orthonormal basis. The tensor $M^{\mathrm{p}0}_{ii_1\ldots i_{n-1}}$ is given, using Eqs. (94) and (95) in Eq. (78), by

$$\begin{aligned} M^{\mathrm{p}0}_{ii_1\ldots i_{n-1}} &= \frac{(2n)!}{2^n n!}\frac{n}{n+1}\int d^3\xi\,\left[\boldsymbol{\xi}\times\mathbf{j}(\boldsymbol{\xi})\right]_i\xi_{i_1}\cdots\xi_{i_{n-1}} \\ &= \frac{(2n)!}{2^n n!}\frac{n}{n+1}I\oint_C(\mathbf{l}\times d\mathbf{l})_i l_{i_1}\cdots l_{i_{n-1}} \\ &= \frac{(2n)!}{2^n n!}\frac{n}{n+1}I\int_0^{2\pi}\left(\hat{\mathbf{n}}R^2d\alpha\right)_i l_{i_1}\cdots l_{i_{n-1}} \\ &= \frac{(2n)!}{2^n n!}\frac{n}{n+1}IR^2\hat{n}_i\int_0^{2\pi}d\alpha\, l_{i_1}\cdots l_{i_{n-1}} \\ &= \frac{(2n)!}{2^n n!}\frac{n}{n+1}IR^2\hat{n}_iW_{i_1\ldots i_{n-1}}, \end{aligned} \tag{96}$$

where

$$W_{i_1\ldots i_{n-1}} \equiv \int_0^{2\pi} d\alpha\, l_{i_1}\cdots l_{i_{n-1}}. \tag{97}$$

The tensor $W_{i_1\ldots i_k}$ is symmetric and orthogonal to $\hat{\mathbf{n}}$:

$$\hat{n}_i W_{ii_2\ldots i_k} = 0. \tag{98}$$

Also, due to the cylindrical symmetry of the the problem, the only direction upon which $W_{i_1\ldots i_k}$ depends is $\hat{\mathbf{n}}$. From these facts it follows that $W_{i_1\ldots i_k}$ is proportional to the tensor obtained by symmeterizing the tensor $(\delta_{i_1 i_2} - \hat{n}_{i_1}\hat{n}_{i_2})(\delta_{i_3 i_4} - \hat{n}_{i_3}\hat{n}_{i_4})\ldots(\delta_{i_{k-1}i_k} - \hat{n}_{i_{k-1}}\hat{n}_{i_k})$. Note that $W_{i_1\ldots i_k} = 0$ for odd k. $W_{i_1\ldots i_k}$ may be expressed as

$$W_{i_1\ldots i_k} = R^{(k)} \frac{\partial^k}{\partial x_{i_1}\ldots\partial x_{i_k}} F\left(r^2 - (\mathbf{r}\cdot\hat{\mathbf{n}})^2\right)\Big|_{\mathbf{r}=0}, \tag{99}$$

where F is any function whose derivatives are non-zero and where $R^{(k)}$ is a constant of proportionality that depends on the particular choice for F. The constant $R^{(k)}$ can be determined from the value of one component of the tensor $W_{i_1\ldots i_k}$, *e. g.*, that component for which $i_1 = i_2 = \cdots = i_k = 1$:

$$W_{1\ldots 1} = R^k \int_0^{2\pi} d\alpha\, \cos^k\alpha = \begin{cases} \dfrac{2\pi R^k}{2^k}\dbinom{k}{k/2}, & \text{for even } k, \\ 0, & \text{for odd } k. \end{cases} \tag{100}$$

Choosing F to be the exponential function, we then obtain

$$\frac{\partial^k}{\partial x_1^k} e^{r^2-(\mathbf{r}\cdot\hat{\mathbf{n}})^2}\Big|_{\mathbf{r}=0} = \begin{cases} \dfrac{k!}{(k/2)!}, & \text{for even } k, \\ 0, & \text{for odd } k, \end{cases} \tag{101}$$

so that

$$R^{(k)} = \frac{2\pi R^k}{2^k}\binom{k}{k/2}\frac{(k/2)!}{k!} = \frac{2\pi R^k}{(k/2)!2^k} \tag{102}$$

and [8]

$$W_{i_1\ldots i_k} = \frac{2\pi R^k}{(k/2)!2^k}\frac{\partial^k}{\partial x_{i_1}\ldots\partial x_{i_k}} e^{r^2-(\mathbf{r}\cdot\hat{\mathbf{n}})^2}\Big|_{\mathbf{r}=0}. \tag{103}$$

Using Eq. (103) in Eq. (96) then gives

$$M^{\mathrm{p}0}_{ii_1\ldots i_{n-1}} = \frac{4\pi I R^{n+1}}{[(n-1)/2]!2^n}\frac{(2n)!}{2^n n!}\frac{n}{n+1}\hat{n}_i \times \frac{\partial^{n-1}}{\partial x_{i_1}\ldots\partial x_{i_{n-1}}} e^{r^2-(\mathbf{r}\cdot\hat{\mathbf{n}})^2}\Big|_{\mathbf{r}=0}. \tag{104}$$

Note that $M^{\mathrm{p}0}_{ii_1\ldots i_{n-1}} = 0$ for even n.

To obtain the traceless symmetric part $M^{\mathrm{s}0}_{ii_1\ldots i_{n-1}}$ of the tensor $M^{\mathrm{p}0}_{ii_1\ldots i_{n-1}}$ we use the fact that any rank n traceless symmetric tensor that depends *only* on a direction $\hat{\mathbf{n}}$ is proportional to the spherical tensor $Q^{(n)}_{i_1\ldots i_n}(\hat{\mathbf{n}})$. Evaluation of the constant of proportionality gives

$$M^{\mathrm{s}0}_{ii_1\ldots i_{n-1}} = \begin{cases} (-1)^{\left(\frac{n-1}{2}\right)} 4\pi I R^{n+1}\dfrac{nn!}{2^n(n+1)}\dbinom{n-1}{\frac{n-1}{2}} \\ \qquad \times Q^{(n)}_{ii_1\ldots i_{n-1}}(\hat{\mathbf{n}}), \quad \text{for } n \text{ odd}, \\ 0, \qquad \text{for } n \text{ even}. \end{cases} \tag{105}$$

The magnetic multipole moment vector $\mathbf{M}^{(n)}$ obtained from Eqs. (105) and (74) is a linear combination of $\hat{\mathbf{r}}$ and $\hat{\mathbf{n}}$:

$$\mathbf{M}^{(n)} = C_1\hat{\mathbf{r}} + C_2\hat{\mathbf{n}} = \frac{\begin{vmatrix} \hat{\mathbf{r}}\cdot\mathbf{M}^{(n)} & \hat{\mathbf{r}}\cdot\hat{\mathbf{n}} \\ \hat{\mathbf{n}}\cdot\mathbf{M}^{(n)} & \hat{\mathbf{n}}\cdot\hat{\mathbf{n}} \end{vmatrix}}{\begin{vmatrix} \hat{\mathbf{r}}\cdot\hat{\mathbf{r}} & \hat{\mathbf{r}}\cdot\hat{\mathbf{n}} \\ \hat{\mathbf{n}}\cdot\hat{\mathbf{r}} & \hat{\mathbf{n}}\cdot\hat{\mathbf{n}} \end{vmatrix}}\hat{\mathbf{r}} + \frac{\begin{vmatrix} \hat{\mathbf{r}}\cdot\hat{\mathbf{r}} & \hat{\mathbf{r}}\cdot\mathbf{M}^{(n)} \\ \hat{\mathbf{n}}\cdot\hat{\mathbf{r}} & \hat{\mathbf{n}}\cdot\mathbf{M}^{(n)} \end{vmatrix}}{\begin{vmatrix} \hat{\mathbf{r}}\cdot\hat{\mathbf{r}} & \hat{\mathbf{r}}\cdot\hat{\mathbf{n}} \\ \hat{\mathbf{n}}\cdot\hat{\mathbf{r}} & \hat{\mathbf{n}}\cdot\hat{\mathbf{n}} \end{vmatrix}}\hat{\mathbf{n}}. \tag{106}$$

Using Eqs. (74), (105), (23), and (26), the scalar products $\hat{\mathbf{r}}\cdot\mathbf{M}^{(n)}$ and $\hat{\mathbf{n}}\cdot\mathbf{M}^{(n)}$ are given, for n odd, by

$$\hat{\mathbf{r}}\cdot\mathbf{M}^{(n)} = (-1)^{\frac{n-1}{2}}\frac{4\pi I R^{n+1}}{(n+1)2^n}\binom{n-1}{\frac{n-1}{2}} nP_n(\hat{\mathbf{n}}\cdot\hat{\mathbf{r}}), \tag{107}$$

$$\hat{\mathbf{n}}\cdot\mathbf{M}^{(n)} = (-1)^{\frac{n-1}{2}}\frac{4\pi I R^{n+1}}{(n+1)2^n}\binom{n-1}{\frac{n-1}{2}} nP_{n-1}(\hat{\mathbf{n}}\cdot\hat{\mathbf{r}}). \tag{108}$$

Using Eqs. (107) and (108) in Eq. (106) and then using Eqs. (81) and (82) gives the following explicit formulas for the n-th order magnetic multipole moment vector, vector potential, and magnetic field of the circular current loop in terms of the Legendre polynomials $P_{n-1}(\hat{\mathbf{n}}\cdot\hat{\mathbf{r}})$, $P_n(\hat{\mathbf{n}}\cdot\hat{\mathbf{r}})$, and $P_{n+1}(\hat{\mathbf{n}}\cdot\hat{\mathbf{r}})$:

$$\mathbf{M}^{(n)}(\hat{\mathbf{r}}) = \begin{cases} (-1)^{\frac{n-1}{2}}\dfrac{4\pi I R^{n+1}}{(n+1)2^n}\dbinom{n-1}{\frac{n-1}{2}}\dfrac{n}{1-(\hat{\mathbf{n}}\cdot\hat{\mathbf{r}})^2}\left\{\hat{\mathbf{n}}\left[P_{n-1}(\hat{\mathbf{n}}\cdot\hat{\mathbf{r}}) - (\hat{\mathbf{n}}\cdot\hat{\mathbf{r}})P_n(\hat{\mathbf{n}}\cdot\hat{\mathbf{r}})\right]\right. \\ \qquad \left. +\hat{\mathbf{r}}\left[P_n(\hat{\mathbf{n}}\cdot\hat{\mathbf{r}}) - (\hat{\mathbf{n}}\cdot\hat{\mathbf{r}})P_{n-1}(\hat{\mathbf{n}}\cdot\hat{\mathbf{r}})\right]\right\}, & \text{for odd } n, \\ \mathbf{0}, & \text{for even } n, \end{cases} \tag{109}$$

$$\mathbf{A}^{(n)}(\mathbf{r}) = \begin{cases} (-1)^{\frac{n-1}{2}} \mu_0 I \dfrac{R^{n+1}}{r^{n+1}} \dbinom{n-1}{\frac{n-1}{2}} \dfrac{\hat{\mathbf{n}} \times \hat{\mathbf{r}}}{(n+1)2^n} \dfrac{n}{1-(\hat{\mathbf{n}}\cdot\hat{\mathbf{r}})^2} \left[P_{n-1}(\hat{\mathbf{n}}\cdot\hat{\mathbf{r}}) - (\hat{\mathbf{n}}\cdot\hat{\mathbf{r}})P_n(\hat{\mathbf{n}}\cdot\hat{\mathbf{r}})\right], & \text{for odd } n, \\ \mathbf{0}, & \text{for even } n, \end{cases} \tag{110}$$

$$\mathbf{B}^{(n)}(\mathbf{r}) = \begin{cases} (-1)^{\frac{n-1}{2}} \mu_0 I \dfrac{R^{n+1}}{r^{n+2}} \dbinom{n-1}{\frac{n-1}{2}} \dfrac{1}{(n+1)2^n} \dfrac{n}{1-(\hat{\mathbf{n}}\cdot\hat{\mathbf{r}})^2} \left\{\hat{\mathbf{r}}(n+1)\left[P_n(\hat{\mathbf{n}}\cdot\hat{\mathbf{r}}) - (\hat{\mathbf{n}}\cdot\hat{\mathbf{r}})P_{n+1}(\hat{\mathbf{n}}\cdot\hat{\mathbf{r}})\right]\right. \\ \qquad \left. -\hat{\mathbf{n}} n \left[P_{n-1}(\hat{\mathbf{n}}\cdot\hat{\mathbf{r}}) - (\hat{\mathbf{n}}\cdot\hat{\mathbf{r}})P_n(\hat{\mathbf{n}}\cdot\hat{\mathbf{r}})\right]\right\}, & \text{for odd } n, \\ \mathbf{0}, & \text{for even } n. \end{cases} \tag{111}$$

For $n = 1$, Eqs. (109), (110), and (111) reduce to the standard dipole results:

$$\mathbf{M}^{(1)}(\mathbf{r}) = I\pi R^2 \hat{\mathbf{n}}, \tag{112}$$

$$\mathbf{A}^{(1)}(\mathbf{r}) = \frac{\mu_0 I R^2}{4} \frac{\hat{\mathbf{n}} \times \hat{\mathbf{r}}}{r^2}, \tag{113}$$

$$\mathbf{B}^{(1)}(\mathbf{r}) = \frac{\mu_0 I R^2}{4} \frac{3(\hat{\mathbf{n}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \hat{\mathbf{n}}}{r^3}. \tag{114}$$

An analysis of the circular loop using toroidal coordinates was carried out in Ref. 7. The exact form of the vector potential $\mathbf{A}$ was found to involve a Legendre function of the second kind with fractional index $Q_{\frac{1}{2}}$ and the magnetic field $\mathbf{B}$ involves $Q_{\frac{1}{2}}$ and its derivative. The study of the field near the current loop was especially interesting. Two components of $\mathbf{B}$ were found to diverge as the distance d from the current loop approaches zero. The component of $\mathbf{B}$ that curls around the current diverges as $1/d$. This divergence is also present in the case of an infinite straight wire. The component of $\mathbf{B}$ normal to the current diverges as $\sin(\theta)/R \times \ln(d/R)$, where θ is a toroidal coordinate. This logarithmic divergence vanishes if one first takes the limit $R \to \infty$ to obtain an infinite straight wire. It also vanishes in the plane of the current loop, where $\theta = 0$ corresponds to the disk bounded by the current loop and $\theta = \pi$ corresponds to that part of the plane which is exterior to the current loop.

The behavior of the terms in the multipole expansion can also be analyzed near the current. Using Stirling's expansion we can show that both types of divergence, $1/d$ and $\ln d$, are present but our predictions are not as precise as in Ref. 7 because the divergent terms must be summed to obtain the behavior of the complete field. The multipole expansion is not the best tool with which to analyze the properties of the fields close to the sources. The lack of precise predictions close to the sources is a general property of all multipole expansions rather than a deficiency of our method.

7. Conclusions

Equations (36) and (37) give exact, closed formulas that show how the electric scalar potential $\varphi^{(n)}$ and electric field $\mathbf{E}^{(n)}$ of an n-th order electric multipole depend on the charge density ρ only through the electric multipole moment vector $\boldsymbol{\rho}^{(n)}$. Equations (75) and (76) give exact, closed formulas that show how the magnetic vector potential $\mathbf{A}^{(n)}$ and magnetic field $\mathbf{B}^{(n)}$ of an n-th order magnetic multipole depend on the current density $\mathbf{j}$ only through the magnetic multipole moment vector $\mathbf{M}^{(n)}$. The derivation of Eq. (76) for the magnetic field was complicated because of the need to use additional constraints imposed by electric current conservation. Equations (36), (37), (75), and (76) are valid for time-independent sources ρ and $\mathbf{j}$ and for observation points that are farther away from the origin than are the sources.

For the example of the circular loop carrying the constant current I, Eqs. (109), (110), and (111), respectively, give explicit formulas for the n-th order magnetic multipole moment vector $\mathbf{M}^{(n)}(\hat{\mathbf{r}})$, vector potential $\mathbf{A}^{(n)}(\mathbf{r})$, and magnetic field $\mathbf{B}^{(n)}(\mathbf{r})$ in terms of the Legendre polynomials $P_{n-1}(\hat{\mathbf{n}}\cdot\hat{\mathbf{r}})$, $P_n(\hat{\mathbf{n}}\cdot\hat{\mathbf{r}})$, and $P_{n+1}(\hat{\mathbf{n}}\cdot\hat{\mathbf{r}})$.

Our tensor methods used rotational covariance to its full extent. We also used the cartesian coordinates of spherical tensors rather than spherical harmonics.

Equations (76) and (74) which relate the magnetic field $\mathbf{B}^{(n)}(\mathbf{r})$ to the $\mathbf{r}$-independent magnetic multipole moment tensor $M^{s0}_{ji_1\ldots i_{n-1}}$ are, apart from the factor $\mu_0\epsilon_0 = 1/c^2$, identical to Eqs. (37) and (35) which relate the electric field $\mathbf{E}^{(n)}(\mathbf{r})$ to the $\mathbf{r}$-independent electric multipole moment tensor $\rho^{s0}_{ji_1\ldots i_{n-1}}$. It follows that, in analogy to Eq. (36), the magnetic field $\mathbf{B}^{(n)}(\mathbf{r})$ is also given as the negative gradient of a magnetic scalar potential [9],

$$\mathbf{B}^{(n)}(\mathbf{r}) = -\nabla\left[\frac{\mu_0}{4\pi} \frac{\hat{\mathbf{r}}\cdot\mathbf{M}^{(n)}(\hat{\mathbf{r}})}{r^{n+1}}\right], \tag{115}$$

and that, in analogy to Eq. (75), the electric field $\mathbf{E}^{(n)}(\mathbf{r})$ is also given as the curl of an electric vector potential,

$$\mathbf{E}^{(n)}(\mathbf{r}) = \nabla \times \left[\frac{1}{4\pi\epsilon_0} \frac{\boldsymbol{\rho}^{(n)}(\hat{\mathbf{r}}) \times \hat{\mathbf{r}}}{r^{n+1}}\right], \tag{116}$$

except, of course, for $n = 0$, in which case an electric monopole vector potential does not exist.

The facts that $\varphi^{(n)}$ and $\mathbf{E}^{(n)}$ depend on ρ only through a vector $\boldsymbol{\rho}^{(n)}$ and the facts that $\mathbf{A}^{(n)}$ and $\mathbf{B}^{(n)}$ depend on $\mathbf{j}$ only through a vector $\mathbf{M}^{(n)}$ follow from the facts that $\rho^{s0}_{i_1\ldots i_n}$ and $M^{s0}_{i_1\ldots i_n}$ are traceless symmetric tensors together with the fact that Maxwell's equations are linear. The only scalars and vectors that can be constructed from the traceless symmetric tensors $\rho^{s0}_{i_1\ldots i_n}$ and $M^{s0}_{i_1\ldots i_n}$ and the unit vector $\hat{\mathbf{r}}$ and that are linear in the sources are

$$\rho_i^{(n)} = \frac{1}{n!}\rho^{\mathrm{s}0}_{ii_1\ldots i_{n-1}}\hat{x}_{i_1}\ldots\hat{x}_{i_{n-1}}, \qquad \text{vector,} \tag{117}$$

$$\boldsymbol{\rho}^{(n)}\cdot\hat{\mathbf{r}} = \frac{1}{n!}\rho^{\mathrm{s}0}_{i_1\ldots i_n}\hat{x}_{i_1}\ldots\hat{x}_{i_n}, \qquad \text{scalar,} \tag{118}$$

$$\left[\boldsymbol{\rho}^{(n)}\times\hat{\mathbf{r}}\right]_i = \frac{1}{n!}\varepsilon_{ii_1j}\rho^{\mathrm{s}0}_{i_1\ldots i_n}\hat{x}_j\hat{x}_{i_2}\ldots\hat{x}_{i_{n-1}}, \qquad \text{axial vector,} \tag{119}$$

$$M_i^{(n)} = \frac{1}{n!}M^{\mathrm{s}0}_{ii_1\ldots i_{n-1}}\hat{x}_{i_1}\ldots\hat{x}_{i_{n-1}}, \qquad \text{axial vector,} \tag{120}$$

$$\mathbf{M}^{(n)}\cdot\hat{\mathbf{r}} = \frac{1}{n!}M^{\mathrm{s}0}_{i_1\ldots i_n}\hat{x}_{i_1}\ldots\hat{x}_{i_n}, \qquad \text{pseudoscalar,} \tag{121}$$

$$\left[\mathbf{M}^{(n)}\times\hat{\mathbf{r}}\right]_i = \frac{1}{n!}\varepsilon_{ii_1j}M^{\mathrm{s}0}_{i_1\ldots i_n}\hat{x}_j\hat{x}_{i_2}\ldots\hat{x}_{i_{n-1}}, \qquad \text{vector.} \tag{122}$$

From the linearity of Maxwell equations and the facts that $\varphi^{(n)}$ is a scalar, $\mathbf{E}^{(n)}$ and $\mathbf{A}^{(n)}$ are vectors, and $\mathbf{B}^{(n)}$ is a pseudo-vector it then follows that the potentials and fields are given as linear functions of $\boldsymbol{\rho}^{(n)}$ and $\mathbf{M}^{(n)}$ of the following form:

$$\varphi^{(n)} = \left[\boldsymbol{\rho}^{(n)}\cdot\hat{\mathbf{r}}\right] f(r), \tag{123}$$

$$\mathbf{E}^{(n)} = \hat{\mathbf{r}}\left[\boldsymbol{\rho}^{(n)}\cdot\hat{\mathbf{r}}\right] e_1(r) + \boldsymbol{\rho}^{(n)}e_2(r), \tag{124}$$

$$\mathbf{A}^{(n)} = \left[\mathbf{M}^{(n)}\times\hat{\mathbf{r}}\right] a(r), \tag{125}$$

$$\mathbf{B}^{(n)} = \hat{\mathbf{r}}\left[\mathbf{M}^{(n)}\cdot\hat{\mathbf{r}}\right] b_1(r) + \mathbf{M}^{(n)}b_2(r). \tag{126}$$

The functions $f(r)$ and $a(r)$ are determined from Maxwell equations and the functions $e_1(r)$, $e_2(r)$, $b_1(r)$, and $b_2(r)$, are obtained by differentiation.

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Appendix

We will now use gauge invariance to show that only the traceless symmetric part of the tensor $M^{\mathrm{p}0}_{ji_1\ldots i_{n-1}}$ that appears in Eq. (70) contributes to the magnetic field. Gauge invariance states that any term in the vector potential $\mathbf{A}$ that is the gradient of a scalar function does not contribute to the magnetic field.

According to Theorem 2, the traceless partially symmetric tensor $M^{\mathrm{p}0}_{ji_1\ldots i_{n-1}}$ may be expressed in terms of the symmetric tensor

$$M^{\mathrm{s}}_{ji_1\ldots i_{n-1}} = \frac{1}{n}M^{\mathrm{p}0}_{ji_1\ldots i_{n-1}} + \frac{1}{n}\sum_{k=1}^{n-1}M^{\mathrm{p}0}_{i_kji_1\ldots i_{k-1}i_{k+1}\ldots i_{n-1}} \tag{A.1}$$

and the traceless partially symmetric tensor

$$M^{\mathrm{p}0}_{li_1\ldots i_{n-2}} = \frac{1}{2n}\varepsilon_{ji_{n-1}l}(M^{\mathrm{p}0}_{ji_1\ldots i_{n-1}}M^{\mathrm{p}0}_{i_{n-1}i_1\ldots i_{n-2}j}) \tag{A.2}$$

according to

$$M^{\mathrm{p}0}_{ji_1\ldots i_{n-1}} = M^{\mathrm{s}}_{ji_1\ldots i_{n-1}} + \sum_{k=1}^{n-1}\varepsilon_{ji_kl}M^{\mathrm{p}0}_{li_1\ldots i_{k-1}i_{k+1}\ldots i_{n-1}}. \tag{A.3}$$

Plugging Eq. (A.3) into Eq. (70) and using Eq. (9) gives

$$\begin{aligned} A_i^{(n)} &= \frac{\mu_0}{4\pi}\frac{1}{r^{n+1}}\frac{2^n n!}{(2n)!}Q^{(n)}_{i_1\ldots i_n}(\hat{\mathbf{r}})\left[\varepsilon_{iji_n}M^{\mathrm{s}}_{ji_1\ldots i_{n-1}} + \sum_{k=1}^{n-1}(\delta_{il}\delta_{i_ki_n} - \delta_{ii_k}\delta_{li_n})M^{\mathrm{p}0}_{li_1\ldots i_{k-1}i_{k+1}\ldots i_{n-1}}\right] \\ &= \frac{\mu_0}{4\pi}\frac{1}{r^{n+1}}\frac{2^n n!}{(2n)!}Q^{(n)}_{i_1\ldots i_n}(\hat{\mathbf{r}})\left[\varepsilon_{iji_n}M^{\mathrm{s}}_{ji_1\ldots i_{n-1}} - \sum_{k=1}^{n-1}\delta_{ii_k}\delta_{li_n}M^{\mathrm{p}0}_{li_1\ldots i_{k-1}i_{k+1}\ldots i_{n-1}}\right] \end{aligned} \tag{A.4}$$

where, for the last equality, the term

$$\frac{\mu_0}{4\pi}\frac{1}{r^{n+1}}\frac{2^n n!}{(2n)!}Q^{(n)}_{i_1\ldots i_n}(\hat{\mathbf{r}})\sum_{k=1}^{n-1}\delta_{il}\delta_{i_ki_n}M^{\mathrm{p}0}_{li_1\ldots i_{k-1}i_{k+1}\ldots i_{n-1}} \tag{A.5}$$

vanishes because $Q^{(n)}_{i_1\ldots i_n}$ is traceless. Using Eq. (22), the term

$$-\frac{\mu_0}{4\pi}\frac{1}{r^{n+1}}\frac{2^n n!}{(2n)!}Q^{(n)}_{i_1\ldots i_n}(\hat{\mathbf{r}})\sum_{k=1}^{n-1}\delta_{ii_k}\delta_{li_n}M^{\mathrm{p}0}_{li_1\ldots i_{k-1}i_{k+1}\ldots i_{n-1}} = \frac{\partial}{\partial x_i}\left(\frac{\mu_0}{4\pi}\frac{2^n n!}{(2n)!}\frac{n-1}{n}\frac{1}{r^n}Q^{(n-1)}_{i_1\ldots i_{n-1}}(\hat{\mathbf{r}})M^{\mathrm{p}0}_{i_{n-1}i_1\ldots i_{n-2}}\right) \quad (A.6)$$

is seen to be the gradient of a scalar. Gauge invariance states that this term does not contribute to the magnetic field $\mathbf{B}$. Omitting this term from the vector potential gives

$$A^{(n)}_i = \frac{\mu_0}{4\pi}\frac{1}{r^{n+1}}\frac{2^n n!}{(2n)!}Q^{(n)}_{i_1\ldots i_n}(\hat{\mathbf{r}})\varepsilon_{iji_n}M^{\mathrm{s}}_{ji_1\ldots i_{n-1}}. \quad (A.7)$$

Now, according to Theorem 1, the symmetric tensor $M^{\mathrm{s}}_{ji_1\ldots i_{n-1}}$ can be expressed in terms of its traceless symmetric part $M^{\mathrm{s}0}_{ji_1\ldots i_{n-1}}$ plus terms containing symmetric tensors of lower rank and (products of) the δ tensor:

$$M^{\mathrm{s}}_{ji_1\ldots i_{n-1}} = M^{\mathrm{s}0}_{ji_1\ldots i_{n-1}} + \frac{1}{2n-1}\left(\sum_{l=1}^{n-1}\delta_{ji_l}M^{\mathrm{s}}_{mmi_1\ldots i_{l-1}i_{l+1}\ldots i_{n-1}} + \sum_{k=1}^{n-2}\sum_{l=k+1}^{n-1}\delta_{i_k i_l}M^{\mathrm{s}}_{jmmi_1\ldots i_{k-1}i_{k+1}\ldots i_{l-1}i_{l+1}\ldots i_{n-1}}\right) + \cdots. \quad (A.8)$$

When Eq. (A.8) is inserted into Eq. (A.7) those terms containing the tensor δ do not contribute because $Q^{(n)}_{i_1\ldots i_n}(\hat{\mathbf{r}})$ is traceless and symmetric. The final form of the vector potential is, therefore, given by

$$A^{(n)}_i = \frac{\mu_0}{4\pi}\frac{1}{r^{n+1}}\frac{2^n n!}{(2n)!}Q^{(n)}_{i_1\ldots i_n}(\hat{\mathbf{r}})\varepsilon_{iji_n}M^{\mathrm{s}0}_{ji_1\ldots i_{n-1}}. \quad (A.9)$$

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‡. Traces cannot be defined for scalars (tensors of rank 0) or vectors (tensors of rank 1).

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$$W_{i_1\ldots i_n} = \int_0^{2\pi} d\alpha\, \frac{\partial^n}{\partial x_{i_1}\ldots\partial x_{i_n}} e^{\mathbf{r}\cdot\mathbf{l}}\Big|_{\mathbf{r}=\mathbf{0}} = \frac{\partial^n}{\partial x_{i_1}\ldots\partial x_{i_n}}\int_0^{2\pi} d\alpha\, e^{R\sqrt{(\mathbf{r}\cdot\hat{\mathbf{t}}_1)^2+(\mathbf{r}\cdot\hat{\mathbf{t}}_2)^2}\sin\left(\alpha+\tan^{-1}\frac{\mathbf{r}\cdot\hat{\mathbf{t}}_1}{\mathbf{r}\cdot\hat{\mathbf{t}}_2}\right)}\Big|_{\mathbf{r}=\mathbf{0}}$$

$$= \frac{\partial^n}{\partial x_{i_1}\ldots\partial x_{i_n}} 2\int_0^{\pi} d\alpha\, e^{R\sqrt{r^2-(\mathbf{r}\cdot\hat{\mathbf{n}})^2}\cos\alpha}\Big|_{\mathbf{r}=\mathbf{0}} = \frac{\partial^n}{\partial x_{i_1}\ldots\partial x_{i_n}} 2\pi\sum_{k=0}^{\infty}\frac{\left(\frac{R}{2}\sqrt{r^2-(\mathbf{r}\cdot\hat{\mathbf{n}})^2}\right)^{2k}}{(k!)^2}\Bigg|_{\mathbf{r}=\mathbf{0}}$$

$$= \frac{2\pi}{(\frac{n}{2})!}\left(\frac{R}{2}\right)^n \frac{\partial^n}{\partial x_{i_1}\ldots\partial x_{i_n}} e^{r^2-(\mathbf{r}\cdot\hat{\mathbf{n}})^2}\Big|_{\mathbf{r}=\mathbf{0}},$$

where the second equality relies on a trigonometric identity, the third equality relies on the fact that a phase shift has no effect when a periodic function is integrated over one full period, and the fifth equality relies on the fact that only the $k = n/2$ term in the sum contributes.

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