Discretizing the deformation parameter in the $su_q(2)$ quantum algebra

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Recibido el 3 de abril de 1997; aceptado el 5 de junio de 1997

Inspired in recent works of Biedenharn [1, 2] on the realization of the q-algebra $su_q(2)$, we show in this note that the condition $[2j+1]_q = N_q(j) =$ integer, implies the discretization of the deformation parameter α, where $q = e^{\alpha}$. This discretization replaces the continuum associated to α by an infinite sequence $\alpha_1, \alpha_2, \alpha_3, \ldots$, obtained for the values of j, which label the irreps of $su_q(2)$. The algebraic properties of $N_q(j)$ are discussed in some detail, including its role as a trace, which conducts to the Clebsch-Gordan series for the direct product of irreps. The consequences of this process of discretization are discussed and its possible applications are pointed out. Although not a necessary one, the present prescription is valuable due to its algebraic simplicity especially in the regime of appreciable values of α.

Keywords: $su_q(2)$, quantum algebras, parameter discretization

Inspirados por trabajos recientes de Biedenharn [1, 2] sobre la realizacion de la q-algebra $su_q(2)$, mostramos en esta nota que la condición $[2j+1]_q = N_q(j) =$ entero, implica la discretización del parámetro de deformación α, donde $q = e^{\alpha}$. Esta discretización substituye el continuo asociado a α por una sucesión $\alpha_1, \alpha_2, \alpha_3, \ldots$obtenida para valores de j que rotulan las irreps de $su_q(2)$. Las propriedades algebráicas de $N_q(j)$ son discutidas con algún detalle, incluso su papel como un trazo, que conduce a la serie de Clebsh-Gordan para el producto directo de irreps. Las consecuencias de este processo de discretización son discutidas y sus posibles aplicaciones son indicadas. Aunque no necesaria, la presente prescripción es de interés debido a su sencillez algebráica, especialmente en el régimen de valores apreciables de α.

Descriptores: $su_q(2)$, algebras cuánticas, discretización de parámetros

PACS: 12.15Ff, 11.30Na

1. Introduction

Quantum algebras are by now known to play a distinguished role in several fields of theoretical physics [1, 2]. Some years ago Biedenharn [1] extended the Jordan-Schwinger procedure to the $su_q(2)$ quantum algebra by means of a pair of mutually commuting q-harmonic oscillators a_{iq} and $\bar{a}_{iq}$ ($i = 1$, 2). He constructed a basis $|j, m\rangle_q$ for $(2j+1)$ dimensional irreps of $su_q(2)$, $\mathcal{D}_q^{(j)}$, for every $j = 0$, 1/2, 1, ... with m running in the range $-j \leq m \leq j$ by integer steps.

The action of the generators of $su_q(2)$ on $|j, m\rangle_q$ obey, as is well known, the following relations:

$$J_{\pm}|j,m\rangle_q = ([j \mp m]_q[j \pm m+1]_q)^{1/2}|j, m\pm 1\rangle_q, \quad (1)$$

$$J_z|j,m\rangle_q = m|j,m\rangle_q. \quad (2)$$

It is remarkable that Eqs. (1) and (2) are similar to those of $su(2)$, except for the appearence of the q-number brackets, characteristic of the q-deformed case. In this note we wish to study a problem in the framework of the above $su_q(2)$ algebra, namely, we wish to discuss the meaning and algebraic consequences of the condition

$$[2j+1]_\alpha = N_\alpha(j) = \text{integer}. \quad (3)$$

The bracket in the left-hand side of Eq. (3) is the q-number defined by

$$[x]_\alpha = \frac{e^{x\alpha/2} - e^{-x\alpha/2}}{e^{\alpha/2} - e^{-\alpha/2}} = \frac{\sinh(x\alpha/2)}{\sinh(\alpha/2)}, \quad (4)$$

where the parameter α is defined by $q = e^{\alpha}$, with q a real positive number. The condition (3) says that in the right-hand side $N_\alpha(j)$ is an integer number. Thus, it implies a *discretization* of α, obtained for the values of j defining irreps of the $su_q(2)$ algebra. Instead of a continuous α we have now a discrete set of values of α: $\alpha_1, \alpha_2, \alpha_3, \ldots$. In the next Sections we show the consequences of this discretization and some of its useful aspects in possible applications. This paper is organized as follows: In Sect. 2 we discuss the properties of $N_\alpha(j)$ and the consequences of the α discretization. Section 3 discusses $N_\alpha(j)$ as a trace (or character), see Eq. (30), and develops its consequences [see Eq. (32)], and the useful relations, given by Eqs. (32)–(37), with $N_\alpha(j)$ integer. Finally, Sect. 4 contains our concluding remarks.

2. Properties of $N_\alpha(j)$

Firstly, we recall that $N_\alpha(j)$ can be written in the following equivalent forms:

$$i)\ N_\alpha(j) = [2j+1]_\alpha, \quad (5)$$

$$ii)\ N_\alpha(j) = \sum_{m=-j}^{j} e^{m\alpha}, \quad (6)$$

$$iii)\ N_\alpha(j) = \frac{\sinh((j+1/2)\alpha)}{\sinh(\alpha/2)}. \quad (7)$$

Equation (5) is the defining equation for $N_\alpha(j)$ and shows that the defined positive integer number N_α corresponds to what is called q-dimension of representation, namely $[2j+1]_q$. Equations (6) and (7) follow readily from Eq. (3) in terms of α. Equation (6) follows from the finite-series sum for the q-numbers of an integer n:

$$[n]_q = \sum_{l=-(n-1)}^{n-1} q^{l/2} \tag{8}$$

The expression for $N_\alpha(j)$ given in Eq. (7) will be useful to obtain the equations which follow, with a dependence on hyperbolic functions. From Eqs. (5) and (6) one gets

$$[1]_\alpha = N_\alpha(0) = 1, \quad \forall \alpha. \tag{9}$$

And using Eqs. (6) and (7), one has, for $N_\alpha(j)$,

$$[2]_\alpha = N_\alpha(1/2) = 2\cosh(\alpha/2), \tag{10}$$

$$[3]_\alpha = N_\alpha(1) = 2\cosh\alpha + 1, \tag{11}$$

$$[4]_\alpha = N_\alpha(3/2) = 2\cosh(3\alpha/2) + 2\cosh(\alpha/2), \tag{12}$$

$$[5]_\alpha = N_\alpha(2) = 2\cosh(2\alpha) + 2\cosh(\alpha) + 1, \tag{13}$$

$$[6]_\alpha = N_\alpha(5/2) = 2\cosh(5\alpha/2) + 2\cosh(3\alpha/2) + 2\cosh(\alpha/2), \tag{14}$$

and so on.

Several relations involving $N_\alpha(j)$ arise from Eqs. (9)–(14). For instance, one has

$$N_\alpha(1/2) = \sqrt{N_\alpha(1) + 1}. \tag{15}$$

Furthermore, by taking $j = 1$, one gets, from Eq. (11),

$$\cosh\alpha = \frac{1}{2}[N_\alpha(1) - 1]. \tag{16}$$

Inspection of the above equations involving cosh shows that, if $[3]_\alpha = N_\alpha(1)$ = integer number, then for all integer values of j one has [odd number] = integer number, while if $[2]_\alpha = N_\alpha(1/2)$ = integer number, then one also has [even number] = integer number, corresponding to half-integer values of j.

Thus, we can list the values of α which satisfy condition (3) by inverting Eq. (16), which reads

$$\alpha = \cosh^{-1}\left[\frac{1}{2}(N_\alpha(1) - 1)\right], \tag{17}$$

and by just taking the values of $N_\alpha(1)$ which are integer numbers ≥ 3.

Equation (17) gives, for real α,

$$\alpha = \pm\ln\left(x + \sqrt{x^2 - 1}\right), \tag{18}$$

with $x \equiv (1/2)[N_\alpha(1) - 1]$ an integer or half-integer number ≥ 1. Notice that, as $q = e^\alpha$ one has

$$q = x + \sqrt{x^2 - 1} \quad \text{or} \quad (x + \sqrt{x^2 - 1})^{-1} \tag{19}$$

We will consider here the positive values of α. From Eqs. (17) and (18) one obtains the results displayed in Table I.

The first 25 solutions or "roots" α_i which satisfy the condition of Eq. (3) are listed in Table I. From that list one can note that some values of α_i are of the form $2\alpha_j$, like $\alpha_6 = 2\alpha_2$, $\alpha_{13} = 2\alpha_3$, $\alpha_{22} = 2\alpha_4, \ldots$. Besides, these values of α correspond to those in which $N(1/2)$ is of the form $\sqrt{\text{integer}}$ = integer. The same fact will occur for $j = 3/2$, $5/2, \ldots$ at the same "roots" α. Instead, for j = integer, one has $N(j)$ = integer too. In general, for $N_\alpha(1/2)$, with α of the form $2\alpha_j$, one has $N_{\alpha_i=2\alpha_j}(1/2) = \sqrt{\text{integer}} = \sqrt{(j+1)^2} = j+1$. For instance, $\alpha_6 = 2\alpha_2$, correspondingly, $N_{\alpha_6}(1/2) = \sqrt{9} = 3$.

As another example: when dealing with triplets ($j = 1$), we found

$$Y_\alpha \equiv \frac{[2]_\alpha}{[1/2]^2_\alpha} = [3]^2_{\alpha'=\alpha/2} - 1 = N^2_{\alpha'=\alpha/2}(1) - 1. \tag{20}$$

This quantity, Y_α, is related to the trace of mass matrices for the $j = 1$ representations (see Ref. 3). Notice that for any α of the form $2\alpha_j$ one then has Y_α as an integer number, too. Some results for $\alpha_i = 2\alpha_j$ are given in Table II.

In Table III we show the values of $[2j+1]_\alpha = N_\alpha(j)$ for the first 6 "roots" α. Notice that when $[2]_\alpha$ = integer, then $[4]_\alpha$ and $[6]_\alpha$ are integers too, as we already mentioned.

We note that if we enumerate the discrete values of α in our Tables by means of an index i (writing α_i), we have

$$[2]_{\alpha_i} = N_{\alpha_i}(1/2) = \sqrt{N_{\alpha_i}(1) + 1} = \sqrt{i+3}, \tag{21}$$

as

$$[3]_{\alpha_i} = N_{\alpha_i}(1) = i + 2. \tag{22}$$

If one wishes to extend Table III for higher values of j and α_i, the following relations will also be useful:

$$[4]_{\alpha_i} = N_{\alpha_i}(3/2) = (i+1)\sqrt{i+3}, \tag{23}$$

$$[5]_{\alpha_i} = N_{\alpha_i}(2) = (i^2 + 3i + 1), \tag{24}$$

$$[6]_{\alpha_i} = N_{\alpha_i}(5/2) = (i+2)i\sqrt{i+3}. \tag{25}$$

From these relations one sees that every time the number $(i+3)$ is a perfect square, then $[2]_\alpha, [4]_\alpha, [6]_\alpha, \ldots$ = integer numbers. Furthermore, the number Y_α, defined in (20), can be written as

$$Y_{\alpha_i} = \sqrt{i+3}\left(\sqrt{i+3} + 2\right), \tag{26}$$

and then, in the case $\sqrt{i+3}$ = integer, Y_{α_i} is an integer too, which explains the results for Y_α in Table II. Besides, we should mention that for the $\alpha_i = 2\alpha_j$ one has $i = (j+1)^2 - 3$, with the index $j = 1, 2, 3, \ldots$ (as can be checked from the values in the first column of Table II) and then one finds, for instance

$$Y_{\alpha_i = 2\alpha_j} = (j+1)(j+3), \tag{27}$$

with $j = 1, 2, 3, \ldots$.

TABLE I. List of solutions of condition (3) for real α with N until 25. These solutions were obtained for integer j, from the case of $j = 1$. For half-integer j, they satisfy $N_\alpha = (\text{integer})^{1/2}$.

$N(1)$	$x = \frac{1}{2}(N(1) - 1)$	α_i	$N_\alpha(\frac{1}{2}) = \sqrt{N(1)+1}$
3	1	$\alpha_1 = 0$	$\sqrt{4}$
4	3/2	$\alpha_2 = 0.9624236$	$\sqrt{5}$
5	2	$\alpha_3 = 1.3169579$	$\sqrt{6}$
6	5/2	$\alpha_4 = 1.5667992$	$\sqrt{7}$
7	3	$\alpha_5 = 1.7627472$	$\sqrt{8}$
8	7/2	$\alpha_6 = 1.9248473$	$\sqrt{9}$
9	4	$\alpha_7 = 2.0634371$	$\sqrt{10}$
10	9/2	$\alpha_8 = 2.1846438$	$\sqrt{11}$
11	5	$\alpha_9 = 2.2924317$	$\sqrt{12}$
12	11/2	$\alpha_{10} = 2.3895264$	$\sqrt{13}$
13	6	$\alpha_{11} = 2.4778887$	$\sqrt{14}$
14	13/2	$\alpha_{12} = 2.5589790$	$\sqrt{15}$
15	7	$\alpha_{13} = 2.6339158$	$\sqrt{16}$
16	15/2	$\alpha_{14} = 2.7035758$	$\sqrt{17}$
17	8	$\alpha_{15} = 2.7686593$	$\sqrt{18}$
18	17/2	$\alpha_{16} = 2.8297350$	$\sqrt{19}$
19	9	$\alpha_{17} = 2.8872709$	$\sqrt{20}$
20	19/2	$\alpha_{18} = 2.9416573$	$\sqrt{21}$
21	10	$\alpha_{19} = 2.9932228$	$\sqrt{22}$
22	21/2	$\alpha_{20} = 3.0422471$	$\sqrt{23}$
23	11	$\alpha_{21} = 3.0889699$	$\sqrt{24}$
24	23/2	$\alpha_{22} = 3.1335985$	$\sqrt{25}$
25	12	$\alpha_{23} = 3.1763131$	$\sqrt{26}$

TABLE II. List of solutions of Eq. (3) for half-integer j. They satisfy $\alpha = 2\alpha'$ and are also solutions for integer j. Numerical results for the first 10 α's of the form $\alpha_i = 2\alpha_j$ are displayed.

$\alpha_i = 2\alpha_j$	$[3]_\alpha = N_\alpha(1)$	$[2]_\alpha = N_\alpha(1/2)$	Y_α
$\alpha_1 = 2\alpha_1 = 0$	3	$\sqrt{4} = 2$	8
$\alpha_6 = 2\alpha_2 = 1.9248473$	8	$\sqrt{9} = 3$	15
$\alpha_{13} = 2\alpha_3 = 2.6339158$	15	$\sqrt{16} = 4$	24
$\alpha_{22} = 2\alpha_4 = 3.1335985$	24	$\sqrt{25} = 5$	35
$\alpha_{33} = 2\alpha_5 = 3.5254943$	35	$\sqrt{36} = 6$	48
$\alpha_{46} = 2\alpha_6 = 3.8496946$	48	$\sqrt{49} = 7$	63
$\alpha_{61} = 2\alpha_7 = 4.1268741$	63	$\sqrt{64} = 8$	80
$\alpha_{78} = 2\alpha_8 = 4.3692876$	80	$\sqrt{81} = 9$	99
$\alpha_{97} = 2\alpha_9 = 4.5848633$	99	$\sqrt{100} = 10$	120
$\alpha_{118} = 2\alpha_{10} = 4.7790528$	120	$\sqrt{121} = 11$	143

3. $N_\alpha(j)$ as a trace

After all these numerical remarks we wish to discuss the meaning of $N_\alpha(j)$ from another point of view. As described by Biedenharn [2], the q-dimension is defined by $\sum_{rep} q^{J_z}$, and for *irreps* labelled by j it is given by $[2j+1]_q$. Thus, according to (3), $N_\alpha(j)$ was identified with the q-dimension.

On the other side, we can take $j = 1$ and one generator, say J_x, of the SU(2) group, which is represented by the 3×3 matrix

$$J_x = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}. \tag{28}$$

TABLE III. Values of $[2j+1]_\alpha$ for the first 6 "roots" α_i.

α_i	$[2]_\alpha = N_\alpha(1/2)$	$[3]_\alpha = N_\alpha(1)$	$[4]_\alpha = N_\alpha(3/2)$	$[5]_\alpha = N_\alpha(2)$	$[6]_\alpha = N_\alpha(5/2)$
$\alpha_1 = 0$	$\sqrt{4}$	3	$2\sqrt{4}$	5	$3\sqrt{4}$
$\alpha_2 = 0.9624236$	$\sqrt{5}$	4	$3\sqrt{5}$	11	$8\sqrt{5}$
$\alpha_3 = 1.3169579$	$\sqrt{6}$	5	$4\sqrt{6}$	19	$15\sqrt{6}$
$\alpha_4 = 1.5667992$	$\sqrt{7}$	6	$5\sqrt{7}$	29	$24\sqrt{7}$
$\alpha_5 = 1.7627472$	$\sqrt{8}$	7	$6\sqrt{8}$	41	$35\sqrt{8}$
$\alpha_6 = 1.9248473$	$\sqrt{9}$	8	$7\sqrt{9}$	55	$48\sqrt{9}$

We wish to exponentiate $e^{\alpha J_x}$. To this end, we diagonalize J_x and apply the Cayley-Hamilton theorem [4]. One obtains

$$e^{\alpha J_x} = I + (\sinh\alpha)J_x + (\cosh\alpha - 1)J_x^2. \tag{29}$$

As Tr $J_x = 0$, Tr $J_x^2 = 2$, we obtain the trace

$$\mathrm{Tr}\, e^{\alpha J_x} = 1 + 2\cosh\alpha, \tag{30}$$

a result which coincides with $N_\alpha(1)$. A similar result is valid for $j = 1/2$. Of course, the same holds for the other two generators J_y and J_z, as their traces are the same. Then, from a similarity transformation one can rewrite $\sum_{rep} e^{\alpha J_z}$ as Tr $e^{\alpha J}$, which is equal to $N_\alpha(j)$, as we have seen for the case $j = 1$. The same has to be valid for the other generators and the result, of course, can be generalized. Thus, we write

$$\mathrm{Tr}\, e^{\alpha J_i} = N_\alpha(j). \tag{31}$$

The main point is that here $N_\alpha(j)$ is a trace (not a dimension) and is expressed in function of the generators of the usual SU(2) through analogous relations of those of Eqs. (29) and (30).

The above remark has obvious consequences. For example, if we multiply two $N_\alpha(j)$'s, corresponding to values j_1 and j_2, we reobtain, for this specific case, the well known Clebsch-Gordan summation for the *direct product*

$$N_\alpha(j_1)N_\alpha(j_2) = \sum_{j=|j_1-j_2|}^{j_1+j_2} N_\alpha(j), \tag{32}$$

a result that can be easily verified by using Eqs. (9)–(14). For instance, one has

$$N_\alpha(1/2)N_\alpha(1/2) = N_\alpha(0) + N_\alpha(1), \tag{33}$$

$$N_\alpha(1/2)N_\alpha(1) = N_\alpha(1/2) + N_\alpha(3/2), \tag{34}$$

$$N_\alpha(1)N_\alpha(1) = N_\alpha(0) + N_\alpha(1) + N_\alpha(2), \tag{35}$$

etc. ...

Consider, for example, the case of $\alpha = \alpha_6$. The values of $N_\alpha(j)$ are given in Table III. For α_6 we have $N(0) = 1$, $N(1/2) = 3$, $N(1) = 8$, $N(3/2) = 21$, $N(2) = 55$, as displayed in the last row. Then, in Eqs. (33)–(35), one gets

$$N_{\alpha 6}(1/2)N_{\alpha 6}(1/2) = 3 \times 3 = 9 = 1 + 8, \tag{36}$$

$$N_{\alpha 6}(1/2)N_{\alpha 6}(1) = 3 \times 8 = 24 = 3 + 21, \tag{37}$$

$$N_{\alpha 6}(1)N_{\alpha 6}(1) = 8 \times 8 = 64 = 1 + 8 + 55, \tag{38}$$

and so on.

On the other hand, by taking $j_1 = j_2 = j$, one obtains

$$N_\alpha(j)^2 = [N_\alpha(0) + N_\alpha(1) + \ldots + N_\alpha(2j)]. \tag{39}$$

And Eq. (39) can be rewritten, by means of Eq. (3), as

$$[2j+1]_\alpha^2 = (1 + [3]_\alpha + \ldots + [4j+1]_\alpha), \tag{40}$$

which give rise to a number of interesting relations. We call attention to the fact that most of the q-number identities found in Ref. 2 can be verified from the above equations, where the discretization plays its role for integer numbers N_α.

4. Concluding remarks

We would like to address now to some concluding considerations. Throughout this note we have shown several useful relations for q-numbers which are consequence of the condition that $[2j+1]_\alpha$ = integer. This condition led us to a discretization of the set of values of α. Very practical relations, like Eqs. (21)–(25) for example, allow a quick calculation of $[n]_\alpha$ for any α_i of the enumerated list of α's. We have applied the present scheme in a calculation of fundamental fermion masses using the algebra of the $SU_q(2)$ group as a spectrum generating deformed algebra [3]. The simplifications occuring by the use of the discretization were gratifying. We remark, however, that the values of α for the problem treated in Ref. 3 were typically rather larges, of the order $\alpha = 2.6$–4.8. Notice that for values of α_i in Table I—obtained from discretization—we have $0.96 < \alpha_i < 3.18$. We recall that for instance, for the problem of nuclear rotational bands [5] treated with deformation, values of the order $|\alpha| = 0.030$ were obtained. There is a difference of some orders of magnitude for the values of α in the two problems. We explain this apparent discrepancy by the very different context of each treatment. While here, and in the work of Ref. 3, which is directly based on the $su_q(2)$ algebra, the effect of deformation is large, in other models in hadronic, nuclear and molecular physics the deformed algebras are meant to describe small deviations from an almost exact symmetry, implying much smaller values of the deformation parameter, not necessarily belonging to our "discretized" set.

We point out that for $\alpha = i|\alpha|$, that is, for q a pure phase factor ($q = e^{i|\alpha|}$), the same discretization process could be carried out leading, however, to a different regime, characterized by $\cosh\alpha \to \cosh i|\alpha| = \cos|\alpha|$. Finally, we wish to emphasize that the present prescription of integer values of the q-dimension, in general applications is not a *necessary* one. Nevertheless, due to its algebraic simplicity, it is an useful procedure especially in dealing with problems involving large values of α. This, of course, means that we do not exclude the case of *ordinary* q-dimension in other physical applications. We further remark that one of the useful consequences of the discretization is the possibility of obtaining relations like Eqs. (32)–(40), with $N_\alpha(j)$ integer, in a simple way.

Acknowledgments

We are grateful to Prof. J.A. Castilho Alcarás for helpful discussions and for a careful reading of the manuscript. Thanks are also due to L. Tomio for discussions and for help on closing the manuscript. We are grateful to FAPESP, São Paulo, for peripherical support. One of us, P.L.F., is grateful to the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), Brazil, for a research grant.

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