A derivation of Anderson's equation for the phase slip in superfluids from Kelvin's theorem

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An experimental arrangement has been considered in which a superfluid is forced to pass through a hole where the net force that compels it to flow is the force of the field of gravity. From Kelvin's theorem, it is possible to derive Anderson's equation for the phase slip, where, it regards the phase of the wave function of the state superfluid as a dynamic variable that may be macroscopically identified.

Keywords: Superfluids, Kelvin's theorem, phase slip

Ha sido considerado un arreglo experimental en el cual un superfluido es forzado a pasar a través de un agujero donde la fuerza neta que lo obliga a fluir es la fuerza del campo de gravedad. A partir del Teorema de Kelvin es posible deducir la ecuación de Anderson para el deslizamiento de la fase, donde la fase de la función de onda del estado superfluido es evaluada como una variable dinámica que puede ser macroscópicamente identificada.

Descriptores: Superfluidos, Teorema de Kelvin, deslizamiento de fase

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1. Introduction

One of the most fascinating properties of He-II is superfluidity and not less surprising are the theories that try to explain this, which is a consequence of the macroscopic manifestation of quantum mechanical effects. Concretely, for the case of the operator associated with the particle density, according to Landau [1], its mathematical structure formally coincides with the density of mass in the point of interest. If ρ represents this latter density and $\hat{\rho}$ is the reduced density matrix associated with the particle density, then

$$\rho = m\delta(r - r') = \hat{\rho}(r, r') = \langle \Psi^*(r)\Psi(r')\rangle = m|\Psi(r)|^2, \quad (1)$$

where Ψ is the wave function associated with the state of the system, m is the mass of He^4 and $\langle\rangle$ an average taken over an appropriate ensemble of the system. Strictly speaking this ensemble may be either an equilibrium or a quasi-equilibrium one [2].

From this observation, the commutation properties of the operators associated with the dynamic variables, and assuming that the flow of superfluid is irrotational, Landau derives his quantum hydrodynamic model for the two fluids [1, 3].

In relation to the one particle reduced density matrix $\hat{\rho}$ obtained from the density matrix for a system of N particles, Penrose [4] and Penrose and Onsager [5] postulated that the superfluidity is a state in which the reduced density matrix $\hat{\rho}$ may be factorized as

$$\hat{\rho}(r, r') = \langle \Psi^*(r)\rangle\langle\Psi(r')\rangle + O(\frac{1}{|r - r'|}), \quad (2)$$

and that, furthermore, the property of off-diagonal long-range order for the density matrix (ODLRO) is fulfilled [6]. Therefore,

$$\lim_{|r-r'|\to\infty} \hat{\rho}(r, r') = e^{O(1)} = \text{cte.} \neq 0. \quad (3)$$

In other words, the superfluid is a state which exhibits the Bose-Einstein condensation, and therefore a finite macroscopic fraction of He^4 atoms lies in the ground state.

Anderson [5] re-interpreted Eqs. (2) and (3), by postulating that the average of the field operator $\langle\Psi\rangle$ associated to the wave functions $\{\Psi\}$ that characterize the superfluid state, may be interpreted as a macroscopic dynamic variable, such that:

$$\langle\Psi(r, t)\rangle = f(r, t)e^{i\phi(r,t)}. \quad (4)$$

In words, the average associated with this dynamic variable is considered as being composed of two dynamical variables, namely, an amplitude f(r,t) and the phase ϕ. Moreover, this phase, is interpreted as a dynamical variable conjugated to the number of particles N [7, 8].

From Eqs. (2), (3) and (4), Anderson proceeded in analogy with Josephson's effect in superconductivity, and carrying out his calculations within the frame of the quantum mechanics [7, 8], he derived an equation to evaluate the phase slip, that explains the tunneling effect between two superfluid systems connected by a hole in a membrane. His result reads

$$\frac{\hbar}{m}\left\langle \frac{d}{dt}\nabla\phi \right\rangle = \langle\nabla\mu\rangle, \quad (5)$$

where $\langle d\nabla\phi/dt\rangle$ is the average of the change in time of the gradient of the wave function's phase and $\langle\nabla\mu\rangle$ is the average of the gradient of the chemical potential. Equation (5) has been experimentally verified in the late eighties [9–11]. Notice should be made that in the derivation of this equation the contributions of the fluctuations in energy have been negleted, a problem which we shall not consider here [12, 13].

On the other hand, it is well known that Onsager and Feynman [14] independently postulated, the quantization of the superfluid circulation, namely,

$$\oint V_s \cdot dl = 2\pi n, \tag{6}$$

where the superfluid velocity $\mathbf{V}_s$ is defined as

$$V_s = \frac{\hbar}{m}\nabla\phi, \tag{7}$$

and ϕ is the phase of the wave function that characterizes the superfluid state. It is also well known that this result has been experimentally verified [15].

It is important to remark that Eqs. (6) and (7) are consistent with the irrotationality of the superfluid, on considering that the superfluid is irrotational in multiply connected domains. This implies that the singularities of these domains are the regions where the nuclei of the vortices reside. Furthermore, in Eq. (7) it is clearly seen that the wave function's phase associated with the superfluid state is being taken as a macroscopic dynamic variable.

2. A derivation of Anderson's equation

However, to the author's knowledge, the relationship of classical hydrodynamics and the standard properties exhibiting the quantum mechanical behavior of He-II, [3, 15–19] has been little investigated. If one regards that the superfluid helium behaves like an ideal fluid and macroscopically exhibits quantum mechanical effects such as the Bose-Einstein condensation, one could think that some of these macroscopic manifestations should be derived starting from some well-known theorems of classical hydrodynamic adequately interpret.

Concretely the objetive of this work is to show how, starting from Kelvin's theorem for the conservation of the circulation , it is possible to derive an equation of Anderson's type for the phase slip. For that purpose one starts considering that the superfluid phase behaves as a macroscopic dynamic variable, and introduces into the description the constraints imposed by the experimental arrangement (see Fig. 1 of Ref. 18).

Kelvin's theorem stating the conservation of the circulation establishes [3] that for an ideal fluid (incompressible and inviscid),

$$\frac{d}{dt}\oint V_s \cdot dl = 0, \tag{8}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + V_s \cdot \nabla.$$

Exchanging the integral and the total time derivative [3, 20], and using the Stokes' theorem, we have

$$\oint \frac{d}{dt}V_s \cdot dl = \int_S \nabla \times \frac{d}{dt}V_s \cdot dS. \tag{9}$$

A necessary condition for the surface integral to be zero is that dV_s/dt be irrotational, therefore

$$\frac{d}{dt}V_s \propto \nabla\mu + F, \tag{10}$$

with $F = -\nabla\varphi$. Here φ as we shall see below, is the potential energy of the gravitational field.

The sufficient conditions that justify Eq. 10 are obtained from the experimental arrangement—see Fig. 1 of Ref. 18. Indeed the superfluid is being forced to flow through a hole due to the force of gravity, and there is a change in the number of particles and the energy density, in both sections of the hole.

The change in energy is fixed through the energy necessary to build up or destroy at least one vortex and it is given by [20]

$$\Delta E = \rho_s \kappa \pi R^2, \tag{11}$$

where κ is the circulation, and R is the radius of the vortex. Strictly, in accordance with some models [12, 13], a potential barrier exists which has to be overcome in order to form such a vortex, but for our purposes it is not necessary to consider this refinement.

Specifically, the thermodynamic variable use to measure the change in energy due to a change in the number of particles is the chemical potential μ , and in fact, a gradient of this variable exists through the hole. Thus we have that

$$F = -\nabla\varphi = mg\Delta z = m\frac{d}{dt}V_s + \nabla\mu. \tag{12}$$

In other words, the net force acting on the superfluid that forces it to flow across the hole, the force of gravity, is equal to the change in time of the wave function's phase plus the force due to the change of the number of particles given by a gradient of the chemical potential.

Comparing Eqs. (10) and (12), we have that

$$\frac{\hbar}{m}\frac{d}{dt}\nabla\phi = -\nabla\mu - \nabla\varphi, \tag{13}$$

which is an equation of the Anderson type. Strictly speaking, it is equivalent to a generalization of Euler's equation for an ideal fluid, represented in this case by a superfluid [3, 16].

However the question arises of how to interpret this equation from the standpoint of Kelvin's theorem. The answer is that the requirement of conservation of circulation across the hole due to the flow of the superfluid implies that vortices destruyed on one side are created an the opposite one and viceversa. This conservation implies a change in the phase of the wave function, which is equal to a breaking of the gauge symmetry of the ensemble with respect to which averages are being taken to describe the states of the system. This mechanism is equivalent to a change in the energy of the system in both sides of the hole per unit change in the number of particles, but the total energy is conserved in the whole system. An interesting work in relation with this point is given in Ref. 22, where the numerical solutions of the non-linear Schrödinger equation for the motion in a Bose condensate are studied (Gross- Ginzburg-Pitaevskii equation). There it is shown how such solutions are consistent with Kelvin's theorem.

Evidently, Eq. (12), describes only the average behavior of the superfluid flowing across the hole ignoring the detailed mechanism responsible for the destruction and creation of the vortices around and in the interior of the hole. This question has been dealt in other works [12, 13, 23], using stochastic models by considering the interaction of the system with the walls of the hole in the presence of a potential barrier which has to be overcome in order to form the vortices.

The difference between the derivation of Eq. (13) presented here and other derivations is that we have directly used Euler's equation for a perfect fluid, and explicitly Kelvin's theorem, starting from the validity of Eqs. (6) and (7), as well as the irrotational property of the flow, $\nabla \times dV_s/dt$, which has been physically interpreted from the experimental setup given in the literarure.

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