Local slip planes in accelerated granular media

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We study analytically the local slip planes in non cohesive granular media within a box subjected to uniform rectilinear acceleration. By using the phenomenological Coulomb's method of wedges we found that the angles of the slip planes and the resulting average forces at the wall are strongly changed as a function of the acceleration's magnitude of the system, even if it is so small such that the shape of the free surface is maintained unchanged.

Keywords: Granular media, slip planes

Estudiamos analíticamente los planos locales de deslizamiento en medios granulados no cohesivos dentro de una caja sujeta a aceleración rectilínea uniforme. Usando el método fenomenológico de cuñas de Coulomb encontramos que los ángulos de los planos de deslizamiento y las fuerzas promedio resultantes en la pared varían en función de la magnitud de la aceleración del sistema, aun si ésta es tan pequeña que la forma de la superficie libre no cambia.

Descriptores: Medios granulados, planos de deslizamiento

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1. Introduction

Granular media commonly obey a very complex and unique behavior which only recently has been explored over a wide range of geometrical and dynamical situations [1]. One example concerned with this behavior is related to the stress propagation within the material in the so called static (or quasistatic) regime [2–6]. In fact, on a very small spatial scale, complex stress chains have been observed in confined and unconfined samples of granular material, which allow the transmission of the weight of the grains along selected paths. These chains produce fractal spatial patterns obeying very strong force fluctuations [7]. On the other hand, (at larger scales) in contrast to a liquid, the pressure within a tall enough box filled with dry, non cohesive granular material, does not increase linearly with depth, but saturates at a certain value [8]. In both extreme cases, the theoretical descriptions indeed show that the role of friction and material's inhomogeneity seems to be very important.

When dry granular material is confined in shallow containers, the evaluation of the average force at a wall is a very important problem, which can be solved using the phenomenological Coulomb's method of wedges [2], *i.e.*, by analyzing only a part of the material (wedge) formed by the slip planes, the free surface and the retaining wall (see Fig. 1). The concept of rigid-plastic failure assumes material to be divided itself into two rigid blocks separated by a narrow plastic zone. The size of the plastic zone is more or less ten grain diameters which is often very narrow compared with the typical dimensions of the system. It is, therefore, usually enough to assume that the plastic zone is a plane of negligible width and, being referred to as the yield, or slip, or failure plane.

In order to evaluate the local average force at the retaining wall of a box and the possible failure planes within the material, it was noted by Rankine [9] that two limiting cases can be distinguished: In the first case the wall is pushed under the effect of the granular material self weight. The granular medium is then in the so called active state (Fig. 1b). In the second case the wall can be pushed from outside, compressing the medium. In this latter case, grains exert a passive resistance (Fig. 1c). Although this method is not exact [2], it gives a good qualitative approach to evaluate the forces on the walls. It also permits to obtain the geometry of the possible failures (slip planes) and to understand the dependence of the elastic limit on the material and geometrical parameters. The knowledge of the average forces at the walls of containers during the transportation of grains, may be very useful, for example, for structural designers.

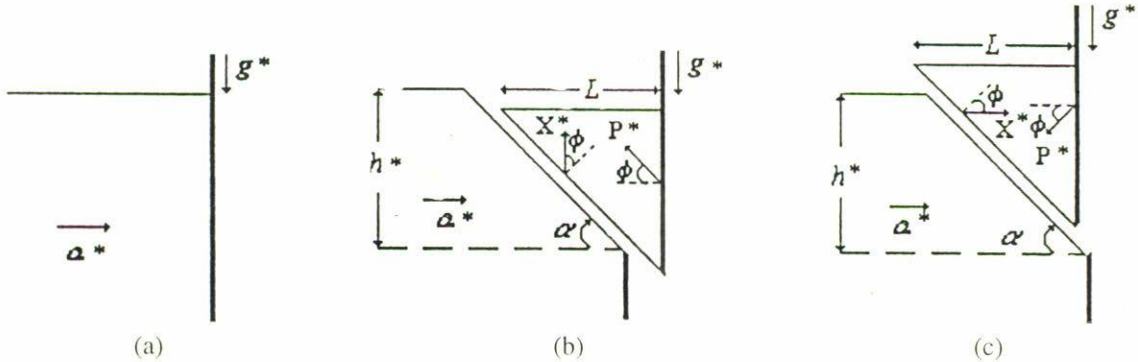

FIGURE 1. Local failures of a retaining wall at the right edge of a box subjected to uniform rectilinear acceleration, a^*. In (a) we show case before the wall has broken. In (b) we show the active state after the wall has broken and in (c) we show the passive state after the wall has broken. The forces X^*, P^* and $\rho^* a^*$ and the angles ϕ and α are defined in the text.

The objective of this paper is to develop an extended version of the Coulomb's method of wedges which takes into account the effect of the acceleration on the slip planes (dynamical aspects). Here, we will assume dry granular systems undergoing small accelerations (relative to the gravity acceleration), in a steady-state regime, such that the free surface can not be deformed due to the motion [10]. This extension is indeed necessary in order to evaluate how the average stresses are influenced by the acceleration process. This work is divided into three parts. In Subsect. 2.1 we show an analysis of the balance of forces, for the active state and for small acceleration compared with gravity. In Subsect. 2.2 we discuss the passive state, also under small accelerations, presenting an analysis of the differences of the average forces and the angles of slip between the active and passive states in slightly accelerated systems. Finally in Sect. 3, we discuss the limitations of the model, presenting also the main conclusions of this work.

2. Local slip planes in a box

In many cases, such as in soil mechanics [3, 4], the *natural* state for the granular medium is purely static, where the complete system can be assumed quiet or under constant velocity. However, in order to understand the dynamic problem indicated in Fig. 1, we show the two possible idealized situations where a fully rough vertical wall (so that $\phi_w = \phi$, where ϕ_w is the angle of wall friction and ϕ is the angle of internal friction) retains granular material with a horizontal free surface subjected to a rectilinear acceleration, a^*. The right-hand side wall supports a local load produced by the material under rigid body motion which can be calculated, as we show later, using the Coulomb's method of wedges. As in the purely static case, the material in the accelerated case will be also assumed to slip in wedges with certain angle of slip or failure. In the following sections we show how this method can be modified in order to take into account the acceleration of the system for the active and passive states. As a first approximation, we do not take into account the dilatancy phenomenon, which induces a change in the volume and therefore a change in the bulk density of the material [3, 4].

2.1. Active state with small acceleration

We follow closely the local phenomenological analysis of Nedderman for the static problem of the retaining wall [2], in order to study the active state due to small acceleration with no free surface deformation. When a box, filled up to a height H^* with dry cohesionless granular material, is moved along the horizontal direction with an uniform acceleration, a^*, the initially flat free surface is not deformed if the acceleration is less than $g^*\mu$, where $\mu = \tan\phi$ is the friction coefficient and g^* is the gravity acceleration [10]. In the active state (Fig. 1b) there are four forces X^*, P^*, W^* and $\rho^* a^*$ acting on the wedge of height h^* and length L^*, bulk density ρ^*, and unknown angle of tilt α. The force X^* is the reaction force of the major block due to the force applied by the wedge, P^* is the reaction force of the wall due to the wedge which forms an angle ϕ respect to the horizontal, W^* is the weight of the wedge which is parallel to the gravity acceleration, and finally $\rho^* a^*$ is the inertial force (per unit volume). Note that the motion of the container is from left to right and therefore the inertial force is acting in the opposite direction. Here, explicitly it was supposed steady-state motion and also the existence of (average) slip planes. The force balance equations, assuming a width b^* of the wedge, are given by

$$P^* \cos\phi + \frac{\rho^* a^* h^{*2} b^*}{2} \cot\alpha = X^* \sin(\alpha - \phi), \qquad (1)$$

$$P^* \sin\phi + X^* \cos(\alpha - \phi) = \frac{\rho^* g^* h^{*2} b^*}{2} \cot\alpha, \qquad (2)$$

for the horizontal and vertical directions, respectively.

Introducing the nondimensional variables $a \to a^*/g^*$, $P \to P^*/\rho^* g^* b^{*3}$, $X \to X^*/\rho^* g^* b^{*3}$, and $h \to h^*/b^*$, the above equations takes the nondimensional form

$$P \cos\phi = \frac{h^2}{2} f(\alpha), \qquad (3)$$

where the nondimensional function $f(\alpha)$ has the form

$$f(\alpha) = \frac{\cot\alpha}{\tan\phi + \cot(\alpha-\phi)} - \frac{a\cot\alpha\cot(\alpha-\phi)}{\tan\phi + \cot(\alpha-\phi)}. \quad (4)$$

This function determines the value of the angle α where the force $P\cos\phi$ has a maximum value and where shear or slip must occur. The physical significance of the maximum of $f(\alpha)$ will be examined in this Subsection.

In order to calculate the maximum value of the dimensionless function, $f(\alpha)$, we introduce in Eq. (4) the variables $t = \tan\alpha$ and $\mu = \tan\phi$. Rearranging terms Eq. (4) transforms to

$$f(t) = \frac{t(1-a\mu) - (a+\mu)}{t(1-\mu^2) + 2t^2\mu}. \quad (5)$$

Therefore, the maximum value of $f(t) = f_{\max}(t^*)$ occurs for a value of $t = t^*$ given by

$$t^* = \frac{a+\mu}{1-a\mu} + \frac{\sqrt{a\mu^5 + \mu^4(1+a^2) + 2a\mu^3 + \mu^2(a^2+1) + a\mu}}{\sqrt{2}\mu(1-a\mu)}, \quad (6)$$

and

$$f_{\max}(t^*) = \frac{(1+a\mu)^2}{(a\mu^3 + 3\mu^2 + 3a\mu + 1) + 2\sqrt{2}\sqrt{a\mu^5 + (1+a^2)\mu^4 + 2a\mu^3 + (a^2+1)\mu^2 + a\mu}}. \quad (7)$$

The explicit dependence of $f_{\max}$ on ϕ, the angle of internal friction, is given then by

$$f_{\max}(\phi) = \frac{(\cos\phi + a\sin\phi)^2}{(a\sin^3\phi\sec\phi + 2\sin^2\phi + 3a\sin\phi\cos\phi + 1) + 2\sqrt{2}\sqrt{a\sin\phi(1+a\sin\phi) + \sin^2\phi}}. \quad (8)$$

The limiting case of $a = 0$ can be obtained easily from Eqs. (7) and (8), producing the well known result [2]

$$f_{\max} = \frac{1}{\left(\sqrt{1+\mu^2} + \sqrt{2}\mu\right)^2}, \\ = \frac{\cos^2\phi}{\left(1+\sqrt{2}\sin\phi\right)^2}. \quad (9)$$

Finally, the normal stress, σ, and the corresponding shear stress, τ, are given by

$$\sigma = \frac{dP_{\max}}{dh}\cos\phi = hf_{\max}, \\ \tau = \sigma\tan\phi = h\mu f_{\max}. \quad (10)$$

These results can be applied when the aspect ratio, deep/length, of the container is so small such that deep/length$<\tan\alpha$, because in deep containers an exponential saturation for the pressure occurs which causes the normal and shear stresses not to be dependent on the dimensionless height, h [8, 11]. The condition $h^*/d^* \gg 1$ must be maintained, making the finite size effects to be negligible. Here d^* is the typical grain's diameter.

In order to show graphically the influence of the acceleration on the averaged stresses and on the local angles of slip planes, we have plotted $f(\alpha)$ in Fig. 2, for several values of the dimensionless acceleration. Here, the active states correspond to the convex curves (concave curves corresponding to the passive state will be treated in next Subsection) and the purely static case ($a = 0$) is given by the upper curve of this family of curves. We note that the physically significative values correspond to the pair (α^*, $f_{\max}$) for each curve. Therefore, in the active state, we have that a smaller acceleration corresponds to larger stresses, *i.e.*, the normal stress necessary to produce an active failure when the system is accelerated is smaller than that in the case purely static. In

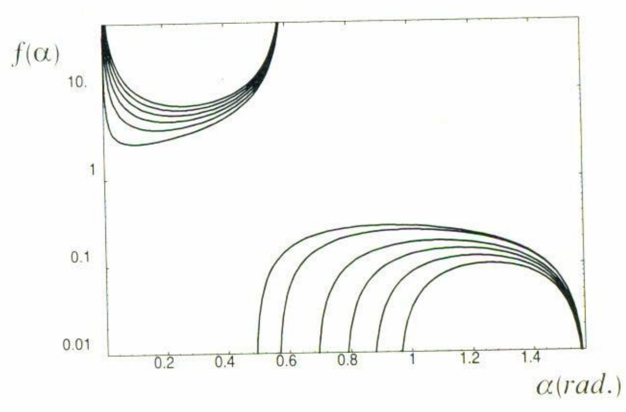

FIGURE 2. Semilog plot of the function f as a function of the angle α for six values of the dimensionless acceleration, a, where $a < \mu = 0.53$, these six values are $a = 0$, 0.1, 0.2, 0.3, 0.4 and 0.5. Both active (convex curves) and passive (concave curves) states were plotted. Upper curve, of the concave curves, corresponds to the passive state with $a = 0$. Also, upper curve, of the convex curves, corresponds to the active state in the limit $a = 0$.

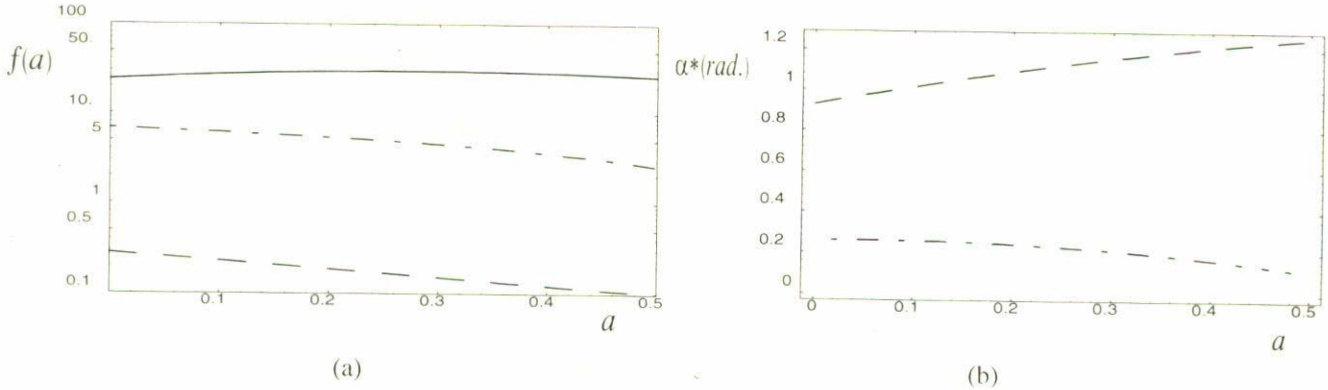

FIGURE 3. (a) Semilog plot of the function, $f_{\max(\min)}$, as a function of the dimensionless acceleration. Curves (— —) corresponds to the active state, curves (— - —) corresponds to the passive state. Continuous line is the ratio $f_{\min}/f_{\max}$. In (b) we show, with similar curves as in (a), the behavior of the angle of fracture, α^*, also as a function of the dimensionless acceleration. The best fit was plotted for each case.

Fig. 3a this behavior is clearly shown. Finally, in Fig. 3b we show the behavior of α^* also as function of a. This plot shows that in the active state, the slip planes have larger inclination respect to the horizontal than that for the static case. All cases here plotted were obtained assuming a material like Ottawa sand which is a material commonly used in experimental tests. For this material $\mu = 0.53$, $\phi = 28^\circ$ and consequently, in all graphics $a < 0.53$.

The significance of the maximum can be visualized by plotting the function $f(\alpha)$ for a = constant (which corresponds to only one curve). In this case if the lateral force at the wall, f_i, is slowly reduced such that $f_i > f_{\max}$, then the first rupture plane that can become active is the one at $f = f_{\max}$, inclined to the horizontal at the angle α^*. The lateral force cannot fall below this value. If the lateral force could be reduced to a value $f_j < f_{\max}$, there would be two possible rupture planes, but any attempt to reduce the total force will simply be matched by activity of the weakest slip wedge at $\alpha = \alpha^*$.

2.2. Passive state with small acceleration

We have already mentioned that the passive case (Fig. 1c) corresponds to the motion of the material near the wall, such that the wall can be pushed from outside compressing the medium. In this case, in accordance with Fig. 1c, we only change the sign of ϕ by $-\phi$ giving

$$f(\alpha) = \left[\frac{\cot\alpha}{\cot(\alpha+\phi) - \tan\phi} - \frac{a\cot\alpha\cot(\alpha+\phi)}{\cot(\alpha+\phi) - \tan\phi}\right]. \tag{11}$$

Similarly, as in the previous Section, if we introduce $t = \tan\alpha$ and $\mu = \tan\phi$. Eq. (11) transforms to

$$f(t) = \frac{t(1+a\mu) - (a-\mu)}{t\,(1-\mu^2) - 2t^2\mu}. \tag{12}$$

This function reaches a minimum value at $t = t^*$ given by

$$t^* = \frac{a-\mu}{1+a\mu} + \frac{\sqrt{-a\mu^5 + \mu^4(1+a^2) - 2a\mu^3 + \mu^2(a^2+1) - a\mu}}{\sqrt{2}\mu(1+a\mu)}, \tag{13}$$

which gives implicitly the value of the angle of slip, α^*, as a function of (a, μ).

Therefore, $f_{\min}(t^*)$ is then given by

$$f_{\min}(t^*) = \frac{(1-a\mu)^2}{(1 - 3a\mu + 3\mu^2 - a\mu^3) + 2\sqrt{2}\sqrt{(1+a^2)\,\mu^4 + (a^2+1)\mu^2 - a\mu - 2a\mu^3 - a\mu^5}}, \tag{14}$$

or

$$f_{\min}(\phi) = \frac{(\cos\phi - a\sin\phi)^2}{(1 + 2\sin^2\phi - 3a\sin\phi\cos\phi - a\sin^3\phi\sec\phi) + 2\sqrt{2}\sqrt{a\sin\phi\,(a\sin\phi - 1) + \sin^2\phi}}. \tag{15}$$

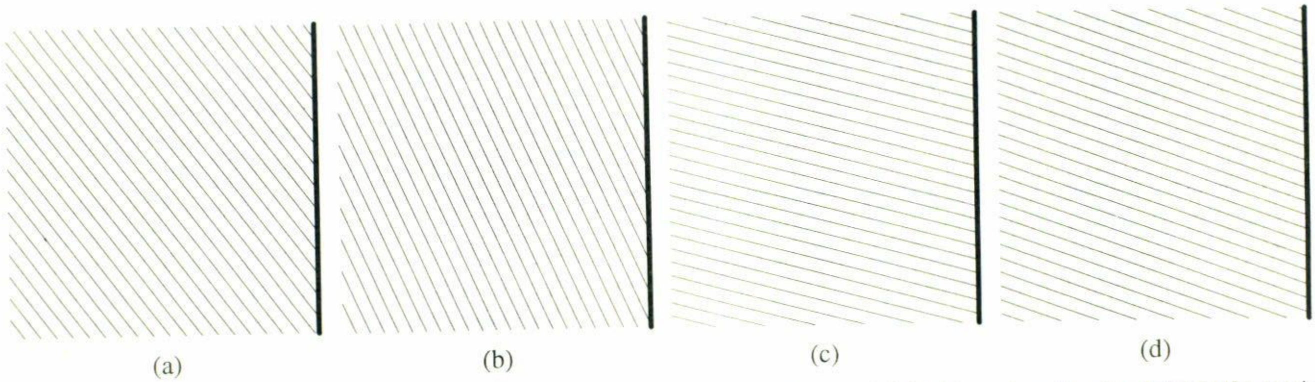

FIGURE 4. Local slip planes in the active state for Ottawa sand with (a) $a = 0$ and (b) $a = 0.3$. In (c) we show the slip planes in the passive state for $a = 0$ and in (d) for $a = 0.3$. Due h^* is arbitrary, in all cases we have a family of slip planes.

For $a = 0$ we easily obtain the well known values of the average slope t^* and the slip angle α^* as

$$t^* = \frac{\sqrt{\mu^2+1}}{\sqrt{2}} - \mu, \qquad \alpha^* = \arctan\left(\frac{\sec\phi}{\sqrt{2}} - \tan\phi\right). \tag{16}$$

and in this limit

$$f_{\min} = \frac{1}{\left(\sqrt{1+\mu^2} - \sqrt{2}\mu\right)^2} = \frac{\cos^2\phi}{(1-\sqrt{2}\sin\phi)^2}. \tag{17}$$

The normal and shear stresses are, respectively,

$$\sigma = \frac{dP_{\min}}{dh}\cos\phi = hf_{\min},\ \tau = \sigma\tan\phi = h\mu f_{\min}. \tag{18}$$

In Fig. 2 we also have plotted f as a function of α. Concave curves correspond to cases with same range of the dimensionless acceleration, however the physically important values correspond to the pair (α^*, $f_{\min}$). Here, similar arguments than that given above for the physical significance of the maximum are valid in order to understand the significance of the minimum. In Fig. 3a and 3b we plotted $f_{\min}$ and α^* as a function of a, respectively. In the passive state the stress decreases (as a nonlinear function of a) if the acceleration increases. Thus, the maximum stress in the passive state occurs when the system is in repose. The continuous line in Fig. 3a shows that for the passive state, always the function $f_{\min}$ overpass the value of $f_{\max}$ obtained for the active state. In Fig. 3b we show that α^*, conversely to the active state, decreases when increases the acceleration. So, the slip planes for systems under acceleration have a smaller slope than that for the static system.

For comparison, in Figs. 4a and 4b we show the local slip planes in the active state for $a = 0$ and $a = 0.3$, respectively. In Figs. 4c and 4d we show the local slip planes in the passive state for similar values, respectively. The influence of the acceleration on the slip planes is easily noted.

3. Remarks and conclusions

The rigid motion of non cohesive granular matter is a very common situation in many technological areas. We have shown that the acceleration induces a non linear change in the slip angle and in the average stresses at the walls, even if the shape of the free surface was maintained unchanged. This state may be reached if the nondimensional acceleration has values in the region $0 < a < \mu$. We also want to note that the region between the values $f_{\min}$ and $f_{\max}$ (Fig. 2), corresponding to the elastic zone of the material (because between both limiting values the material does not yield), clearly is strongly dependent on the acceleration of the system. Otherwise, if the nondimensional acceleration overpass the value of the friction coefficient, μ, the shape of the free surface changes [10], the study of the slip planes in this last case is more complex than that corresponding to small acceleration and it will be treated in a future work.

Finally, cohesive granular materials have, obviously, fractures; the theoretical treatment from a phenomenological point of view for these systems under several conditions of motion has been recently proposed [12]. However, recent studies [13] show that more work is necessary in order to understand the very complex patterns of fracture in these systems because more than only one slip or failure plane should actually occur.

Acknowledgments

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1. H.M. Jaeger and S.R. Nagel, *Science* **255** (1992) 513.
2. R.M. Nedderman, *Statics and Kinematics of Granular Materials*, (Cambridge University Press, London, 1992).
3. V.V. Sokolovskii, *Statics of Granular Media*, (Pergamon Press, London, 1965).
4. A. Schofield and P. Wroth, *Critical State Soil Mechanics*, (McGraw Hill, New York, 1968).
5. R.L. Brown and J.C. Richards, *Principles of Powder Mechanics*, (Pergamon Press, London, 1970).
6. K. Wieghardt, *Ann. Rev. Fluid Mech.* **7** (1975) 89.
7. S.F. Edwards and R.B.S. Oakeshott, *Physica D* **38** (1989) 88.
8. See for a recent review: T. Boutreux, E. Raphaël, and P.G. de Gennes, *Phys. Rev. E* **55** (1997) 5759.
9. W.J.W. Rankine, *Phil. Trans. Roy. Soc. London* **147** (1857) 9.
10. A. Medina and C. Treviño, *Rev. Mex. Fís.* **42** (1996) 193.
11. A. Ngadi, J. Rajchenbach, E. Clément, and J. Duran, Intermittences in the compression process of a model granular medium, in *Powders and Grains 97*, edited by R.P. Beheringer and J.T. Jenkins (A.A. Balkema, Rotterdam, 1997.)
12. See, for example: J.M. Valverde, A. Ramos, A. Castellanos, and P.K. Watson, "The shear fracture of a tilted layer of cohesive powder", and F.C. Genovese, P.K. Watson, A. Castellanos, and A. Ramos, "A disc centrifuge for the study of flow processes in cohesive granular materials", in *Powders and Grains 97*, edited by R.P. Beheringer and J.T. Jenkins (A.A. Balkema, Rotterdam, 1997.)
13. A. Medina and C. Treviño, "On an universal description of the fracture patterns in rotating cohesive granular media", To be submitted to *Phys. Rev. E* (1997).