A new wide-range approximation of modified Bessel functions in terms of elementary functions

D. Kh. Morozov
Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México
Apartado postal 70-543, 04510 México, D.F., Mexico

V.V. Voitsekhovich
Instituto de Astronomia, Universidad Nacional Autónoma de México
Apartado postal 70-264, 04510 Mexico, D.F., Mexico

Recibido el 6 de enero de 1997; aceptado el 9 de febrero de 1998

A new approximation for modified Bessel functions in terms of elementary functions is proposed. In contrast with the existing approximations, the expression derived is valid for any positive values of arguments and orders of modified Bessel functions. The errors of approximation are investigated numerically. It is shown that the magnitude of error of the approximation decreases rapidly with an increase of the order, having a maximum for Bessel function of order zero. The results obtained are compared witn tabulated ones and illustrated graphically. A wide range of validity and relatively small magnitude of errors makes the approximation useful for applications.

Keywords: Modified Bessel functions; elementary functions

Se propone una nueva aproximación en términos de funciones elementales para las funciones de Bessel modificadas. En contraste con las aproximaciones existentes, la expresión derivada es válida para cualquier valor positivo de los argumentos, así como cualquier orden de las funciones de Bessel modificadas. La propagación del error se ha investigado numéricamente. Se demuestra que la magnitud del error de la aproximación decrece rápidamente con el incremento del orden, teniendo un máximo para la función de Bessel de orden cero. Los resultados obtenidos se comparan con los tabulados y se ilustran gráficamente. El amplio rango de validez, y la relativamente pequeña magnitud de los errores, hace esta aproximación útil para una gran variedad de aplicaciones.

Descriptores: Funciones de Bessel modificadas; funciones elementales

PACS: 02.30.Gp; 02.30.Mv

1. Introduction

The modified Bessel functions $I_\nu(x)$ are finding wide applications in different areas of physics. These functions satisfy the following second-order ordinary differential equation [1]:

$$\left[x^2\frac{d^2}{dx^2}+x\frac{d}{dx}-\left(x^2+\nu^2\right)\right]I_\nu(x)=0.$$

They are related to the first-kind Bessel functions $J_\nu(x)$ as [1]

$$I_\nu(x)=e^{-\nu\pi i/2}J_\nu(ix).$$

The functions $I_\nu(x)$ are well-investigated theoretically [2] as well as there is a number of successful numerical algorithms for their calculation [3].

The Taylor's series expansion of $I_\nu(x)$ has the form

$$I_\nu(x)=\left(\frac{x}{2}\right)^\nu\sum_{k=0}^{\infty}\frac{\left(x^2/4\right)^k}{k!\Gamma\left(\nu+k+1\right)},$$

where Γ denotes the Gamma-function. This series converges quickly while $x\ll\nu$ [4].

On the other hand, there is the asymptotic expansion [1]

$$I_\nu(x)\approx\frac{e^x}{\sqrt{2\pi x}}\left[1-\frac{4\nu^2-1}{8x}+\frac{\left(4\nu^2-1\right)\left(4\nu^2-9\right)}{2!\left(8x\right)^2}-\frac{\left(4\nu^2-1\right)\left(4\nu^2-9\right)\left(4\nu^2-25\right)}{3!\left(8x\right)^3}+\dots\right],$$

which is applicable when $\nu^2\gg x$.

Another ascending series and asymptotic expansions associated with the modified Bessel functions are known: asymptotic expansion for large arguments and fixed order, asymptotic expansion for large orders and fixed argument, Debye's asymptotic expansions, uniform asymptotic expansions, etc. (for more information see Ref. 1). All the above representations are applicable inside bounded intervals of x and ν. On the other hand, Pade approximations were proposed [4,5] which allows one to approximate $I_\nu(x)$ for any value of the argument and order. However, despite of the wide range of validity, the last approximations are cumbersome for analytical calculations.

The aim of this paper is a presentation of the simple and wide-ranged continuous approximation for the modified Bessel functions convenient for analytical calculations. Such approximation is useful for physical applications involving these functions where analytical results are of main interest: evaluation of integrals, extreme analysis, differential equations with coefficients including modified Bessel functions, etc.

Our analysis is restricted by only real, non-negative values of the argument x and order ν. Applying the integral representation for modified Bessel functions, we derive the needed expression for arbitrary x and $\nu \gg 1$ without any fitting parameters. Then we show that the validity range of this expression can be extended for the arbitrary ν using some small corrections. In order to show the errors of approximation we present the comparison of numerical and approximate calculations of $I_\nu(x)$.

2. Approximate formula

The following integral representation of $I_\nu(x)$ holds [6]:

$$I_\nu(x) = \frac{(x/2)^\nu}{\Gamma(\nu+1/2)\,\Gamma(1/2)} \times \int_{-1}^{1} dt\, \left(1-t^2\right)^{\nu-1/2} \exp(xt)\,, \qquad (1)$$

where Γ denotes the Gamma function.

Let us approximate the integrand in Eq. (1) as follows:

$$\left(1-t^2\right)^{\nu-1/2} \exp(xt) \approx A \exp\left[\frac{(t-t_0)^2}{\sigma^2}\right]. \qquad (2)$$

Choosing t_0 in such a way that the maximum of the approximate function coincides with the maximum of the integrand, we have

$$t_0 = -\frac{2\nu-1}{2x} + \sqrt{1+\left(\frac{2\nu-1}{2x}\right)^2}. \qquad (3)$$

Then, equating the magnitudes of the integrand and approximate function at $t=t_0$ we get the following expression for A:

$$A = \left(1-t_0^2\right)^{\nu-1/2} \exp(xt_0)\,. \qquad (4)$$

Finally, equating the second derivatives of the integrand and approximate function at $t=t_0$ we determine σ:

$$\sigma^2 = \frac{1-t_0^2}{xt_0+\nu+1/2}. \qquad (5)$$

For the case of $\nu \gg 1$ the integration in Eq. (1) can be performed in infinite limits. Substituting Eqs. (2)–(5) into Eq. (1) and performing the integration, we have the following approximation of $I_\nu(x)$ for $\nu \gg 1$:

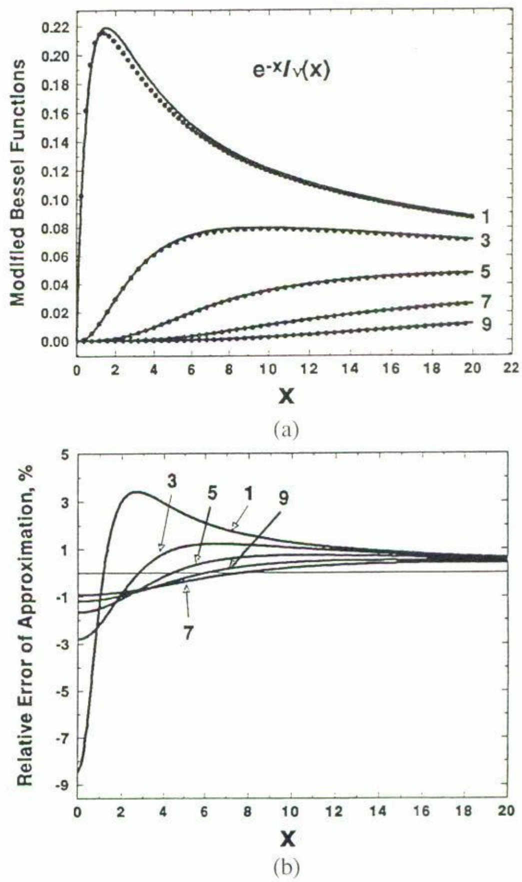


FIGURE 1. Approximation of modified Bessel functions (a) and relative errors of approximation (b) for $\nu = 1, 3, 5, 7, 9$. Explicit and approximate functions are shown by solid and dotted lines, respectively. Approximation given by Eq. (8) has been used. The digits inside the plot show the orders of the functions.

$$I_\nu(x) \approx \frac{\nu^\nu \left(\sqrt{1+\nu^2/x^2} - \nu/x\right)^\nu}{\Gamma(\nu+1/2)\left(x^2+\nu^2\right)^{1/4}} \times \exp\left(-\nu + \sqrt{x^2+\nu^2}\right). \qquad (6)$$

The approximation formula can be simplified making use of the following asymptotic expansion of the Gamma function:

$$\Gamma(\nu+1/2) \approx \sqrt{2\pi}\nu^\nu \exp(-\nu)\,. \qquad (7)$$

Substituting (7) into (6) we get a relatively simple approximation for $I_\nu(x)$:

$$I_\nu(x) \approx \frac{\left(\sqrt{1+\nu^2/x^2} - \nu/x\right)^\nu}{\sqrt{2\pi}\left(x^2+\nu^2\right)^{1/4}} \exp\left(\sqrt{x^2+\nu^2}\right) \qquad (8)$$

Unexpectedly, the approximation (8) appears even more accurate than (6).

The corresponding results are plotted in Fig. 1. Hereafter the routine *bessik* [2] is used for calculation of the accurate magnitudes of $I_\nu(x)$.

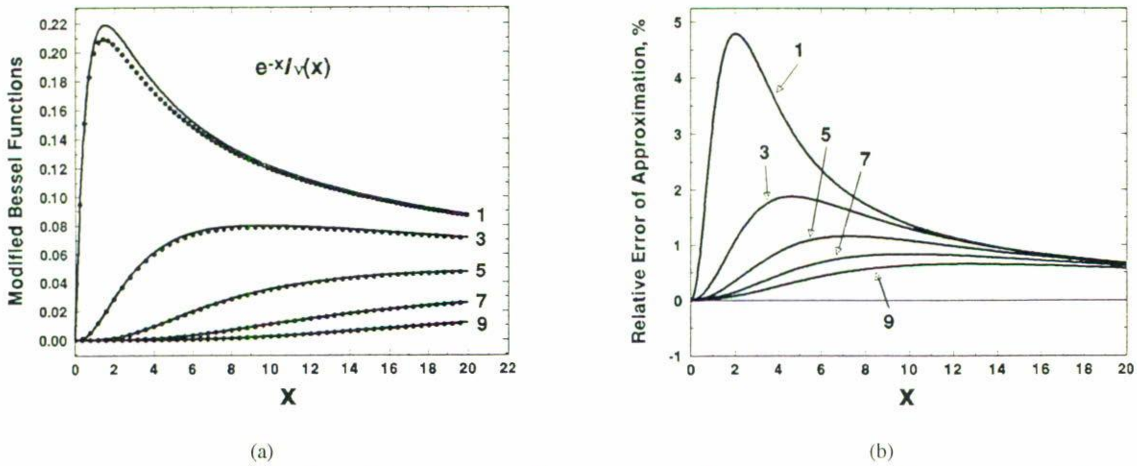


FIGURE 2. Approximation of modified Bessel functions (a) and relative errors of approximation (b) for $\nu = 1, 3, 5, 7, 9$. Explicit and approximate functions are shown by solid and dotted lines, respectively. Approximation given by Eq. (9) has been used. The digits inside the plot show the orders of the functions.

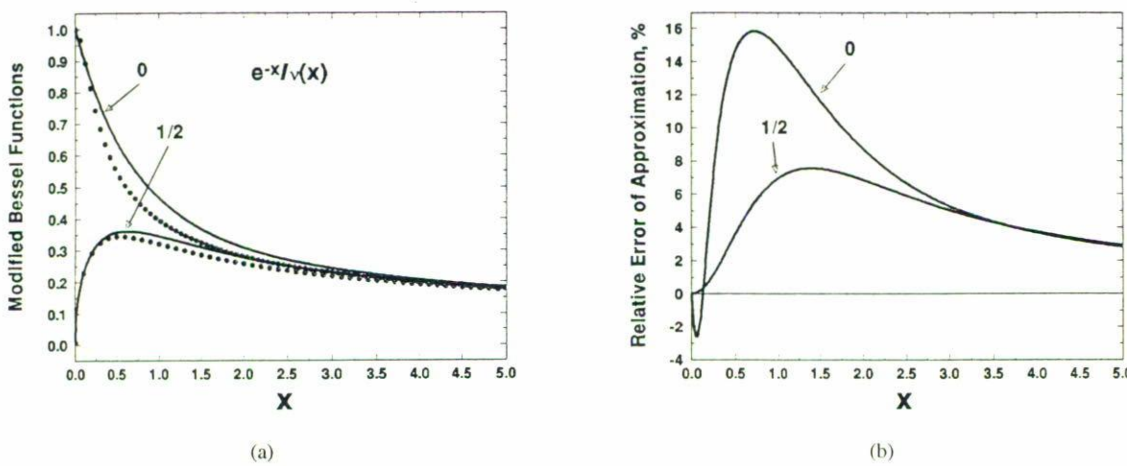


FIGURE 3. Approximation of modified Bessel functions (a) and relative errors of approximation (b) for $\nu = 0, 1/2$. Explicit and approximate functions are shown by solid and dotted lines, respectively. Approximation (9) with one fitting parameter has been used. The digits inside the plot show the orders of the functions.

In principle, Eq. (8) may be used starting from $\nu \geq 1$. As one expected the error grows as ν approaches to unit, but remains suitable for some applications. However this error

can be reduced significantly applying small modifications of Eq. (8); namely, introducing a fitting parameter α, depending on ν only, as follows:

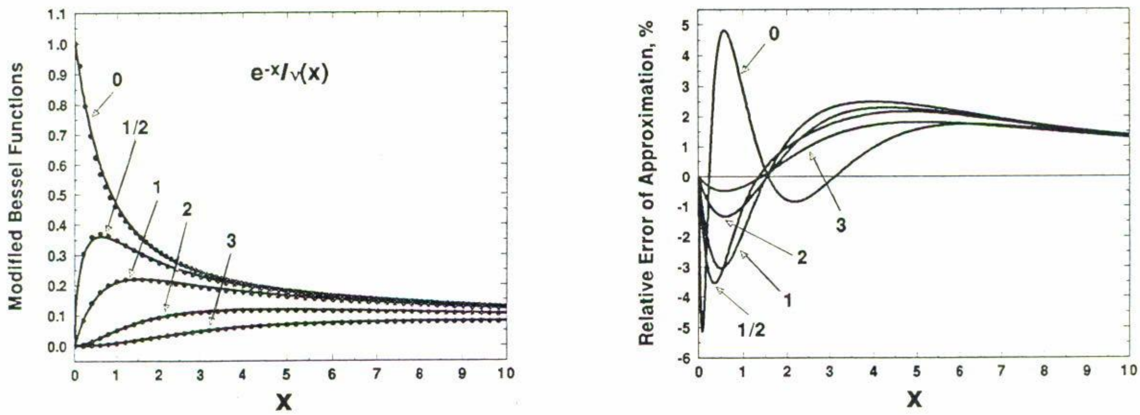
$$I_\nu(x) \approx \frac{\left(\sqrt{1+\nu^2/x^2} - \nu/x\right)^\nu}{\sqrt{2\pi}\,(x^2+\alpha)^{1/4}} \exp\left(\sqrt{x^2+\nu^2}\right),$$

$$\alpha = \left[\frac{e^\nu}{\sqrt{2\pi}} \frac{\Gamma(\nu+1)}{\nu^\nu}\right]^4. \quad (9)$$

The parameter α is chosen from consideration that the approximate and explicit values of $I_\nu(x)$ coincide at $x = 0$. The results are shown in Fig. 2. Unfortunately, the approximations (8) and (9) give relatively high errors for $\nu < 1$ (maximum error approaches to 16% as ν tends to zero, as illustrated in Fig. 3). However, one can reduce these errors using the following considerations. It turns out that for $0 \leq \nu < 1$ the formula (9) gives an appropriate approximation for all non-negative x excluding some interval near to 1, where the error has a sharp peak. Thus, one may introduce some fitting function that corrects the formula (9) for this narrow interval only. It is possible to construct such a function in many ways, and we use the following one:

$$I_\nu(x) \approx \beta \frac{\left(\sqrt{1+\nu^2/x^2} - \nu/x\right)^\nu}{\sqrt{2\pi}\,(x^2+\alpha_1)^{1/4}} \exp\left(\sqrt{x^2+\nu^2}\right),$$

$$\beta = \left[1 + x\exp(-x-\nu)/2.6\right]^{-1}, \quad \alpha_1 = \alpha\beta^4. \quad (10)$$



(a) (b)

FIGURE 4. Approximation of modified Bessel functions (a) and relative errors of approximation for $\nu = 0, 1/2, 1, 2, 3$. Explicit and approximate functions are shown by solid and dotted lines, respectively. Approximation (10) with two fitting parameters has been used. The digits inside the plot show the orders of the functions.

The calculated results, obtained by means of Eq. (10), are plotted in Fig. 4. One can see that the maximum error appears within 5%.

3. Conclusions

It has been shown that an approximation formula suggested for modified Bessel functions $I_\nu(x)$ can be used for all non-negative ν and x. The calculated results have demonstrated that the deviations from explicit values are small and only on a separate intervals on the x semi-axis they can grow, remaining inside 5%. Although the formula has been derived for non-negative arguments, an extension over negative arguments is straightforward.

Acknowledgments

This work was supported by Consejo Nacional de Ciencia y Tecnología (México) projects 1020P-E9507 and 400340-5-241OPE.

1. M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions*, (Dover Publications, Inc., New York, 1965).
2. G.N. Watson, *Theory of Bessel Functions* , (Cambridge U. Press, Cambridge, 1948).
3. W.H. Press, B.P. Flannery, S.A. Teukolsky, and W.T. Vetterling, *Numerical Recipes in C* , (Cambridge University Press, Cambridge, 1988).
4. Y.L. Luke, *Mathematical functions and their approximations*, (Academic Press Inc., New York, 1975).
5. G.A. Baker, Jr, *Essentials of Pade approximations*, (Academic Press Inc., New York, 1975).
6. I.S. Gradshtein and I.M. Ryzhik, *Table of Integrals, Series and Products*, (Academic Press, New York, 1980), p. 958.