The presence of external forces on the decay of unstable states. Description in one, two and three variables

P. Orea and J.I. Jiménez-Aquino[1]
Depertamento de Física, Universidad Autónoma Metropolitana, Iztapalapa
Apartado postal 55–534, 09340 México, D.F., Mexico
[1] *e-mail: ines@xanum.uam.mx*

Recibido el 26 de septiembre de 1997; aceptado el 17 de marzo de 1998

The standard formulation of quasideterministic (SFQD) approach and its connection with the nonlinear relaxation times (NLRT) to characterize the decay of nonlinear unstable systems in the presence of constant external force, has only been developed in terms of two physical variables and applied for detection of weak optical signals in a laser [1]. The purpose of this work is to improve the above formalism, by proposing a more general nonlinear Langevin equation with constant external force, and apply such a formalism to the study of nonlinear unstable systems of one, two and three variables. Only for the case of one variable, we study the effect of Gaussian stochastic fluctuations of time-dependent external forces, on the dynamical relaxation of such systems.

Keywords: Standard formulation of quasideterministic (SFQD) approach; nonlinear relaxation times

La formulación estandard de la teoría cuasideterminista (SFQD: standard formulation of quasideterministic approach) y su conexión con los tiempos de relajación no lineales (NLRT: nonlinear relaxation times) para caracterizar el decaimiento de sistemas inestables no lineales en presencia de fuerza externa constante, ha sido desarrollado solamente en términos de dos variables físicas y aplicado para la detección de señales ópticas débiles en un laser [1]. El propósito de este trabajo es mejorar el formalismo anterior, al proponer una ecuación de Langevin no lineal más general con fuerza externa constante, y aplicar dicho formalismo al estudio de sistemas inestables no lineales de una, dos y tres variables. Solamente para el caso de una variable, estudiamos el efecto de fluctuaciones estocásticas gaussianas de fuerzas externas dependientes del tiempo, en la relajación dinámica de tales sistemas.

Descriptores: Formulación estandard de la teoría cuasideterminista; tiempos de relajación no lineales

PACS: 05.40.+j

1. Introduction

The standard formulation of quasideterministic (SFQD) approach has been developed in the context of Langevin equation and has also been connected basically with two time scales: One is the so called passage time (PT) distribution and the other is the nonlinear relaxation time (NLRT), to characterize the dynamical relaxation of unstable systems. We adopt the terminology of *standard formulation* for those situations in which the description of the transient stochastic dynamics is made in terms of the Langevin equation whose associated systematic force is necessary derived from a potential function, such as it can be corroborated in an amount of works [1–7]. The study of the decay of unstable states through other descriptions has also been proposed by other autors [8, 9].

Recently a generalized matricial method of SFQD approach and its corresponding connection with the NLRT, has been proposed to characterize the transient stochastic dyanamics of multivariate unstable systems [10, 11]. It is shown, in this matricial scheme, that the associated systematic force of the Langevin equation is not already derived from a potential function.

On the other hand, the inmediate application of the study of transient stochastic dynamics in the context of SFQD approach, has been found in laser systems in which some of their statistical properties such as, the switch-on time and the detection of weak optical signals have been well characterized via NLRT [1] and PT distribution [2, 12]. This stochastic description is essentially formulated in terms of two physical variables.

In this work and also in the context of SFQD approach, we improve the formalism proposed in Ref. 1 and characterize the dynamical relaxation of nonlinear unstable systems of one, two and three variables under the action of constant external forces. We compare the results for the case of two variables with that of Refs. 1 and 2. We also study the presence of Gaussian time-dependent stochastic fluctuations of external forces in the dynamical relaxation of one variable unstable systems. In this case, we assume a time-dependent phase diffusion model as that proposed in Refs. 13 and 14, and analyze the effect of small intensity of the fluctuating phase.

In Sect. 2 of this work, we exhibit the connection between SFQD approach and NLRT through a more general nonlinear Lagevin equation with constant external force. In Sect. 3, we show the analytical results of NLRT for those cases of one, two and three variables. We know that the results in the case of two variables have already been reported in Ref. 1, however the results for the cases of one and three variables are new. In Sect. 4, we characterize the decay of unstable states of nonlinear systems, in the case of one variable with time-dependent external force. Concluding remarks are given in Sect. 5.

2. Standard formulation of QD approach and NLRT

The nonlinear Langevin equation for the i-th component of vector $\mathbf{x} = (x_1, x_2, x_3)$, submited to a constant external force f_{ei} can be written as

$$\dot{x}_i = a\, x_i \; + \; n(r) x_i \; + \; f_{ei} \; + \; \xi_i(t); \qquad i = 1, 2, 3\,, \tag{1}$$

where $a > 0$ is the control parameter, $n(r)$ is a scalar function of nonlinear contributions, and f_{ei} is the i-th component of the constant external force $\mathbf{F}_e$. The fluctuating force or noise $\xi(t)$ is a Gaussian white noise with zero mean value and correlation function

$$\langle \xi_i(t) \xi_j(t') \rangle = 2\epsilon\, \delta_{ij} \delta(t - t'); \qquad j = 1, 2, 3\,, \tag{2}$$

and ϵ is the intensity of noise.

The usual definition of NLRT associated with the moment $\langle r^l(t) \rangle$ of order l reads as [1]

$$T \equiv \frac{1}{M_0} \int_0^{\infty} [\langle r^l(t) \rangle - \langle r^l \rangle_{st}]\, dt\,, \tag{3}$$

where r is the square modulus of vector $\mathbf{x}$, that is $r = x_1^2 + x_2^2 + x_3^2$. The quantity M_0 is defined as $M_0 \equiv \langle r^l(0) \rangle - \langle r^l \rangle_{st}$. The quantity $\langle r^l(0) \rangle$ is the initial value of $\langle r^l(t) \rangle$ at time $t = 0$, and the quantity $\langle r^l \rangle_{st}$ its value in the steady state. In this work we will suppose that $\langle r^l(0) \rangle = 0$ which physically means to consider fixed initial conditions.

The connection between the time scale (3) and the QD approach is achieved through the solution of the linear deterministic equation for the variable r [1]. Then, this linear equation reads

$$\dot{r} = 2ar\,, \tag{4}$$

and its solution is

$$r(t) = r(0)\, e^{2at}\,. \tag{5}$$

Deterministically at time $t = 0$, the process $r(t)$ has the value $r(0)$ and the system is located in the origin of coordinates $(0, 0, 0)$ corresponding to the instability point of the potential $U(r) = -ar^2/2$. We know from the deterministic point of view that the system will stay on that point, unless we assume the hypothesis of statistical fluctuations of the initial conditions around the instability point, this is $r(0) = h^2$, where h is a set of random variables. So, because of fluctuations of the initial conditions the process (5) becomes a cuasiterministic process of the form

$$r(t) = h^2 e^{2at}\,, \tag{6}$$

where h^2 accounts for fluctuantions of the initial conditions responsible for the dynamical relaxation of the system around the unstable state. For $t > 0$, as it is shown in (6), the system grows as the time goes to infinty, unless it is stoped at some reference value r_{st} such that $r_{st} = r(t_i)$ where t_i is a random variable. Therefore, the whole process (6) including the order of moment can be written as

$$r^l(t) = h^{2l} e^{2al} \theta(t_i - t) + r_{st}^l \theta(t - t_i)\,, \tag{7}$$

with $\theta(x)$ the step function.

With the substitution of the process (7) into Eq. (3) we obtain

$$T_L = \langle t_i \rangle - C\,, \tag{8}$$

where the constant $C = [1 - \langle h^{2l} \rangle / r_{st}^l]/2al$. The first term is known as mean first passage time (MFPT) distribution in the decay of unstable states. The constant C is the contribution of the NLRT and accounts for the time scale close to the potential barrier r_{st}.

The justification of the hypothesis of fluctuations of initial conditions, as well as the statistics of the random variable h is essentially obtained from QD approach. This approach starts basically with the linear Langevin equation associated with (1) given by

$$\dot{x}_i = ax_i + f_{ei} + \xi_i(t); \qquad i = 1, 2, 3\,, \tag{9}$$

and the formal solution reads

$$x_i(t) = h_i(t) e^{at}\,, \tag{10}$$

where

$$h_i(t) = \int_0^t e^{-as} [f_{ei} + \xi_i(s)] ds\,. \tag{11}$$

The basic ideas of QD approach is to show that in the limit of long times, such that $at \gg 1$ the process (11) becomes a Gaussian random variable. This is so, because for small value of $\xi_i(t)$ we can guarantee that

$$\lim_{t \to \infty} \frac{dh_i(t)}{dt} = \lim_{t \to \infty} e^{-at} [f_{ei} + \xi_i(t)] \to 0\,, \tag{12}$$

and therefore $h_i(\infty)$ becomes a constant that we can call h_i. Under these circumstances the process (11) can be replaced, in the limit of long times, by a quasideterministic process $x_i(t) = h_i e^{at}$, or in terms of variable r

$$r(t) = h^2 e^{2at}\,, \tag{13}$$

which is similar to (6) where the variable $h = \sqrt{h_1^2 + h_2^2 + h_3^2}$ plays the role of an effective initial condition.

On the other hand, since h_i is a Gaussian random variable, the probability density $P(h_i)$ is determined only with the first two moments $\langle h_i\rangle$ and $\langle h_i^2\rangle$. It is clear from (11) that $\langle h_i\rangle = f_{ei}/a$ and the second moment $\langle h_i^2\rangle = f_{ei}^2/a^2 + \epsilon/a$. Therefore the variance of the Gaussian random variable h_i, is equals to

$$\sigma^2 \equiv \langle h_i^2\rangle - \langle h_i\rangle^2 = \frac{\epsilon}{a}\,. \tag{14}$$

The joint probability density for the space of variables $\mathbf{h} = (h_1, h_2, h_3)$ is the Gaussian distribution

$$P(h_1,h_2,h_3) = cte \exp\left[-\alpha^2 \sum_{i=1}^{3}(h_i - \langle h_i\rangle)^2\right], \tag{15}$$

where $\alpha^2 = 1/(2\sigma^2)$ and σ^2 is given by (14). The marginal probability density $P(h)$ can be constructed with the help of a new space of variables defined as $\mathbf{u} = (u_1, u_2, u_3)$ and using the transformation

$$P(h_1,h_2,h_3)dh_1dh_2dh_3 \to P(u_1,u_2,u_3)dV\,, \tag{16}$$

with $dV = J(\mathbf{u})d\mathbf{u}$ is the volume element in the space $\mathbf{u}$ and $J(\mathbf{u})$ is the Jacobian of the transformation given by

$$J(\mathbf{u}) = \begin{vmatrix} \frac{\partial h_1}{\partial u_1} & \frac{\partial h_1}{\partial u_2} & \frac{\partial h_1}{\partial u_3} \\ \frac{\partial h_2}{\partial u_1} & \frac{\partial h_2}{\partial u_2} & \frac{\partial h_2}{\partial u_3} \\ \frac{\partial h_3}{\partial u_1} & \frac{\partial h_3}{\partial u_2} & \frac{\partial h_3}{\partial u_3} \end{vmatrix} \tag{17}$$

If we take the variable $u_1 = h$, then the joint probability density in the space (h, u_2, u_3) will be written in the formal way as

$$P(h,u_2,u_3)dV = cte \exp\left[-\alpha^2(h^2 + b^2 - 2\mathbf{b}\cdot\mathbf{h})\right]dV, \tag{18}$$

where we define b as the modulus of vector $\mathbf{b} \equiv (\langle h_1\rangle, \langle h_2\rangle, \langle h_3\rangle)$ and the dot means the scalar product. Finally the marginal probability density $P(h)$ can be obtained by calculating the Jacobian and integrating over the rest the variables (u_2, u_3).

2.1. The nonlinearities in NLRT

The time scale (8) characterizes the dynamical relaxation in the linear regime of the nonlinear systems (1). However, the effect of nonlinear contributions present in the deterministic force of the Langevin equation (1), is taken into account through the general definition of unstable states in terms of the variable r which is given by [1]

$$\dot{r} = v(r) = \frac{r(r_{st} - r)}{C_0 + rg(r)}\,, \tag{19}$$

with $C_0 = r_{st}/2a$, and $g(r) \geq 0$ is a ponlynomial of the form $g(r) = \sum_{n=0}^{m} g_n r^n$. In the case of (1) the nonlinear deterministic equation is

$$\dot{x}_i = ax_i + n(r)x_i\,. \tag{20}$$

For the r variable we have

$$\dot{r} = 2ar + 2rn(r)\,. \tag{21}$$

This equation is compatible with (19), according to the explicit form of the function $n(r)$. So that, for nonlinear systems of the form (1) the NLRT associated with the quantity $\langle r(t)^l\rangle$ can be written in terms of a quadrature, by substituting Eq. (19) into Eq. (3), that is

$$T = -\frac{1}{r_{st}^l}\left\langle \int_{h^2}^{r_{st}} \frac{(r^l - r_{st}^l)}{v(r)} dr \right\rangle, \tag{22}$$

or

$$T = T_L + C + \mathbf{I}_{NL}\,, \tag{23}$$

with the definition

$$\mathbf{I}_{NL} \equiv \frac{1}{r_{st}}\left\langle \int_{h^2}^{r_{st}} \left[[1 + S(r)]g(r) + C_0\frac{S(r)}{r}\right] dr \right\rangle, \tag{24}$$

where $S(r)$ is a polynomial function given by $S(r) = \sum_{k=1}^{l-1}(r/r_{st})^k$. Clearly this time scale characterizes the complete relaxation of the nonlinear system towards its corresponding steady state, characterized by the value r_{st}^l.

3. Description in one, two and three variables

The linear characterization of those nonlinear systems of one, two and three variables, by meas of NLRT associated with $\langle r^l\rangle$, in absence of external force is a known result [1]. Here we define this time scale as T_L^0, and it is given for each variable, in the limit of small noise, as

$$T_L^0 = \frac{1}{2a}\left\{ \ln(\alpha^2 r_{st}) - \psi\left(\frac{n}{2}\right) - \frac{1}{l}\right\}; \quad n = 1,2,3, \tag{25}$$

with $\psi(x)$ is the digamma function and $\alpha^2 = 1/2\sigma^2 = a/2\epsilon$.

The complete dynamical characterization of nonlinear systems (1), is obtained through the time scale (23). In the absence of external force it is also known and given for each variable, in the limit of small noise intensity as

$$T_0 = \frac{1}{2a}\left\{ \ln(\alpha^2 r_{st}) - \psi\left(\frac{n}{2}\right)\right\} + \mathbf{I}_{NL}; \quad n = 1,2,3, \tag{26}$$

where $\mathbf{I}_{NL}$ is the corresponding integral term of (24), which is clearly the nonlinear contribution of NLRT and type dependent model.

3.1. The NLRT for the case of one variable, in presence of external force

For this case the probability density $P(h)$ reads

$$P(h) = \frac{2\alpha}{\sqrt{\pi}} e^{2\alpha^2 bh} e^{-\alpha^2(h^2+b^2)}, \tag{27}$$

with $b = F_e/a$, being $F_e = f_{e1}$. The corresponding time scale in the linear regime, defined as T_L^e, and in the limit of small noise intensity, is

$$T_L^e = T_L^0 + \frac{\sqrt{\pi}}{2a} \sum_{m=1}^{\infty} \frac{(-1)^m}{m\left(m-1/2\right)!} (\alpha^2 b^2)^m + \mathcal{O}(\epsilon) + \mathcal{O}(b^2) + \mathcal{O}(\epsilon b^2), \tag{28}$$

where the series power is convergent for values of $\alpha^2 b^2 \leq 1$, and T_L^0 is the same as (25) if $n = 1$.

The complete dynamical relaxation in the presence of external force, according to (23) is then

$$T_e = T_0 + \frac{\sqrt{\pi}}{2a} \sum_{m=1}^{\infty} \frac{(-1)^m}{m\left(m-1/2\right)!} (\alpha^2 b^2)^m, \tag{29}$$

and T_0 is the same as (26) if $n = 1$.

3.2. The NLRT for the case of two variables, in presence of external force

For this case, we obtain the following result for the marginal probability [1]

$$P(h) = 2\alpha^2 h I_0(2\alpha^2 bh) e^{-\alpha^2(h^2+b^2)}, \tag{30}$$

where now $b = F_e/a$, with $F_e = \sqrt{f_{e1}^2 + f_{e2}^2}$. The NLRT associated with $\langle r^l \rangle$ in the linear regime, for small noise intensity, is

$$T_L^e = T_L^0 + \frac{1}{2a} \sum_{m=1}^{\infty} \frac{(-1)^m}{mm!} (\alpha^2 b^2)^m + \mathcal{O}(\epsilon) + \mathcal{O}(b^2) + \mathcal{O}(\epsilon b^2). \tag{31}$$

In this case the sum can explicitly be reduced to $\sum_{m=1}^{\infty} (-1)^m x^m/mm! = -[E_1(x) + \gamma + \ln(x)]$, being $E_1(x)$ is the integral exponential function and γ is the Euler constant [15, 16]. The time scale T_L^0 is given by (25) if $n = 2$.

The corresponding NLRT for the complete process, according to (23) will be

$$T_e = T_0 + \frac{1}{2a} \sum_{m=1}^{\infty} \frac{(-1)^m}{mm!} (\alpha^2 b^2)^m, \tag{32}$$

taking into account that T_0 is the same as (26) if $n = 2$.

The result (31) reduces to that of Ref. 2 for a laser system, if we take into T_L^0 the order of moment $l = 1$ and the modulus $r = E^2$, being E^2 the intensity of the laser. On the other hand, the result (32) reduces to that of Ref. 1 for the corresponding laser model, if we take the scalar function $n(r) = F/[1 + (F/A)r]$, where A and F are the parameters of the laser. In this case the integral term reads, for very weak optical signal, as $\mathbf{I}_{NL} = (F/k - 1)/2a$, where $a = F - k$ and k is other parameter of the system.

3.3. The NLRT for the case of three variables, in presence of external force

For this system, we obtain after some algebra, the marginal probability density

$$P(h) = \frac{2\alpha}{\sqrt{\pi} b} h \sinh(2\alpha^2 bh) e^{-\alpha^2(h^2+b^2)}, \tag{33}$$

with $b = F_e/a$, and $F_e = \sqrt{f_{e1}^2 + f_{e2}^2 + f_{e3}^2}$. In this case the linear time scale, for small noise intensity, is now given by

$$T_L^e = T_L^0 + \frac{\sqrt{\pi}}{2a} \sum_{m=1}^{\infty} \frac{(-1)^m}{2m(m+1/2)!} (\alpha^2 b^2)^m + \mathcal{O}(\epsilon) + \mathcal{O}(b^2) + \mathcal{O}(\epsilon b^2), \tag{34}$$

being T_L^0 the same as (25) with $n = 3$.

The nonlinear characterization of the process (1) is again obtained from (23). This result reduces to

$$T_e = T_0 + \frac{\sqrt{\pi}}{2a} \sum_{m=1}^{\infty} \frac{(-1)^m}{2m(m+1/2)!} (\alpha^2 b^2)^m. \tag{35}$$

Here T_0 is obtained from (26) with $n = 3$.

In the three cases, the term $\alpha^2 b^2 = F_e^2/2a\epsilon$, which is proportional to F_e^2/ϵ. The condition of convergence $\alpha^2 b^2 \leq 1$ means that the intensity of the external force must be less or of the same order as internal noise ϵ. This was precisely shown in Refs. 2 and 3, in which very weak optical signals of the same order as the intrinsic noise level but 10^8 times smaller than the steady-state intensity of the laser, can be detected using the laser as a supergenerative receiver.

4. Description in one variable with time-dependent external force

Here, we assume an oscilating external force $f_1(t)$ with time-dependent fluctuating phase of the form $f_{e1} = F_e \cos[\Phi(t)]$. In this case, the Langevin equation is the same as (1) except it now reads

$$\dot{x} = ax + n(r)x + F_e \cos\Phi(t) + \xi(t), \tag{36}$$

where $\xi(t)$ is again Gaussian with the same properties given in Sect. 2. The phase fluctuating is taken as a Gaussian time-dependent phase diffusion model with zero mean value and correlation function [13]

$$\langle \Phi(t)\Phi(t')\rangle = 2D\min(t-t'), \tag{37}$$

and D is the intensity of the fluctuating phase.

In the case of time-dependent external force, the formalism of QD approach is similar to that case of constant force, given in Sect. 2. Here we just summarize the results obtained for the ramdom variable h. These are

$$\langle h\rangle = \frac{F_e}{a+D}, \tag{38}$$

$$\langle h^2\rangle = \frac{\epsilon}{a} + \frac{F_e^2}{a(a+2D)}. \tag{39}$$

So that, the variance reads

$$\sigma^2 = \frac{\epsilon}{a} + \frac{F_e^2}{a(a+2D)} - \frac{F_e^2}{(a+D)^2}, \tag{40}$$

and the probability density $P(h)$ will be the same as (27), except that the parameter b is given by $b = F_e/(a+D)$ and $\alpha^2 = 1/(2\sigma^2)$ with σ^2 given by (40). The expected result is that if $D = 0$, we restore the results in the constant case, that is $\sigma^2 = \epsilon/a$ and $\alpha^2 b^2 = F_e^2/(2a\epsilon)$. However, if we take the limit of small intensity of the fluctuating phase, such that $D \ll a$, it can be shown that (40) reduces to $\sigma^2 = \epsilon/a$ and

$$\alpha^2 b^2 = \frac{F_e^2}{2a\epsilon} - \frac{F_e^2}{a^2\epsilon}D. \tag{41}$$

Therefore, in this limit of approximation, the intensity of initial fluctuations σ^2 has the same expression as that of the constant case, whereas the term $\alpha^2 b^2$, according to (41), has a very small correction with respect to the constant case.

Finally, in the limits of approximation $\epsilon \ll a$ and $D \ll a$, togheter with the condition $F_e^2/(2a\epsilon) \leq 1$, the time scales in the linear and nonlinear regimes have the similar structure as that of (28) and (29) respectively.

5. Concluding remarks

The influence of internal fluctuations and external forces on the dynamical relaxation of nonlinear unstable systems, by mean of NLRT, is essentially given at early times of the dynamical evolution, or well, in the linear region of the unstable potential. However, the complete dynamical relaxation has also been well characterized.

Here we have done an effort by improving the formalism of Ref. 1 and at the same time to characterize the decay dynamics of nonlinear unstable systems of one and three variables, under the action of external forces, since the calculation of both series power in the time scales (28), and (34) are not inmediate results.

The nonlinear contribution $\mathbf{I}_{NL}$ depends on the analytical structure of the scalar function $n(r)$, and therefore the time scales (29), (32) and (35) are quite general results. In the case of laser system of Refs. 1 and 2, the term $\mathbf{I}_{NL} = (F/k-1)/2a$.

In the case of one variable systems, we have shown that the effect of small intensity of the fluctuating phase on the decay dynamics of those systems, takes place not in the intensity of the initial fluctuations σ^2, but only in the the parameter $\alpha^2 b^2$, according to (41).

Finally, the application of the results in the cases of one and three variables, to certain physical systems is subject of investigation.

Acknowledgments

Financial support from Consejo Nacional de Ciencia y Tecnología (CONACyT) México is acknowledged.

1. J.I. Jiménez-Aquino and J.M. Sancho, *Phys. Rev. E* **47** (1993) 1558.
2. S. Balle, F. de Pascuale, and M. San Miguel, *Phys. Rev. A* **41** (1990) 5012.
3. G. Vemuri and R. Roy, *Phys. Rev. A* **39** (1989) 2539.
4. I. Littler *et al.*, *Phys. Rev. A* **41** (1990) 4131.
5. J.I. Jiménez-Aquino, *J. Phys. A* **27** (1994) 4745.
6. J.M. Sancho and M. San Miguel, *Phys. Rev. A* **39** (1989) 2722.
7. M.C. Torren and M. San Miguel, *Phys. Rev. A* **38** (1988) 245.
8. M. Suzuki, Y. Liu, and T. Tsumo, *Physica A* **138** (1986) 433.
9. F. Haake, J.W. Haus, and R. Glauber, *Phys. Rev. A* **23** (1981) 3255.
10. J.I. Jiménez-Aquino, *Physica A* **229** (1996) 444.
11. J.I. Jiménez-Aquino, *Physica A* **245** (1997) 503.
12. P. Jung, G. Vemuri, and R. Roy, *Optics Comm.* **78** (1990) 58.
13. J. Dellunde, J.M. Sancho, and M. San Miguel, *Opt. Comm* **109** (1994) 435.
14. C.W. Gardiner, *Handbook of stochastic method*, (Springer, Berlin, 1983).
15. *Handbook of Mathematical Functions*, edited by M. Abramowitz and I.A. Stegun, (Dover, New York, 1972).
16. *Table of integrals series and products*, edited by I.S. Gradshteyn and I.M. Ryzhik, (Academic Press, New York, 1965).