

The Faddeev-Popov term reviewed

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Some textbooks and reports claim that the Jacobian $\Delta_f[A]$ which arises in the discussion of the Faddeev-Popov method to quantize non-Abelian gauge theories and which is given by the derivative of the gauge fixing conditions over the gauge group parameters, is gauge invariant. Other references however prove the opposite. In this brief report we present a discussion about this matter.

Keywords: Functional integration; gauge invariance; gauge transformation

Algunos textos mencionan que el jacobiano $\Delta_f[A]$, el cual surge en la discusión del método de Faddeev-Popov para cuantizar teorías de norma no abelianas y está dado por la derivada de las condiciones que fijan la norma con respecto a los parámetros del grupo de simetría, es invariante de norma. Otras referencias muestran lo contrario. En este trabajo se presenta una discusión sobre este hecho.

Descriptores: Integral funcional; invariancia de norma; transformación de norma

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1. Introduction

Already thirty years ago L.D. Faddeev and V.N. Popov introduced their prescription [1] to quantize non-Abelian gauge theories, according to which the gauge fixing conditions give rise to a system of anticommutating scalar ghost fields which enter only as internal lines in Feynman loops.

In non-Abelian gauge theories, considering only the gauge bosons, the vacuum-to-vacuum amplitude $\langle 0, +\infty | 0, -\infty \rangle \equiv +\langle 0 | 0 \rangle_-$ is expressed by the functional integration [2]

$$+\langle 0 | 0 \rangle_- \sim \int \mathcal{D}A^\mu e^{iS[A^\mu]}, \quad (1)$$

where $\mathcal{D}A^\mu = \prod_{a,x} dA_a^\mu(x)$ and the action $S \equiv \int d^4x \mathcal{L}$ are invariant under the gauge transformation

$$A_\mu \rightarrow A_\mu^\theta = U^\dagger A_\mu U + iU^\dagger (\partial_\mu U), \quad (2)$$

with $U \equiv e^{i\theta}$ and $\theta \equiv \theta_a T_a$ (setting the coupling constant equal to one). The generators T_a of the symmetry group satisfy the algebra

$$[T_a, T_b] = iC_{abc} T_c. \quad (3)$$

An immediate problem arises because of the divergent nature of the functional integration (1), which is due to the gauge invariance of the action. Hence an infinity factor should be factorized and removed before implementing

the perturbative expansion. The trick designed by Faddeev-Popov, for this purpose, begins with the introduction of the Jacobian

$$\Delta_f[A_\mu] = \left(\int \mathcal{D}\theta \delta[f[A_\mu^\theta]] \right)^{-1}, \quad (4)$$

with $\mathcal{D}\theta = \prod_{a,x} d\theta_a(x)$, so that we can write the expression (1) as

$$+\langle 0 | 0 \rangle_- \sim \int \mathcal{D}A_\mu e^{iS[A_\mu]} \Delta_f[A_\mu] \int \mathcal{D}\theta \delta[f[A_\mu^\theta]]. \quad (5)$$

$f[A_\mu^\theta(x)] = 0$ is called the gauge fixing condition and it should have a solution $\theta(x)$ for a given A_μ [3]. If θ_n is a zero of $f[A_\mu^\theta]$, we obtain

$$\Delta_f[A] = \left. \frac{\delta f}{\delta \theta} \right|_{\theta_n}. \quad (6)$$

Some textbooks and reports [5] claim that the Jacobian $\Delta_f[A]$ is gauge invariant when they are explaining the quantization of non-Abelian gauge theories. Some other references [6] state without proof that this determinant is gauge invariant. The argument of Refs. 5 about the gauge invariance of $\Delta_f[A]$ goes as follows:

$$\begin{aligned} \Delta_f^{-1}[A_g] &= \int \mathcal{D}g' \delta(F[A_{g'}]) \\ &= \int \mathcal{D}(g'g) \delta(F[A_{g'g}]) \\ &= \int \mathcal{D}(g''g) \delta(F[A_{g''}]) = \Delta_f^{-1}[A]. \end{aligned} \quad (7)$$

Although the last three equalities are correct, the first one is wrong since it assumes that the group measure $\mathcal{D}g$ is equal to the parameter measure [which enters in the definition given in Eq. (4)],

$$\mathcal{D}g = \prod_{a,x} d\theta_a(x) = \mathcal{D}\theta.$$

In this note we show that this Jacobian is not gauge invariant, and we give an example. At the end we explain how this result is in agreement with Ref. 7.

After the gauge transformation $A \rightarrow A^{-\theta}$ the Jacobian $\Delta_f[A]$ defined in the Eq.(4) transforms as

$$\begin{aligned} \Delta_f^{-1}[A^{-\theta}] &= \int \mathcal{D}\theta'' \delta \left[F[(A_\mu^{-\theta})^{\theta''}] \right] \\ &= \int \mathcal{D}\theta' \left| \frac{\delta\theta''}{\delta\theta'} \right| \delta \left[f[A_\mu^{\theta'}] \right] \\ &= \Delta_f^{-1}[A] \left| \frac{\delta\theta''}{\delta\theta'} \right|_{\theta'=\theta_n}, \end{aligned} \tag{8}$$

where θ, θ' and θ'' are related by

$$e^{i\theta''} = e^{i\theta} e^{i\theta'} \equiv e^{i\theta \# i\theta'},$$

and the exponent $x \# y$ is given by an infinite Baker-Campbell-Hausdorff series of multiple commutators [4]

$$\begin{aligned} x \# y &= x + y + \frac{1}{2}[x, y] \\ &+ \frac{1}{12} \left([x, [x, y]] + [y, [y, x]] \right) + \dots \end{aligned} \tag{9}$$

Therefore the variation $i\delta\theta''$, with respect to θ' is given by

$$\begin{aligned} i\delta\theta'' &= i\theta \# i(\theta' + \delta\theta') - i\theta \# i\theta' \\ &= i\delta\theta' - \frac{1}{2}[\theta, \delta\theta'] - \frac{i}{12}[\theta, [\theta, \delta\theta']] + \dots, \end{aligned} \tag{10}$$

so that in terms of the components of θ we can write

$$\frac{\delta\theta''_a}{\delta\theta'_b} = \delta_{ab} + \frac{1}{2}C_{abc}\theta_c + \frac{1}{12}C_{ace}C_{dbe}\theta_c\theta_d + \dots \tag{11}$$

For example in SU(2) we have $C_{abc} = \epsilon_{abc}$,

$$\begin{aligned} \left| \frac{\delta\theta''_a}{\delta\theta'_b} \right|_{\theta'_n} &= 1 + \frac{1}{144} (\theta_1^2 + \theta_2^2 + \theta_3^2) \\ &\times [\theta_1^2 + \theta_2^2 + \theta_3^2 + 12] + \dots \end{aligned} \tag{12}$$

and obviously the Jacobian $\Delta_f[A]$ is not gauge invariant.

Note that to get Eq. (8) we have integrated over all parameters of the symmetry group instead of over all group elements. We would like to stress that the references [5] get Eq. (8) without the determinant $|\delta\theta''/\delta\theta'|$. They have integrated over all group elements.

Now we show that we can obtain the result (8) from Eq. (37) of Zaidi's paper [7]. First, let us explain how he obtains this equation.

Let us consider the functional integral $\int F[q]\mathcal{D}q$ and suppose that we want to change the function $q(x)$ in the functional integral by another function $q'(x)$ given by

$$q(x) = \int K(x, y)q'(y) dy,$$

with $K(x, y) = K(y, x)$.

Now we wish to find out what happens in the functional integral. For this aim we must seek the relationship between the two measures $\mathcal{D}q$ and $\mathcal{D}q'$. If we expand $q(x)$ and $q'(x)$ in terms of an orthonormal set of functions $\{\phi(x)\}$, we obtain

$$\mathcal{D}q = \prod_{i=1}^{\infty} dq_i = \left| \frac{\partial q_i}{\partial q'_j} \right| \mathcal{D}q',$$

hence

$$\int F[q][dq] = \left| \frac{\partial q_i}{\partial q'_j} \right| \int F[Kq'] [dq']. \tag{13}$$

Equation (13) is Eq. (37) in Zaidi's paper. q_i and q'_j are the coefficients in the expansion of $q(x)$ and $q'(x)$, respectively.

If we consider $\theta''(x)$ and $\theta'(x)$ instead of $q(x)$ and $q'(x)$, respectively, and expand them in terms of the set of generators $\{T_a\}$ of the gauge transformation as

$$\theta''(x) = \sum_a \theta''_a(x) T_a,$$

$$\theta'(x) = \sum_b \theta'_b(x) T_b,$$

with $F[q] = \Delta_f[A]$, we obtain Eq. (8) from Eq. (13). In this case $\theta''(x)$ and $\theta'(x)$ are related by $e^{i\theta''} = e^{i\theta} e^{i\theta'} \equiv e^{i\theta \# i\theta'}$. Also we can see that the equivalent expression to $K(x, y)$ is not symmetric.

We also can obtain the result (8) from Eq. (15.5.17) of Weinberg's book [7]. First, let us mention that Eq. (15.5.1) of this reference,

$$I = \int \mathcal{D}\phi g[\phi] B[f[\phi]] |F[\phi]|,$$

is equal to Eq. (5) with the following correspondence:

$$g[\phi] = e^{iS[A^\mu]},$$

$$B[f[\phi]] = \delta[F[A_\mu]],$$

$$|F[\phi]| = \Delta_f[A],$$

where the F-matrix is

$$F_{\alpha x, \beta y}[\phi] = \left. \frac{\delta f_\alpha[\phi_\lambda; x]}{\delta \lambda_\beta(y)} \right|_{\lambda=0}. \tag{14}$$

If we consider the gauge transformation with parameters $\rho^\alpha(x; \Lambda, \lambda)$ in the ϕ fields as the product of the gauge transformation with parameters $\Lambda^\alpha(x)$ followed by the gauge transformation with parameters $\lambda^\alpha(x)$, we obtain

$$F_{\alpha x, \beta y}[\phi_\Lambda] = \int J_{\alpha x, \gamma z}[\phi, \Lambda] R_{\beta y}^{\gamma z}[\Lambda] d^4 z, \tag{15}$$

with

$$\begin{aligned} J[\phi, \Lambda] &= \left. \frac{\partial f_\alpha[\phi_\rho; x]}{\partial \rho^\gamma(z)} \right|_{\rho=\Lambda}, \\ R_{\beta y}^{\gamma z} &= \left. \frac{\partial \rho^\gamma(z; \Lambda, \lambda)}{\partial \lambda^\beta(y)} \right|_{\lambda=0}, \end{aligned} \tag{16}$$

hence

$$|F[\phi_\Lambda]| = |J[\phi, \Lambda]| |R[\Lambda]|. \tag{17}$$

Equation (17) is Eq. (15.5.17) in Weinberg's book, and it is equal to Eq. (8). Weinberg has introduced a weight-function $\rho(\Lambda)$ as

$$\rho(\Lambda) = \frac{1}{|R[\Lambda]|}, \tag{18}$$

thus, $\rho(\Lambda)$ is $|\delta\theta''/\delta\theta'|^{-1}$.

Finally, if we use the gauge invariance of the action S and the measure $\mathcal{D}A^\mu$ combined with (8), after the gauge trans-

formation $A \rightarrow A^{-\theta}$, the integral (5) can be written as

$$\begin{aligned} +\langle 0|0\rangle_- &\sim \int \mathcal{D}A^\mu e^{iS[A_\mu]} \delta[f[A_\mu]] \Delta_f[A_\mu] \\ &\times \int \mathcal{D}\theta \left| \frac{\delta\theta''}{\delta\theta'} \right|_{\theta'=\theta_n}^{-1}, \end{aligned} \tag{19}$$

so that removing the entire factor $\int \mathcal{D}\theta \left| \frac{\delta\theta''}{\delta\theta'} \right|^{-1}$ we write (19) as

$$+\langle 0|0\rangle_- \sim \int \mathcal{D}A_\mu \delta[f[A_\mu]] \Delta_f[A_\mu] e^{iS[A_\mu]}. \tag{20}$$

which is the Faddeev-Popov prescription.

We notice that the contribution of $\Delta_f[A]$ is contained in the infinity factor which is removed from the integration. Thus, the expression (20) is obtained independently of whether $\Delta_f[A]$ is gauge invariant or not. If $\Delta_f[A]$ were gauge invariant, then one would remove just $\int \mathcal{D}\theta$.

In conclusion, we have showed that the Jacobian $\Delta_f[A]$ is not gauge invariant and given an example. Also we have explained how this result can be obtained from Refs. 7.

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