On the connection between the Kepler problem and the isotropic harmonic oscillator in classical mechanics

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Using the relationship between the cartesian and the parabolic coordinates, it is shown that the Kepler problem in two dimensions can be related with the isotropic harmonic oscillator in two dimensions in such a way that the Hermann-Bernoulli-Laplace-Runge-Lenz vector and the angular momentum, as well as the dynamical symmetry group generated by them, are obtained from the constants of the motion of the oscillator and its symmetry group. All possible values of the energy are considered and it is shown that the orbits in the Kepler problem are easily obtained from those of the harmonic oscillator.

Keywords: Kepler problem; isotropic harmonic oscillator

Se muestra que, por medio de la relación entre las coordenadas cartesianas y las parabólicas, el problema de Kepler en dos dimensiones se puede relacionar con el oscilador armónico isótropo en dos dimensiones de tal manera que el vector de Hermann-Bernoulli-Laplace-Runge-Lenz y el momento angular, así como el grupo de simetría dinámico que generan, se obtienen de las constantes de movimiento del oscilador y su grupo de simetría. Se consideran todos los posibles valores de la energía y se muestra que las órbitas en el problema de Kepler se obtienen fácilmente de las del oscilador armónico.

Descriptores: Problema de Kepler; oscilador armónico isótropo

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1. Introduction

The Kepler problem for bounded motion and the isotropic harmonic oscillator are related to each other in various ways. They correspond to the only two central potentials whose orbits are all closed for bounded motion, in both cases the orbits are ellipses and one can find constants of the motion that satisfy the same Poisson bracket relations (see, *e.g.*, Refs. 1 and 2 and the references cited therein). Furthermore, by means of the Kustaanheimo-Stiefel transformation, the Kepler problem can be related to a four-dimensional isotropic harmonic oscillator with a constraint (see, *e.g.*, Ref. 2 and the references cited therein).

In this paper we show that, by expressing the Hamiltonian of the two-dimensional Kepler problem in parabolic coordinates, one can relate the Kepler problem with energy $E < 0$ to a two-dimensional isotropic harmonic oscillator (TIHO), whose frequency depends on E. This relationship allows us to derive the usual constants of the motion of the Kepler problem from those of the TIHO and to see how the SU(2) symmetry of the TIHO leads to the SO(3) symmetry of the two-dimensional Kepler problem for bounded motion (see also Refs. 3 and 4). A similar analysis is given for the two-dimensional Kepler problem with positive or zero energy, showing that, in the first case, there exists a relationship with the analogue of the TIHO where the force is repulsive instead of attractive and, in the second case, with a TIHO with zero frequency, *i.e.*, a free particle. We show explicitly that the Hamiltonian of a particle in two dimensions with a repulsive central potential proportional to r^2 possesses a SU(1,1) hidden symmetry and that, therefore, the Kepler problem with positive energy has a SO(2,1) dynamical symmetry. We also show that the orbits in the Kepler problem are the images under the complex-variable function $f(z) = z^2/2$ of the orbits corresponding to a central potential proportional to r^2 (see also Refs. 3–5).

2. The two-dimensional isotropic harmonic oscillator revisited

The Hamiltonian of a two-dimensional isotropic harmonic oscillator (TIHO) is given in cartesian coordinates by

$$H = \frac{1}{2M}(p_x^2 + p_y^2) + \frac{M\omega^2}{2}(x^2 + y^2), \qquad (1)$$

where ω is the angular frequency of the oscillator. Since the potential depends only on $x^2 + y^2 = r^2$, the angular momentum

$$L_z = xp_y - yp_x \qquad (2)$$

is a constant of the motion.

The Hamilton-Jacobi equation for the Hamiltonian (1) is separable in cartesian coordinates; in fact, this equation reads

$$\frac{1}{2M}\left[\left(\frac{\partial S}{\partial x}\right)^2+\left(\frac{\partial S}{\partial y}\right)^2\right] + \frac{M\omega^2}{2}(x^2+y^2)+\frac{\partial S}{\partial t}=0 \quad (3)$$

and looking for a complete solution of Eq. (3) of the form $S=f(x)+g(y)-Et$, one gets

$$\frac{1}{2M}\left(\frac{df}{dx}\right)^2+\frac{M\omega^2}{2}x^2=\frac{E}{2}+\lambda,$$
$$\frac{1}{2M}\left(\frac{dg}{dy}\right)^2+\frac{M\omega^2}{2}y^2=\frac{E}{2}-\lambda, \quad (4)$$

where λ is a separation constant whose meaning can be obtained by subtracting Eqs. (4)

$$2\lambda=\frac{1}{2M}\left[\left(\frac{df}{dx}\right)^2-\left(\frac{dg}{dy}\right)^2\right]+\frac{M\omega^2}{2}(x^2-y^2)$$

or, taking into account that $df/dx=\partial S/\partial x=p_x$ and $dg/dy=\partial S/\partial y=p_y$,

$$\lambda=\frac{1}{4M}(p_x^2-p_y^2)+\frac{M\omega^2}{4}(x^2-y^2). \quad (5)$$

Thus, Eq. (5) gives a second constant of the motion and, by virtue of Poisson's theorem, the Poisson bracket of L_z and λ is also a constant of the motion. By means of a straightforward computation one finds that

$$\{L_z,\lambda\}=\frac{1}{M}p_xp_y+M\omega^2xy. \quad (6)$$

The use of Poisson's theorem does not lead to further constants of the motion. In fact, denoting the functions L_z, λ and $\{L_z,\lambda\}$ by $2S_3$, $-\omega S_1$ and $-\omega S_2$, respectively, *i.e.*,

$$S_1\equiv\frac{1}{4M\omega}(p_y^2-p_x^2)-\frac{M\omega}{4}(y^2-x^2),$$
$$S_2\equiv-\frac{1}{2M\omega}p_xp_y+\frac{M\omega}{2}xy, \quad (7)$$
$$S_3\equiv\frac{1}{2}(xp_y-yp_x),$$

one finds that

$$\{S_i,S_j\}=\sum_{k=1}^{3}\epsilon_{ijk}S_k \quad (8)$$

(the definitions (7) differ slightly from those employed in Refs. 1 and 6).

Equations (8) imply that the S_i are generating functions of a group of transformations in phase space isomorphic to SU(2) (the group of 2×2 unitary matrices with unit determinant) or SO(3) (the group of 3×3 real orthogonal matrices with determinant $+1$, which represents the rotations in three dimensions). As is well known, the groups SU(2) and SO(3) are locally isomorphic; there is a two-to-one homomorphism of SU(2) onto SO(3) such that if $U\in$ SU(2), then U and $-U$ correspond to the same element of SO(3). It turns out that the group generated by the functions (7) is isomorphic to SU(2) (see, *e.g.*, Ref. 6), which is related to the fact that if in the transformation generated by S_i, the parameter is set equal to 2π, one obtains minus the identity transformation.

From Eqs. (7) and (8) it follows that $\{L_z,S_1\}=2S_2$ and $\{L_z,S_2\}=-2S_1$, which means that

$$(A_{ij})\equiv\begin{pmatrix}S_1 & S_2\\ S_2 & -S_1\end{pmatrix}$$

and

$$(B_{ij})\equiv\begin{pmatrix}S_2 & -S_1\\ -S_1 & -S_2\end{pmatrix} \quad (9)$$

transform under rotations in the plane as the cartesian components of a symmetric tensor. The eigenvectors of (A_{ij}) coincide with the axes of the orbit of the oscillator (which is an ellipse centered at the origin), whereas the eigenvectors of (B_{ij}) bisect the axes of the orbit.

3. Equivalence of the Kepler problem with negative energy and the TIHO

The Hamiltonian for the Kepler problem in two dimensions, written in cartesian coordinates, is

$$H=\frac{1}{2M}(p_x^2+p_y^2)-\frac{k}{\sqrt{x^2+y^2}}, \quad (10)$$

where k is a positive constant. In order to show a connection between the Kepler problem and the TIHO, we express the Hamiltonian (10) in terms of the parabolic coordinates u, v, defined by $x=\frac{1}{2}(u^2-v^2)$, $y=uv$,

$$H=\frac{1}{2M}\frac{1}{u^2+v^2}(p_u^2+p_v^2)-\frac{2k}{u^2+v^2}, \quad (11)$$

where p_u and p_v are the canonical momenta conjugate to u and v, respectively. Hence, the hypersurface in phase space defined by $H=E$ corresponds to

$$\frac{1}{2M}(p_u^2+p_v^2)-E(u^2+v^2)=2k. \quad (12)$$

In other words, $H=E$ is equivalent to the condition $h_E=2k$, where

$$h_E\equiv\frac{1}{2M}(p_u^2+p_v^2)-E(u^2+v^2)$$

is an auxiliary Hamiltonian (conjugate to a fictitious time) that depends parametrically on E and is of the form of Eq. (1) if $E<0$, with u and v in place of x and y, and $M\omega^2/2$ replaced by $-E$ (see also Refs. 3–5). Therefore, making these substitutions in Eqs. (7) one obtains three generating functions of canonical transformations that leave the hypersurface $h_E = 2k$ or, equivalently, the hypersurface $H = E$, invariant. This means that we have at once three constants of the motion for the Kepler problem with negative energy, which satisfy the Poisson bracket relations (8), by simply replacing x, y, p_x, p_y and ω in Eqs. (7) by u, v, p_u, p_v and $\sqrt{-2E/M}$, respectively. This gives

$$S_1 = \frac{1}{\sqrt{-2ME}}\left[\frac{1}{4}(p_v^2 - p_u^2) - \frac{ME}{2}(v^2 - u^2)\right],$$
$$S_2 = -\frac{1}{\sqrt{-2ME}}\left[\frac{1}{2}p_u p_v - MEuv\right], \tag{13}$$
$$S_3 = \frac{1}{2}(up_v - vp_u).$$

(It may be noticed that, in order to obtain the equations for a four-dimensional isotropic harmonic oscillator from the Kepler problem in three dimensions by means of the Kustaanheimo-Stiefel transformation, it is also necessary to introduce a fictitious time. A similar change of the independent variable was introduced in Ref. 7.)

The constants of the motion (13) can now be written in terms of the cartesian coordinates using the fact that

$$p_u = \frac{\partial L}{\partial \dot{u}} = \frac{\partial L}{\partial \dot{x}}\frac{\partial \dot{x}}{\partial \dot{u}} + \frac{\partial L}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \dot{u}}$$
$$= p_x\frac{\partial x}{\partial u} + p_y\frac{\partial y}{\partial u}$$
$$= up_x + vp_y$$

and, similarly, $p_v = -vp_x + up_y$. Substituting these relations into Eqs. (13) we get, for instance, $S_3 = \frac{1}{2}[u(-vp_x + up_y) - v(up_x + vp_y)] = \frac{1}{2}(u^2 - v^2)p_y - uvp_x = xp_y - yp_x$. In this manner we find that

$$S_1 = \frac{1}{\sqrt{-2ME}}\left[p_y(xp_y - yp_x) - \frac{Mkx}{r}\right],$$
$$S_2 = \frac{1}{\sqrt{-2ME}}\left[-p_x(xp_y - yp_x) - \frac{Mky}{r}\right], \tag{14}$$
$$S_3 = xp_y - yp_x,$$

where we have used that $E = (p_x^2 + p_y^2)/2M - k/r$.

Even though the constants of the motion (14) have been obtained from Eqs. (7) by means of coordinate transformations, the group generated by the functions (14) is not SU(2) but SO(3). Since $x = \frac{1}{2}(u^2 - v^2)$, $y = uv$,

$$p_x = \frac{up_u - vp_v}{u^2 + v^2}, \qquad p_y = \frac{up_v + vp_u}{u^2 + v^2}, \tag{15}$$

(u, v, p_u, p_v) and $(-u, -v, -p_u, -p_v)$ correspond to the same point (x, y, p_x, p_y) of the phase space, therefore, for a given element of SO(3), the action of the two corresponding SU(2) transformations, U and $-U$, on the variables (u, v, p_u, p_v) yields the same result on the variables (x, y, p_x, p_y). In particular, the 2π rotation generated by S_i, being minus the identity transformation on the variables (u, v, p_u, p_v), corresponds to the identity transformation on (x, y, p_x, p_y). [Note that the factor 1/2 connecting S_3 and L_z in Eqs. (7), which is necessary in order to obtain the relations (8), is absent in Eqs. (14); whereas in the latter case a 2π rotation generated by S_3 is the identity, in the former case it takes a 4π rotation to obtain the identity transformation.]

From Eqs. (8) and (14) we now have that $\{L_z, S_1\} = \{S_3, S_1\} = S_2$ and $\{L_z, S_2\} = \{S_3, S_2\} = -S_1$, which means that (S_1, S_2) and $(-S_2, S_1)$ transform under rotations in the plane as the cartesian components of a vector. In fact, $(S_1, S_2) = (A_x, A_y)/\sqrt{-2ME}$, where

$$\mathbf{A} \equiv \mathbf{p} \times \mathbf{L} - \frac{Mk\mathbf{r}}{r} \tag{16}$$

is the Hermann-Bernoulli-Laplace-Runge-Lenz (HBLRL) vector, which points in the direction of the point of closest approach of the orbit to the origin (see, *e.g.*, Refs. 1 and 2). Clearly, the vector $(-S_2, S_1)$ is orthogonal to (S_1, S_2). Thus, as in the case of the TIHO, S_1 and S_2 determine the orientation of the orbit.

The fact that the orbits in the Kepler problem with negative energy are ellipses with a focus at the origin follows from the fact that the orbits of the TIHO are ellipses centered at the origin. Indeed, by writing the complex variables $u + iv$ and $x + iy$ in polar form as $u + iv = \rho e^{i\phi}$ and $x + iy = re^{i\theta}$, we see that $x + iy = \frac{1}{2}(u^2 - v^2) + iuv = \frac{1}{2}(u + iv)^2 = \frac{1}{2}\rho^2 e^{2i\phi}$, therefore

$$r = \tfrac{1}{2}\rho^2, \qquad \theta = 2\phi \tag{17}$$

(see also Refs. 4 and 5). From Eq. (12) it follows that the orbits in the Kepler problem expressed in terms of u and v are ellipses centered at the origin. Considering, for example, an ellipse with semi-axes a, b ($a \geq b$) such that the semimajor axis coincides with the v axis, we have $u^2/b^2 + v^2/a^2 = 1$ or, equivalently, $(\rho^2 \cos^2\phi)/b^2 + (\rho^2 \sin^2\phi)/a^2 = 1$. Making use of Eqs. (17) one finds that this last equation amounts to $2r[\cos^2(\theta/2)/b^2 + \sin^2(\theta/2)/a^2] = 1$, hence

$$r = \frac{\dfrac{a^2b^2}{a^2 + b^2}}{1 + \dfrac{a^2 - b^2}{a^2 + b^2}\cos\theta}, \tag{18}$$

which is the equation of an ellipse with one focus at the origin and semi-axes $(a^2 + b^2)/4$ and $ab/2$. The second equation in (17) implies that an ellipse in the variables u and v whose major axis forms an angle γ with the u axis, corresponds to an ellipse in the variables x and y whose major axis forms an angle 2γ with the x axis.

4. Dynamical symmetry of the Kepler problem with positive or zero energy

According to the preceding results, one can guess that the Kepler problem in two dimensions with positive energy is related with a TIHO with imaginary frequency, whose Hamiltonian is obtained from Eq. (1) replacing ω^2 by $-\omega^2$:

$$H = \frac{1}{2M}(p_x^2 + p_y^2) - \frac{M\omega^2}{2}(x^2 + y^2), \tag{19}$$

where ω is a real constant. It is convenient to introduce the two-component complex vector

$$\psi \equiv \begin{pmatrix} p_y + ip_x \\ M\omega(x+iy) \end{pmatrix} \tag{20}$$

(*cf.* Ref. 6), in terms of which the Hamiltonian (19) can be written as

$$H = \frac{1}{2M}\psi^\dagger \eta \psi, \tag{21}$$

where $\psi^\dagger$ is the adjoint of ψ and

$$\eta \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{22}$$

Proceeding as in section 2, one can obtain the following three constants of the motion [*cf.* Eqs. (7)]

$$\begin{aligned} K_1 &\equiv \frac{1}{4M\omega}(p_y^2 - p_x^2) - \frac{M\omega}{4}(y^2 - x^2), \\ K_2 &\equiv -\frac{1}{2M\omega}p_x p_y + \frac{M\omega}{2}xy, \\ K_3 &\equiv \frac{1}{2}(xp_y - yp_x). \end{aligned} \tag{23}$$

These functions satisfy the relations

$$\{K_1, K_2\} = -K_3, \qquad \{K_2, K_3\} = K_1, \qquad \{K_3, K_1\} = K_2, \tag{24}$$

which means that the group of canonical transformations generated by the K_i is isomorphic to SU(1,1) (the group of 2×2 complex matrices with unit determinant, U, such that $U^\dagger \eta U = \eta$), SL(2,**R**) (the group of 2×2 real matrices with unit determinant) or SO$^\uparrow$(2,1) (the group of 3×3 real matrices with unit determinant, $L = (L^i{}_j)$, such that $L^{\mathrm{t}} g L = g$, where $g \equiv \mathrm{diag}(1, 1, -1)$, and $L^3{}_3 > 0$, which represents the "proper orthochronous Lorentz transformations" in two space dimensions).

Making use of the 2×2 complex matrices

$$\tau_1 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \tau_2 \equiv \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \qquad \tau_3 \equiv \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \tag{25}$$

which satisfy the commutation relations

$$[\tau_1, \tau_2] = -2\tau_3, \quad [\tau_2, \tau_3] = 2\tau_1, \quad [\tau_3, \tau_1] = 2\tau_2 \tag{26}$$

(*cf.* Eqs. (24)), the constants of the motion (23) can be expressed in the form

$$K_i = \frac{1}{4M\omega}\mathrm{Im}(\psi^{\mathrm{t}} \epsilon \tau_i \psi), \tag{27}$$

where Im denotes the imaginary part and

$$\epsilon \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{28}$$

The matrices τ_i satisfy the condition $\tau_i^\dagger \eta + \eta\, \tau_i = 0$; therefore, they form a basis of the Lie algebra of SU(1,1). Since the only nonvanishing Poisson brackets between the components of ψ and their conjugates are $\{\psi_1, \overline{\psi_2}\} = -2iM\omega = -\{\psi_2, \overline{\psi_1}\}$, using Eq. (27), one can show that the Poisson bracket relations (24) follow from Eqs. (26).

The rate of change of any function f under the one-parameter group of canonical transformations generated by a function G is given by $df/ds = \{f, G\}$, hence, from Eq. (27) it follows that under the transformations generated by K_i, $d\psi/ds = \frac{1}{2}\tau_i\psi$, therefore,

$$\psi(s) = \exp\left(\frac{s\tau_i}{2}\right)\psi(0), \tag{29}$$

which, taking into account that the τ_i form a basis of the Lie algebra of SU(1,1), means that the symmetry group of the Hamiltonian (19) generated by the constants of motion (23) is isomorphic to SU(1,1) (which, in turn, is isomorphic to SL(2,**R**)). Note that it can be directly verified that if $U \in \mathrm{SU}(1,1)$, then the transformation $\psi \mapsto U\psi$ leaves the Hamiltonian (21) invariant; the preceding computations show that these transformations are generated by the functions K_i and that they are canonical transformations. [Note also that the Hamiltonian (19) is actually invariant under the group SO(2,2), but not all these transformations are canonical. The groups SO(2,2) and SU(1,1) turn out to be related since SO(2,2) is locally isomorphic to $\mathrm{SU}(1,1) \times \mathrm{SU}(1,1)$. Similarly, the Hamiltonian (1) is invariant under the group SO(4), but not all these transformations are canonical. The group SO(4) is related with SU(2), which is a dynamical symmetry group of (1), being locally isomorphic to $\mathrm{SU}(2) \times \mathrm{SU}(2)$.]

In the present case, the symmetry axes of the orbit, which is a hyperbola centered at the origin, coincide with the eigenvectors of the symmetric matrix

$$\begin{pmatrix} K_1 & K_2 \\ K_2 & -K_1 \end{pmatrix}$$

[*cf.* Eqs. (9)].

We now turn to the Kepler problem with positive energy. The hypersurface in phase space defined by $H = E$ also corresponds to Eq. (12), which now is of the form (19) with x, y and $M\omega^2/2$ replaced by u, v and E, respectively. Then, from Eqs. (23) we obtain the constants of the motion

$$K_1 = \frac{1}{\sqrt{2ME}}\left[\frac{1}{4}(p_v^2 - p_u^2) - \frac{ME}{2}(v^2 - u^2)\right],$$
$$K_2 = -\frac{1}{\sqrt{2ME}}\left[\frac{1}{2}p_u p_v - MEuv\right], \tag{30}$$
$$K_3 = \frac{1}{2}(up_v - vp_u).$$

and by comparison with Eqs. (13) and (14) we find that, in terms of the cartesian coordinates,

$$K_1 = \frac{1}{\sqrt{2ME}}\left[p_y(xp_y - yp_x) - \frac{Mkx}{r}\right] = \frac{A_x}{\sqrt{2ME}},$$
$$K_2 = \frac{1}{\sqrt{2ME}}\left[-p_x(xp_y - yp_x) - \frac{Mky}{r}\right] = \frac{A_y}{\sqrt{2ME}},$$
$$K_3 = xp_y - yp_x, \tag{31}$$

where A_x and A_y are the nonvanishing components of the HBLRL vector (16). Among other things, Eqs. (31) imply that the HBLRL vector is conserved.

Owing to the fact that (u, v, p_u, p_v) and $(-u, -v, -p_u, -p_v)$ correspond to the same point (x, y, p_x, p_y), a transformation $U \in \mathrm{SU}(1,1)$ and its negative, $-U$, produce the same effect on a point (x, y, p_x, p_y), of the phase space and since there exists a two-to-one homomorphism of SU(1,1) onto $\mathrm{SO}^\uparrow(2,1)$ such that, for $U \in \mathrm{SU}(1,1)$, U and $-U$ correspond to the same $\mathrm{SO}^\uparrow(2,1)$ transformation (see, *e.g.*, Ref. 8), it follows that the group generated by the constants of the motion (31) is isomorphic to $\mathrm{SO}^\uparrow(2,1)$.

When $E > 0$, from Eq. (12) it follows that the orbits expressed in terms of u and v are hyperbolas centered at the origin. Assuming that the orbit is given by $u^2/b^2 - v^2/a^2 = 1$ or $(\rho^2\cos^2\phi)/b^2 - (\rho^2\sin^2\phi)/a^2 = 1$, using Eqs. (17) one finds that this equation amounts to

$$r = \frac{\dfrac{a^2b^2}{a^2 - b^2}}{1 + \dfrac{a^2 + b^2}{a^2 - b^2}\cos\theta}, \tag{32}$$

which means that the orbits in the Kepler problem with positive energy are hyperbolas with one focus at the origin.

Finally, we shall consider the Hamiltonian (1) with $\omega = 0$

$$H = \frac{1}{2M}(p_x^2 + p_y^2). \tag{33}$$

From Eqs. (5) and (6) it follows that in this case

$$N_1 \equiv \frac{1}{4M}(p_y^2 - p_x^2), \qquad N_2 \equiv -\frac{1}{2M}p_x p_y,$$
$$N_3 \equiv \frac{1}{2}(xp_y - yp_x), \tag{34}$$

are constants of the motion and one finds that

$$\{N_1, N_2\} = 0, \qquad \{N_2, N_3\} = N_1,$$
$$\{N_3, N_1\} = N_2. \tag{35}$$

Therefore, the group of canonical transformations generated by the N_i is isomorphic to SE(2), the Euclidean group of the plane, formed by the rigid motions of the Euclidean plane. The orbits are straight lines and the orientation of the orbit is determined by the eigenvectors of the matrix

$$\begin{pmatrix} N_1 & N_2 \\ N_2 & -N_1 \end{pmatrix}$$

(one eigenvector is parallel to the orbit and the other is orthogonal to it).

Turning now to the Kepler problem in two dimensions with zero energy, using the parabolic coordinates one gets Eq. (12) with $E = 0$, which is of the form (33) with p_u and p_v in place of p_x and p_y. Hence, from Eqs. (34) it follows that

$$N_1 \equiv \frac{1}{4M}(p_v^2 - p_u^2), \qquad N_2 \equiv -\frac{1}{2M}p_u p_v,$$
$$N_3 \equiv \frac{1}{2}(up_v - vp_u), \tag{36}$$

are constants of the motion for the Kepler problem with zero energy. In terms of the cartesian coordinates, the functions (36) take the form

$$N_1 = \frac{1}{M}\left[p_y(xp_y - yp_x) - \frac{Mkx}{r}\right] = \frac{A_x}{M},$$
$$N_2 = \frac{1}{M}\left[-p_x(xp_y - yp_x) - \frac{Mky}{r}\right] = \frac{A_y}{M}, \tag{37}$$
$$N_3 = xp_y - yp_x,$$

where we have made use of the fact that $E = 0$. Thus, again, we obtain the HBLRL vector starting from constants of the motion for the TIHO (in the present case with zero frequency).

According to the preceding results, in terms of u and v, the orbits correspond to straight lines. Considering an orbit given by $u = a$ or $\rho\cos\phi = a$, using Eqs. (17) one obtains $r = a^2/[2\cos^2(\theta/2)]$ or, equivalently

$$r = \frac{a^2}{1 + \cos\theta}. \tag{38}$$

Thus, the orbits in the Kepler problem with zero energy are parabolas with the focus at the origin.

5. Concluding remarks

We have shown that the separability of the Hamilton-Jacobi equation in an appropriate coordinate system allows us to find constants of the motion associated with hidden symmetries.

Whereas it is easy to find constants of the motion for the Kepler problem and to obtain the Poisson bracket relations between them, it is a difficult task to identify the corresponding symmetry group and to find the explicit expression for its action on phase space; however, making use of the relationship between the Kepler problem and the TIHO presented here, one can give an explicit expression for the action of SO(3) as a dynamical symmetry group of the Kepler problem with negative energy.

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