On the dynamical symmetry of the quantum Kepler problem

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Using the fact that the Schrödinger equation for the two-dimensional Kepler problem with negative energy is equivalent to an integral equation on the unit sphere in the three-dimensional space, the eigenfunctions and the generators of a dynamical symmetry group for this problem are obtained from the usual spherical harmonics and the angular momentum operators on the sphere. It is shown that if the spherical harmonics are eigenfunctions of L_y, instead of L_z, the corresponding eigenfunctions of the Schrödinger equation are separable in parabolic coordinates. It is also shown that in the case of zero energy, the Schrödinger equation for the Kepler problem in two or three dimensions is equivalent to an integral equation on the two- or three-dimensional Euclidean space, respectively.

Keywords: Kepler problem; dynamical symmetry group

Usando el hecho de que la ecuación de Schrödinger para el problema de Kepler en dos dimensiones con energía negativa equivale a una ecuación integral sobre la esfera de radio 1 en el espacio tridimensional, se obtienen las eigenfunciones y los generadores de un grupo de simetría dinámica para este problema a partir de los armónicos esféricos usuales y los operadores de momento angular sobre la esfera. Se muestra que si los armónicos esféricos son eigenfunciones de L_y, en lugar de L_z, las eigenfunciones correspondientes de la ecuación de Schrödinger son separables en coordenadas parabólicas. Se muestra también que en el caso de energía cero, la ecuación de Schrödinger para el problema de Kepler en dos o tres dimensiones equivale a una ecuación integral sobre el espacio Euclideano de dimensión dos o tres, respectivamente.

Descriptores: Problema de Kepler; grupo de simetría dinámica

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1. Introducción

It is a well known fact that in the Kepler problem (in classical or quantum mechanics), besides the angular momentum, there exists another conserved vector, known as the Hermann-Bernoulli-Laplace-Runge-Lenz (HBLRL) vector; whereas the conservation of the angular momentum follows from the invariance of the potential under rotations, the conservation of the HBLRL vector is associated with a "hidden symmetry", that is, with transformations that mix the position and momentum variables, leaving the Hamiltonian invariant (see, *e.g.*, Refs. 1–5).

In the case where the energy is negative, the commutation relations (or the Poisson brackets) between the components of the angular momentum and of the HBLRL vector coincide with those of a basis for the generators of rotations in the four-dimensional Euclidean space (or of the rotations in three dimensions, if one considers the Kepler problem in two dimensions) (see, *e.g.*, Refs. 3–10). Fock [1] showed that the corresponding Schrödinger equation possesses, in effect, a symmetry group isomorphic to the group of rotations in four dimensions by transforming this equation into an equation on the unit sphere in the four-dimensional space, where the symmetry becomes obvious. This transformation allows to find the energy levels and to express the eigenfunctions in terms of hyperspherical surface harmonics [1, 3, 4].

When the energy is positive, the commutation relations between the components of the angular momentum and of the HBLRL vector correspond to those of a set of generators of the Lorentz group and the Schrödinger equation can be transformed into an equation on a two-sheeted hyperboloid in Minkowski space where the global action of the Lorentz group can be seen explicitly [1, 4]. Finally, when the energy is equal to zero, the commutation relations of the angular momentum and the HBLRL vector coincide with those of a set of generators of the Euclidean group in three dimensions and in Ref. 4 it is shown that the Schrödinger equation can be transformed into an equation on a paraboloid. However, since the rotations in four dimensions are the isometries of the unit sphere and the Lorentz transformations are the isometries of an hyperboloid in Minkowski space, one would expect that in the case of zero energy the Schrödinger equation could be transformed into an equation on the Euclidean space, where the symmetry of the Schrödinger equation would be manifest.

In this paper we use the fact that the Schrödinger equation for the two-dimensional Kepler problem with negative energy can be transformed into an equation on the unit sphere in three-dimensional space to find the energy levels and the eigenfunctions explicitly, obtaining a relationship between the generating functions of the associated Legendre functions and of the associated Laguerre polynomials. We show that the HBLRL vector (in two dimensions) can be derived from the expressions for the usual angular momentum operators and that the eigenfunctions of one of the components of the HBLRL vector are the separable solutions of the Schrödinger equation for the $1/r$ potential in parabolic coordinates. The analogue of this result in classical mechanics is given in the Appendix. We also consider the Schrödinger equation for the Kepler problem in two and three dimensions with zero energy, showing that this equation can be transformed into one on the two- or three-dimensional Euclidean space, respectively, whose solutions can be easily obtained.

2. The Schrödinger equation for the bound states of the two-dimensional Kepler problem

In this section we give a treatment of the Schrödinger equation for the Kepler problem with negative energy in two dimensions parallel to that given in Refs. 3 and 4 for the three-dimensional case, following a simpler procedure.

2.1. Invariance of the Schrödinger equation under the three-dimensional rotation group

By expressing the solution of the Schrödinger equation for the two-dimensional Kepler problem,

$$-\frac{\hbar^2}{2M}\nabla^2\psi - \frac{k}{r}\psi = E\psi, \tag{1}$$

as a Fourier transform,

$$\psi(\mathbf{r}) = \frac{1}{2\pi\hbar}\int \Phi(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{r}/\hbar}\, d^2\mathbf{p}, \tag{2}$$

using the fact that $\int (1/r)e^{i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{r}/\hbar}\, d^2\mathbf{r} = 2\pi\hbar/|\mathbf{p}-\mathbf{p}'|$, one obtains the integral equation

$$(p^2 - 2ME)\Phi(\mathbf{p}) = \frac{Mk}{\pi\hbar}\int \frac{\Phi(\mathbf{p}')}{|\mathbf{p}-\mathbf{p}'|}\, d^2\mathbf{p}', \tag{3}$$

where $p \equiv |\mathbf{p}|$. Throughout this section, we shall consider bound states only, for which $E < 0$. Then, by means of the stereographic projection, the vector $\mathbf{p}$ can be replaced by a unit vector $\mathbf{n} = (n_x, n_y, n_z)$ according to [1, 3]

$$\mathbf{p} = (p_x, p_y) = p_0\frac{(n_x, n_y)}{1-n_z}, \tag{4}$$

where

$$p_0 \equiv \sqrt{-2ME}, \tag{5}$$

or, equivalently,

$$\mathbf{n} = (n_x, n_y, n_z) = \frac{(2p_0p_x, 2p_0p_y, p^2 - p_0^2)}{p^2 + p_0^2}. \tag{6}$$

Under the correspondence between $\mathbf{p}$ and $\mathbf{n}$ given by Eqs. (4) and (6), the plane is mapped onto the unit sphere and making use of the spherical coordinates θ, ϕ, of $\mathbf{n}$, from Eq. (4) we find that

$$\begin{aligned}\mathbf{p} &= \frac{p_0}{1-\cos\theta}(\sin\theta\cos\phi, \sin\theta\sin\phi)\\ &= p_0\cot(\theta/2)(\cos\phi, \sin\phi),\end{aligned} \tag{7}$$

therefore,

$$p = p_0\cot\left(\frac{\theta}{2}\right), \qquad d^2\mathbf{p} = \frac{1}{4}p_0^2\csc^4\left(\frac{\theta}{2}\right)d\Omega, \tag{8}$$

where $d\Omega = \sin\theta d\theta d\phi$ is the solid angle element, and

$$\begin{aligned}|\mathbf{p}-\mathbf{p}'| &= \frac{p_0|\mathbf{n}-\mathbf{n}'|}{(1-n_z)^{1/2}(1-n_z')^{1/2}}\\ &= \frac{1}{2}p_0\csc\left(\frac{\theta}{2}\right)\csc\left(\frac{\theta'}{2}\right)|\mathbf{n}-\mathbf{n}'|,\end{aligned} \tag{9}$$

where $\mathbf{n}'$ is the unit vector corresponding to $\mathbf{p}'$ according to Eq. (6). Substituting Eqs. (5), (8) and (9) into Eq. (3) one gets

$$\csc^3\left(\frac{\theta}{2}\right)\Phi(\mathbf{n}) = \frac{Mk}{2\pi\hbar p_0}\int \frac{\csc^3(\theta'/2)\Phi(\mathbf{n}')}{|\mathbf{n}-\mathbf{n}'|}\, d\Omega',$$

hence, by defining

$$\begin{aligned}\hat{\Phi}(\mathbf{n}) &\equiv 2^{-3/2}p_0\csc^3(\theta/2)\Phi(\mathbf{n})\\ &= p_0\left[\frac{p^2+p_0^2}{2p_0^2}\right]^{3/2}\Phi(\mathbf{p}),\end{aligned} \tag{10}$$

one arrives at the integral equation

$$\hat{\Phi}(\mathbf{n}) = \frac{Mk}{2\pi\hbar p_0}\int \frac{\hat{\Phi}(\mathbf{n}')}{|\mathbf{n}-\mathbf{n}'|}\, d\Omega'. \tag{11}$$

The constant factors included in the definition (10) are chosen in such a way that $\hat{\Phi}$ is dimensionless and $\hat{\Phi}$ is normalized over the sphere if and only if ψ is normalized over the plane [3]. Since the distance between points on the sphere and the solid angle element $d\Omega$ are invariant under rotations of the sphere, Eq. (11) is explicitly invariant under these transformations, thus showing that the rotation group SO(3) is a symmetry group of the original Eq. (1), for $E < 0$. Substituting Eqs. (7), (8) and (10) into Eq. (2) one obtains the wave function $\psi(\mathbf{r})$ in terms of the solution of the integral equation (11)

$$\psi(\mathbf{r}) = \frac{p_0}{2\sqrt{2}\pi\hbar} \int \hat{\Phi}(\theta,\phi) \csc\left(\frac{\theta}{2}\right) \times \exp\{ip_0 \cot(\theta/2)(x\cos\phi + y\sin\phi)/\hbar\}\, d\Omega. \quad (12)$$

The integral equation (11) can be easily solved using the fact that the spherical harmonics form a complete set for the functions defined on the sphere, therefore the function $\hat{\Phi}$ can be expanded in the form

$$\hat{\Phi}(\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\theta,\phi). \quad (13)$$

Substituting Eq. (13) into Eq. (11), making use of the expansion

$$\frac{1}{|\mathbf{n}-\mathbf{n}'|} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} Y^*_{lm}(\theta',\phi') Y_{lm}(\theta,\phi),$$

where θ, ϕ and θ', ϕ' are the spherical coordinates of $\mathbf{n}$ and $\mathbf{n}'$, respectively, and of the orthonormality of the spherical harmonics one obtains

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[1 - \frac{2Mk}{\hbar p_0(2l+1)}\right] a_{lm} Y_{lm}(\theta,\phi) = 0,$$

which implies that, in order to have a nontrivial solution, $2Mk = \hbar p_0(2l+1)$, for some l; hence, according to Eq. (5),

$$E = -\frac{2Mk^2}{\hbar^2(2l+1)^2}, \quad (14)$$

(*cf.* Ref. 10). The $2l+1$ coefficients a_{lm}, $(m = -l, -l+1, \ldots, l)$ are arbitrary and $a_{l'm} = 0$ for all $l' \neq l$. Thus, the degeneracy of the energy level (14) is $2l+1$; all the spherical harmonics of degree l are solutions of Eq. (11), corresponding to the energy (14) and the solutions of the homogeneous integral equation (11) are precisely the eigenfunctions of L^2, the square angular momentum operator.

2.2. Explicit form of the eigenfunctions

According to the preceding results, the solutions of the Schrödinger equation (1), for $E < 0$, are given by Eq. (12),

$$\psi(x,y) = \frac{p_0}{\sqrt{2}\pi\hbar} \int \hat{\Phi}(\theta,\phi) \cos\left(\frac{\theta}{2}\right) \times \exp\{ip_0 \cot(\theta/2)(x\cos\phi + y\sin\phi)/\hbar\}\, d\theta\, d\phi, \quad (15)$$

where $\hat{\Phi}(\theta,\phi)$ is an eigenfunction of L^2. The rotational symmetry of the Hamiltonian (1) suggests the use of polar coordinates, in terms of which Eq. (15) takes the form

$$\begin{aligned}\psi(r,\varphi) &= \frac{p_0}{\sqrt{2}\pi\hbar} \int \hat{\Phi}(\theta,\phi) \cos\left(\frac{\theta}{2}\right) \times \exp\{ip_0 \cot(\theta/2) r\cos(\varphi-\phi)/\hbar\}\, d\theta\, d\phi \\ &= \frac{p_0}{\sqrt{2}\pi\hbar} \int \hat{\Phi}(\theta,\phi) \sum_{m'=-\infty}^{\infty} i^{m'} e^{im'(\varphi-\phi)} \times J_{m'}\left[\frac{p_0 r}{\hbar} \cot\left(\frac{\theta}{2}\right)\right] \cos\left(\frac{\theta}{2}\right) d\theta\, d\phi, \quad (16)\end{aligned}$$

where we have made use of the expansion $e^{ix\sin\theta} = \sum_{m=-\infty}^{\infty} e^{im\theta} J_m(x)$ (see, *e.g.*, Ref. 11, Sect. 4). Taking $\hat{\Phi}(\theta,\phi)$ as a spherical harmonic, $Y_{lm}(\theta,\phi)$, which are the normalized separable eigenfunctions of L^2 in spherical coordinates, from Eq. (16) we obtain

$$\begin{aligned}\psi_{lm}(r,\varphi) &= \frac{p_0}{\sqrt{2}\pi\hbar} \int (-1)^m \left[\frac{2l+1}{4\pi}\frac{(l-m)}{(l+m)}\right]^{1/2} P_l^m(\cos\theta) e^{im\phi} \sum_{m'=-\infty}^{\infty} i^{m'} e^{im'(\varphi-\phi)} J_{m'}\left[\frac{p_0 r}{\hbar}\cot\left(\frac{\theta}{2}\right)\right] \cos\left(\frac{\theta}{2}\right) d\theta\, d\phi \\ &= \frac{p_0}{\hbar}\left[\frac{2l+1}{2\pi}\frac{(l-m)}{(l+m)}\right]^{1/2} (-i)^m \left\{\int_0^\pi P_l^m(\cos\theta) J_m\left[\frac{p_0 r}{\hbar}\cot\left(\frac{\theta}{2}\right)\right] \cos\left(\frac{\theta}{2}\right) d\theta\right\} e^{im\varphi}, \quad (17)\end{aligned}$$

which shows that the separable eigenfunctions of L^2 in spherical coordinates correspond to separable eigenfunctions of the Hamiltonian in polar coordinates. Denoting by I_{lm} the integral between braces in Eq. (17), and introducing an auxiliary parameter t we have

$$\sum_{l=0}^{\infty} (2l+1) I_{lm} t^l = \int_0^\pi \sum_{l=0}^{\infty} (2l+1) P_l^m(\cos\theta) t^l \times J_m\left[\frac{p_0 r}{\hbar}\cot\left(\frac{\theta}{2}\right)\right] \cos\left(\frac{\theta}{2}\right) d\theta$$

therefore, making use of the recurrence relation $(2l+1)\sin\theta P_l^m(\cos\theta) = P_{l+1}^{m+1}(\cos\theta) - P_{l-1}^{m+1}(\cos\theta)$ and of the generating function of the associated Legendre functions,

$$\frac{(2m)!(1-x^2)^{m/2}}{2^m m!(1-2tx+t^2)^{m+1/2}} = \sum_{k=0}^{\infty} P_{m+k}^m(x)t^k$$

(see, *e.g.*, Ref. 12), for $m \geq 0$, we find

$$\sum_{l=0}^{\infty}(2l+1)I_{lm}t^l = \int_0^{\pi} \frac{(1-t^2)t^m}{\sin\theta}$$
$$\times \frac{(2m+2)!(1-\cos^2\theta)^{(m+1)/2}}{2^{m+1}(m+1)!(1-2t\cos\theta+t^2)^{m+3/2}}$$
$$\times J_m\left[\frac{p_0 r}{\hbar}\cot\left(\frac{\theta}{2}\right)\right]\cos\left(\frac{\theta}{2}\right) d\theta.$$

Replacing the variable θ by $\mu \equiv \cot(\theta/2)$ one finds that

$$\sum_{l=0}^{\infty}(2l+1)I_{lm}t^l = \int_0^{\infty} \frac{(1-t^2)t^m}{(1-t)^{2m+3}} \frac{(2m+2)!}{(m+1)!}$$
$$\times \frac{\mu^{m+1}J_m\left[(p_0 r/\hbar)\mu\right]}{\left[\mu^2 + \left(\frac{1+t}{1-t}\right)^2\right]^{m+3/2}} d\mu.$$

The last integral can be evaluated by first differentiating with respect to s the equation

$$\int_0^{\infty} e^{-xs}x^m J_m(x)\, dx = \frac{(2m)!}{2^m m!(s^2+1)^{m+1/2}}$$

(see, *e.g.*, Ref. 11, Sect. 15), which yields

$$\int_0^{\infty} e^{-xs}x^{m+1} J_m(x)\, dx = \frac{(2m+1)!s}{2^m m!(s^2+1)^{m+3/2}}. \quad (18)$$

Using now the fact that $\int_0^{\infty} f(x)J_n(xy)x\,dx = g(y)$ amounts to $\int_0^{\infty} g(x)J_n(xy)x dx = f(y)$ (see, *e.g.*, Ref. 11, Sect. 13), from Eq. (18) one gets

$$\int_0^{\infty} \frac{(2m+1)!}{2^m m!}\frac{s x^{m+1}J_m(xy)}{(x^2+s^2)^{m+3/2}}\, dx = y^m e^{-ys}$$

thus

$$\sum_{l=0}^{\infty}(2l+1)I_{lm}t^l = 2^{m+1}t^m(p_0 r/\hbar)^m$$
$$\times e^{-p_0 r/\hbar}\frac{e^{-(2p_0 r/\hbar)t/(1-t)}}{(1-t)^{2m+1}}$$

and recalling that

$$\frac{e^{-xz/(1-z)}}{(1-z)^{k+1}} = \sum_{n=0}^{\infty} L_n^k(x)z^n,$$

where L_n^k denote the associated Laguerre polynomials (see, *e.g.*, Ref. 13), it follows that

$$\sum_{l=0}^{\infty}(2l+1)I_{lm}t^l = 2^{m+1}e^{-p_0 r/\hbar}\left(\frac{p_0 r}{\hbar}\right)^m t^m$$
$$\times \sum_{k=0}^{\infty} L_k^{2m}\left(\frac{2p_0 r}{\hbar}\right)t^k$$

therefore

$$I_{lm} = \frac{2^{m+1}}{2l+1}e^{-p_0 r/\hbar}\left(\frac{p_0 r}{\hbar}\right)^m L_{l-m}^{2m}\left(\frac{2p_0 r}{\hbar}\right)$$

and the normalized eigenfunctions of the Hamiltonian, for $m \geq 0$, are given by

$$\psi_{lm}(r,\varphi) = \frac{p_0}{\hbar}\left[\frac{2l+1}{2\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2}$$
$$\times (-i)^m \frac{2^{m+1}}{2l+1}e^{-p_0 r/\hbar}\left(\frac{p_0 r}{\hbar}\right)^m L_{l-m}^{2m}\left(\frac{2p_0 r}{\hbar}\right). \quad (19)$$

Using the relations $P_l^{-m} = (-1)^m \frac{(l-m)!}{(l+m)!}P_l^m$ and $J_{-m} = (-1)^m J_m$, from Eq. (17) it follows that

$$\psi_{l,-m} = \psi_{lm}^*. \quad (20)$$

2.3. The generators of the symmetry

As we have shown, Eq. (15) gives a correspondence between the solutions of the integral equation (11) and those of the Schrödinger equation (1). As remarked above, Eq. (11) is explicitly invariant under the rotations of the sphere and, as is well known, a set of generators of these rotations are the angular momentum operators

$$\hat{L}_x = i\hbar\left(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right),$$
$$\hat{L}_y = i\hbar\left(-\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right), \quad (21)$$
$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi},$$

where the $\hat{}$ indicates that these operators act on functions defined on the sphere. Then, by means of the correspondence (12) we can find the operators on the wave functions that correspond to the generators of rotations (21).

From Eqs. (2), (7) and (10) it follows that the function $\hat{\Phi}$ on the sphere corresponding to a given wave function ψ is

$$\hat{\Phi}(\mathbf{n}) = \frac{p_0}{4\sqrt{2}\pi\hbar}\frac{1}{\sin^3(\theta/2)}$$
$$\times \int \psi(\mathbf{r})e^{-ip_0\cot(\theta/2)(x\cos\phi+y\sin\phi)/\hbar}\, d^2\mathbf{r}. \quad (22)$$

By applying, for example, $\hat{L}_x$ to both sides of the last equation one obtains

$$\hat{L}_x\hat{\Phi} = \frac{ip_0}{8\sqrt{2}\pi\hbar}\frac{1}{\sin^3(\theta/2)}\int\psi(\mathbf{r})\left[-3\cot\left(\frac{\theta}{2}\right)\sin\phi + \frac{ip_0}{\hbar}\cot^2\left(\frac{\theta}{2}\right)(2x\sin\phi\cos\phi + y\sin^2\phi - y\cos^2\phi) + \frac{ip_0}{\hbar}y\right] \times \exp\{-ip_0\cot(\theta/2)(x\cos\phi + y\sin\phi)/\hbar\}\,d^2\mathbf{r},$$

which can be rewritten as

$$\frac{\hbar}{8\sqrt{2}\pi\sin^3(\theta/2)}\int\psi(\mathbf{r})\left[3\frac{\partial}{\partial y} + 2x\frac{\partial}{\partial x}\frac{\partial}{\partial y} + y\left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2}\right) - \frac{p_0^2}{\hbar^2}y\right]e^{-i\mathbf{p}\cdot\mathbf{r}/\hbar}\,d^2\mathbf{r} = \frac{\hbar}{8\sqrt{2}\pi\sin^3(\theta/2)}\int e^{-i\mathbf{p}\cdot\mathbf{r}/\hbar}\left[\frac{\partial}{\partial y} + x\frac{\partial}{\partial x}\frac{\partial}{\partial y} + y\left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2}\right) - \frac{p_0^2}{\hbar^2}y\right]\psi(\mathbf{r})\,d^2\mathbf{r},$$

where we have integrated by parts. Now, assuming that ψ satisfies Eq. (1), the last term can be replaced according to $p_0^2\psi = \hbar^2\nabla^2\psi + (2Mk/r)\psi$, hence,

$$\hat{L}_x\hat{\Phi} = \frac{p_0}{4\sqrt{2}\pi\hbar\sin^3(\theta/2)}\int e^{-i\mathbf{p}\cdot\mathbf{r}/\hbar}\frac{1}{p_0}\left[\frac{\hbar^2}{2}\left(\frac{\partial}{\partial y} + 2x\frac{\partial}{\partial x}\frac{\partial}{\partial y} - 2y\frac{\partial^2}{\partial x^2}\right) - \frac{Mky}{r}\right]\psi(\mathbf{r})\,d^2\mathbf{r},$$

which is of the form (22), with ψ replaced by $\frac{1}{p_0}\left[\frac{\hbar^2}{2}\left(\frac{\partial}{\partial y} + 2x\frac{\partial}{\partial x}\frac{\partial}{\partial y} - 2y\frac{\partial^2}{\partial x^2}\right) - \frac{Mky}{r}\right]\psi = \frac{1}{p_0}\left[\frac{\hbar^2}{2}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\frac{\partial}{\partial x} + \frac{\hbar^2}{2}\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) - \frac{Mky}{r}\right]\psi$. Thus, under the correspondence given by Eq. (22), restricted to the solutions of Eq. (1) for a fixed value of E, $\hat{L}_x$ corresponds to the operator $\frac{1}{p_0}\left[\frac{\hbar^2}{2}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\frac{\partial}{\partial x} + \frac{\hbar^2}{2}\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) - \frac{Mky}{r}\right]$, which, apart from the constant factor $(1/p_0)$, coincides with A_y, one of the cartesian components of the HBLRL vector

$$\mathbf{A} \equiv \frac{1}{2}(\mathbf{p}\times\mathbf{L} - \mathbf{L}\times\mathbf{p}) - \frac{Mk\mathbf{r}}{r}, \tag{23}$$

where $\mathbf{L} = \mathbf{r}\times\mathbf{p}$ and $\mathbf{p} = -i\hbar\nabla$ (*cf.* Refs. 3 and 14).

In a similar manner, one finds that the operators $\hat{L}_y$ and $\hat{L}_z$ correspond to $-(1/p_0)A_x$ and L_z, respectively. Thus, one concludes that the components of the HBLRL vector (23) are associated with the SO(3) symmetry of Eq. (1) and that the operators $(1/p_0)A_y$, $-(1/p_0)A_x$ and L_z obey the same commutation relations as $\hat{L}_x$, $\hat{L}_y$ and $\hat{L}_z$.

2.4. Separation of variables in parabolic coordinates

The Schrödinger equation (1) is also known to be separable in parabolic coordinates (see, *e.g.*, Ref. 6). In fact, in terms of the parabolic coordinates u and v defined by $x = \frac{1}{2}(u^2 - v^2)$, $y = uv$, Eq. (1) takes the form

$$-\frac{\hbar^2}{2M}\frac{1}{u^2+v^2}\left(\frac{\partial^2\psi}{\partial u^2} + \frac{\partial^2\psi}{\partial v^2}\right) - \frac{2k}{u^2+v^2}\psi = E\psi. \tag{24}$$

Substituting $\psi = U(u)V(v)$ into Eq. (24), one obtains the separated equations

$$-\frac{\hbar^2}{2M}\frac{d^2U}{du^2} - Eu^2U = (k+\lambda)U,$$
$$-\frac{\hbar^2}{2M}\frac{d^2V}{dv^2} - Ev^2V = (k-\lambda)V, \tag{25}$$

where λ is a separation constant. Each of these separated equations, for $E < 0$, has the form of the Schrödinger equation for a harmonic oscillator (*cf.* Refs. 6, 7 and 15); making use of the dimensionless variables $\xi \equiv \sqrt{p_0/\hbar}\,u$ and $\eta \equiv \sqrt{p_0/\hbar}\,v$, from Eqs. (25) we get

$$-\frac{d^2U}{d\xi^2} + \xi^2U = \frac{2M(k+\lambda)}{\hbar p_0}U,$$
$$-\frac{d^2V}{d\eta^2} + \eta^2V = \frac{2M(k-\lambda)}{\hbar p_0}V,$$

therefore, in order to have well-behaved solutions of Eqs. (25),

$$\frac{M(k+\lambda)}{\hbar p_0} = n_1 + \tfrac{1}{2}, \qquad \frac{M(k-\lambda)}{\hbar p_0} = n_2 + \tfrac{1}{2}, \tag{26}$$

where n_1 and n_2 are non-negative integers, and

$$\psi = Ce^{-\xi^2/2}H_{n_1}(\xi)e^{-\eta^2/2}H_{n_2}(\eta) = Ce^{-p_0u^2/2\hbar}H_{n_1}(\sqrt{p_0/\hbar}\,u)e^{-p_0v^2/2\hbar} \times H_{n_2}(\sqrt{p_0/\hbar}\,v), \tag{27}$$

where the H_n are Hermite polynomials and C is a normalization constant [6].

Since $H_n(-x) = (-1)^nH_n(x)$ and the couples (u,v) and $(-u,-v)$ correspond to the same point (x,y), in order to have a single-valued wave function it is necessary that n_1

and n_2 be both odd or even; hence, $n_1 + n_2$ must always be an even number, $2l$, say, and, for a fixed value of l,

$$m_y \equiv (n_1 - n_2)/2, \tag{28}$$

can take the $(2l+1)$ integral values $-l, -l+1, \ldots, l$. Then, adding Eqs. (26) and recalling Eq. (5) we obtain again $E = -2Mk^2/[\hbar^2(2l+1)^2]$ and subtracting Eqs. (26) we find that the separation constant λ must be quantized according to

$$\lambda = \frac{\hbar p_0}{M} m_y. \tag{29}$$

(Note that the existence of $2l + 1$ different values of m_y for a given value of l means that the degeneracy of the energy levels is $2l + 1$.)

The meaning of the separation constant λ (and, hence, of m_y) can be found by multiplying the first equation in (25) by $v^2 V$, the second one by $u^2 U$ and subtracting, which leads to

$$-\frac{\hbar^2}{2M}\frac{1}{u^2+v^2}\left(v^2\frac{\partial^2}{\partial u^2} - u^2\frac{\partial^2}{\partial v^2}\right)\psi + \frac{k(u^2-v^2)}{u^2+v^2}\psi = \lambda\psi$$

or, equivalently,

$$-\frac{1}{M}\left[-\frac{\hbar^2}{2}\frac{\partial}{\partial y}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) - \frac{\hbar^2}{2}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\frac{\partial}{\partial y} - \frac{Mkx}{r}\right]\psi = \lambda\psi, \tag{30}$$

which is just the condition $(-1/M)A_x\psi = \lambda\psi$ [*cf.* Eq. (23)]. Hence, the separable solutions of Eq. (1) in parabolic coordinates are eigenfunctions of $(-1/p_0)A_x$ (which is the operator on the wave functions corresponding to $\hat{L}_y$), with eigenvalue $M\lambda/p_0 = m_y\hbar$. Thus, whereas the functions on the sphere, $\hat{\Phi}$, corresponding [by means of Eq. (22)] to the separable solutions (19) of Eq. (1) in polar coordinates are the eigenfunctions of $\hat{L}_z$ and L^2, those corresponding to the separable solutions (27) of Eq. (1) in parabolic coordinates are the eigenfunctions of $\hat{L}_y$ and L^2.

It should be remarked that, as we have seen in the preceding paragraph, the conservation of the HBLRL vector follows from the separability of the Schrödinger equation (1) in parabolic coordinates. (As shown in the Appendix, in a similar manner, the conservation of the classical HBLRL vector follows from the separability of the Hamilton-Jacobi equation for the two-dimensional Kepler problem.) In this context, the HBLRL vector arises in a very natural way (compare with the derivation of the classical HBLRL vector given, *e.g.*, in Ref. 16).

We close this section pointing out that from the commutation relations of the angular momentum operators (21), $[\hat{L}_i, \hat{L}_j] = i\hbar\varepsilon_{ijk}\hat{L}_k$, it follows, in the usual way, that the action of $\hat{L}_z \pm i\hat{L}_x$ on an eigenfunction of L^2 and $\hat{L}_y$ yields another eigenfunction of L^2 and $\hat{L}_y$ with the same eigenvalue of L^2 and the eigenvalue of $\hat{L}_y$ shifted by $\pm\hbar$. Therefore, owing to the results of the preceding subsection, the operators $L_z \pm (i/p_0)A_y$ (which correspond to $\hat{L}_z \pm i\hat{L}_x$) raise or lower the eigenvalue m_y of the wave functions

$$\psi_{lm_y} = Ce^{-p_0u^2/2\hbar} H_{l+m_y}(\sqrt{p_0/\hbar}\,u)e^{-p_0v^2/2\hbar} \times H_{l-m_y}(\sqrt{p_0/\hbar}\,v) \tag{31}$$

[see Eqs. (27) and (28)] by one unit. In fact, a straightforward computation, using again the relation $p_0^2\psi = \hbar^2\nabla^2\psi + (2Mk/r)\psi$, gives the simple expression

$$L_z \pm \frac{i}{p_0}A_y = \pm\frac{\hbar}{i}\left(\sqrt{\frac{p_0}{2\hbar}}\,u \mp \sqrt{\frac{\hbar}{2p_0}}\frac{\partial}{\partial u}\right) \times \left(\sqrt{\frac{p_0}{2\hbar}}\,v \pm \sqrt{\frac{\hbar}{2p_0}}\frac{\partial}{\partial v}\right). \tag{32}$$

The operators in the right-hand side of the last equation can be recognized as raising or lowering operators corresponding to the linear harmonic oscillators described by Eqs. (25). Letting $a_1 \equiv \sqrt{p_0/2\hbar}\,u + \sqrt{\hbar/2p_0}\,\partial/\partial u$, $a_1^\dagger \equiv \sqrt{p_0/2\hbar}\,u - \sqrt{\hbar/2p_0}\,\partial/\partial u$, $a_2 \equiv \sqrt{p_0/2\hbar}\,v + \sqrt{\hbar/2p_0}\,\partial/\partial v$, $a_2^\dagger \equiv \sqrt{p_0/2\hbar}\,v - \sqrt{\hbar/2p_0}\,\partial/\partial v$, we have

$$\begin{aligned} L_z + \frac{i}{p_0}A_y &= -i\hbar a_1^\dagger a_2, \\ L_z - \frac{i}{p_0}A_y &= i\hbar a_1 a_2^\dagger, \\ -\frac{1}{p_0}A_x &= \frac{\hbar}{2}(a_1^\dagger a_1 - a_2^\dagger a_2). \end{aligned} \tag{33}$$

These equations correspond to the well-known Schwinger's realization of the Lie algebra of the rotation group in terms of creation and annihilation operators.

3. The quantum-mechanical Kepler problem with zero energy

3.1. Explicit SE(2) invariance of the two-dimensional Kepler problem with zero energy

When $E = 0$, Eq. (3) gives

$$p^2\Phi(\mathbf{p}) = \frac{Mk}{\pi\hbar}\int \frac{\Phi(\mathbf{p}')}{|\mathbf{p} - \mathbf{p}'|}\,d^2\mathbf{p}'. \tag{34}$$

Making now the change of variable

$$\mathbf{p} = \frac{2Mk}{\hbar}\frac{\mathbf{q}}{q^2}, \tag{35}$$

where $\mathbf{q}$ is a dimensionless vector in the plane and $q = |\mathbf{q}|$, we get

$$p = \frac{2Mk}{\hbar q},$$
$$d^2\mathbf{p} = \frac{4M^2k^2}{\hbar^2 q^4}\, d^2\mathbf{q},$$
$$|\mathbf{p} - \mathbf{p}'| = \frac{2Mk}{\hbar}\frac{|\mathbf{q} - \mathbf{q}'|}{qq'},$$

and from Eq. (34) it follows that

$$\hat{\Phi}(\mathbf{q}) = \frac{1}{2\pi}\int \frac{\hat{\Phi}(\mathbf{q}')}{|\mathbf{q} - \mathbf{q}'|}\, d^2\mathbf{q}', \tag{36}$$

where

$$\hat{\Phi}(\mathbf{q}) \equiv \frac{2Mk}{\hbar q^3}\Phi(\mathbf{q}) = \frac{\hbar^2 p^3}{4M^2k^2}\Phi(\mathbf{p}). \tag{37}$$

Equation (36) is explicitly invariant under SE(2), the group of rigid transformations of the Euclidean plane onto itself that do not change the orientation.

The homogeneous integral equation (36) can be easily solved. Substituting $\hat{\Phi}(\mathbf{q}) = \int f(\mathbf{s}) e^{i\mathbf{s}\cdot\mathbf{q}}\, d^2\mathbf{s}$ into Eq. (36) we get

$$\int f(\mathbf{s}) e^{i\mathbf{s}\cdot\mathbf{q}}\, d^2\mathbf{s} = \frac{1}{2\pi}\iint \frac{f(\mathbf{s}) e^{i\mathbf{s}\cdot\mathbf{q}'}\, d^2\mathbf{s}\, d^2\mathbf{q}'}{|\mathbf{q} - \mathbf{q}'|}$$
$$= \int \frac{f(\mathbf{s}) e^{i\mathbf{s}\cdot\mathbf{q}}}{s}\, d^2\mathbf{s}$$

hence, $f(\mathbf{s}) \neq 0$ only if $|\mathbf{s}| = 1$; thus, $\hat{\Phi}(\mathbf{q}) = e^{i\mathbf{s}\cdot\mathbf{q}}$ is a solution of Eq. (36) for any (constant) unit vector $\mathbf{s}$. These solutions are separable in cartesian coordinates: $e^{i\mathbf{s}\cdot\mathbf{q}} = e^{is_x x} e^{is_y y}$, with $\mathbf{q} = (x, y)$ and $\mathbf{s} = (s_x, s_y)$. Since $e^{i\mathbf{s}\cdot\mathbf{q}} = \sum_{m=-\infty}^{\infty} i^m e^{-im\alpha} e^{im\theta} J_m(\rho)$, where (ρ, θ) are the polar coordinates of $\mathbf{q}$ and $\mathbf{s} = (\cos\alpha, \sin\alpha)$, it follows that $\hat{\Phi}(\rho, \theta) = J_m(\rho) e^{im\theta}$ is a separable solution in polar coordinates of Eq. (36), for any integral value of m. Hence, the degeneracy of the energy level $E = 0$ is infinite. It may be noticed that the solutions to Eq. (36) coincide with those of the Helmholtz equation $\nabla^2\hat{\Phi} = -\hat{\Phi}$. A somewhat similar result holds in the case of the integral equation (11), which is equivalent to the eigenvalue equation $L^2\hat{\Phi} = l(l+1)\hbar^2\hat{\Phi}$, since $-\hbar^{-2}L^2$ is the Laplace operator of the sphere; however, in the latter case, all non-negative integral values of l are allowed, while in the case of the Laplace operator of the plane, only one of its eigenvalues is relevant, which is related to the fact that E only takes the value zero.

From Eqs. (2), (35) and (37) it follows that the solutions of the Schrödinger equation (1) with $E = 0$ are related with the functions $\hat{\Phi}$ through

$$\hat{\Phi}(\rho, \theta) = \frac{Mk}{\pi\hbar^2\rho^3}\int \psi(x, y)$$
$$\times \exp\{-2iMk(x\cos\theta + y\sin\theta)/\hbar^2\rho\}\, d^2\mathbf{r}, \tag{38}$$

where, again, ρ, θ are the polar coordinates of $\mathbf{q}$. On the other hand, the generators of the rigid motions of the plane can be chosen as

$$\hat{P}_1 = -i\hbar\left(\cos\theta\frac{\partial}{\partial\rho} - \frac{\sin\theta}{\rho}\frac{\partial}{\partial\theta}\right),$$
$$\hat{P}_2 = -i\hbar\left(\sin\theta\frac{\partial}{\partial\rho} + \frac{\cos\theta}{\rho}\frac{\partial}{\partial\theta}\right), \tag{39}$$
$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\theta}.$$

$\hat{P}_1$ and $\hat{P}_2$ generate translations in the x and y directions, respectively, and $\hat{L}_z$ generates rotations about the origin. The commutation relations of the operators (39) are

$$[\hat{P}_1, \hat{P}_2] = 0,$$
$$[\hat{L}_z, \hat{P}_1] = i\hbar\hat{P}_2, \tag{40}$$
$$[\hat{L}_z, \hat{P}_2] = -i\hbar\hat{P}_1.$$

The operators on the wave functions corresponding to the generators of the symmetry transformations (39) can be obtained following the same steps as in Sect. 2.3, making use of Eq. (38). One finds, for instance,

$$\hat{P}_1\hat{\Phi} = \frac{Mk}{\pi\hbar^2\rho^3}\int e^{-i\mathbf{p}\cdot\mathbf{r}/\hbar}\frac{\hbar^3}{2Mk}$$
$$\times\left[-\frac{\partial}{\partial x} + x\left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2}\right) - 2y\frac{\partial}{\partial x}\frac{\partial}{\partial y}\right]\psi(x, y)\, d^2\mathbf{r}$$

hence, recalling that in the present case $\hbar^2\nabla^2\psi + (2Mk/r)\psi = 0$, the operator corresponding to the generator of translations $\hat{P}_1$ is

$$\frac{\hbar^3}{2Mk}\left[-\frac{\partial}{\partial x} + 2x\frac{\partial^2}{\partial y^2} + \frac{2Mk}{\hbar^2 r}x - 2y\frac{\partial}{\partial x}\frac{\partial}{\partial y}\right] =$$
$$\frac{\hbar}{Mk}\left\{\frac{\hbar^2}{2}\left[\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\frac{\partial}{\partial y}\right.\right.$$
$$\left.\left.+\frac{\partial}{\partial y}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right] + \frac{Mkx}{r}\right\} = -\frac{\hbar}{Mk}A_x$$

[see Eq. (23)]. In a similar way, one finds that the operators corresponding to $\hat{P}_2$ and $\hat{L}_z$ are $(-\hbar/Mk)A_y$ and L_z, respectively. Thus, A_x, A_y and L_z generate symmetry transformations of the Schrödinger equation (1), as in the case where E is negative.

The fact that L_z is the operator corresponding to $\hat{L}_z$ under the correspondence (38) implies that, under this relationship, the separable solutions in polar coordinates of the

Schrödinger equation with $E = 0$ correspond to the separable solutions in polar coordinates of Eq. (36) [which are of the form $J_m(\rho)e^{im\theta}$]. On the other hand, since Eq. (30) holds for any value of E, the separable solutions in parabolic coordinates of the Schrödinger equation with $E = 0$ are eigenfunctions of A_x and, therefore, they correspond to eigenfunctions of $\hat{P}_1$, which are the separable solutions in cartesian coordinates of Eq. (36).

3.2. Symmetry of the Schrödinger equation for the three-dimensional Kepler problem with zero energy

Considering now the Schrödinger equation (1) in three dimensions, writing

$$\psi(\mathbf{r}) = \frac{1}{(2\pi\hbar)^{3/2}} \int \Phi(\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{r}/\hbar}\, d^3\mathbf{p},$$

one obtains the integral equation

$$(p^2 - 2ME)\Phi(\mathbf{p}) = \frac{Mk}{\pi^2\hbar} \int \frac{\Phi(\mathbf{p}')}{|\mathbf{p} - \mathbf{p}'|^2}\, d^3\mathbf{p}', \qquad (41)$$

which takes the place of Eq. (3). Making $E = 0$ and $\mathbf{p} = 2Mk\mathbf{q}/(\hbar q^2)$, where $\mathbf{q}$ is a dimensionless vector in three dimensions, one finds that Eq. (41) is equivalent to

$$\hat{\Phi}(\mathbf{q}) = \frac{1}{2\pi^2} \int \frac{\hat{\Phi}(\mathbf{q}')}{|\mathbf{q} - \mathbf{q}'|^2}\, d^3\mathbf{q}', \qquad (42)$$

where

$$\hat{\Phi}(\mathbf{q}) \equiv \left(\frac{2Mk}{\hbar}\right)^{3/2} \frac{\Phi(\mathbf{q})}{q^4} = \left(\frac{\hbar}{2Mk}\right)^{5/2} p^4 \Phi(\mathbf{p}).$$

Equation (42) is manifestly invariant under the rigid transformations of the three-dimensional Euclidean space, which means that this group of transformations constitute a symmetry group of the three-dimensional Kepler problem with zero energy.

It can be easily seen that the functions $\hat{\Phi} = e^{i\mathbf{s}\cdot\mathbf{q}}$, where $\mathbf{s}$ is a constant vector with $|\mathbf{s}| = 1$, and $\hat{\Phi} = j_l(\rho)Y_{lm}(\theta, \phi)$ are separable solutions of Eq. (42) in cartesian coordinates and spherical coordinates, respectively, and these functions are also solutions of the Helmholtz equation $\nabla^2\hat{\Phi} = -\hat{\Phi}$; the existence of these sets of separable solutions corresponds to the separability of the Schrödinger equation (1) in parabolic and spherical coordinates.

4. Concluding remarks

The cases considered in this paper, as well as those treated in Refs. 3 and 4, show the usefulness of exhibiting the underlying symmetry of the quantum Kepler problem, which allows to change the Schrödinger equation by a simpler condition.

The results of Sect. 2 show, among other things, that the usual spherical harmonics are related with the associated Laguerre polynomials and the Hermite polynomials, which form bases for representations of the rotation group; these results also show one of the many connections between the Kepler problem with negative energy and the isotropic oscillator (*cf.* also Ref. 9 and the references cited therein).

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Appendix

The separability of the Schrödinger equation and of the Hamilton-Jacobi equation for the two-dimensional Kepler problem in polar coordinates is a consequence of the invariance of the Hamiltonian under rotations about the center of force (as in the case of any central potential) which, in turn, is equivalent to the conservation of the angular momentum L_z. The $1/r$ potential is distinguished by the existence of another conserved vector—the HBLRL vector—which turns out to be related with the separability of the above-mentioned equations in parabolic coordinates.

The Hamilton-Jacobi equation for the two-dimensional Kepler problem written in the parabolic coordinates u, v, defined by $x = \frac{1}{2}(u^2 - v^2)$, $y = uv$, is

$$\frac{1}{2M}\frac{1}{u^2+v^2}\left[\left(\frac{\partial S}{\partial u}\right)^2 + \left(\frac{\partial S}{\partial v}\right)^2\right] - \frac{2k}{u^2+v^2} + \frac{\partial S}{\partial t} = 0. \qquad (\text{A}.1)$$

Even though both coordinates are non-ignorable, Eq. (A.1) admits separable solutions of the form

$$S(u, v, t) = -Et + f(u) + g(v). \qquad (\text{A}.2)$$

In fact, substituting Eq. (A.2) into Eq. (A.1) one obtains the separated equations

$$\frac{1}{2M}\left(\frac{df}{du}\right)^2 - k - Eu^2 = \lambda,$$
$$\frac{1}{2M}\left(\frac{dg}{dv}\right)^2 - k - Ev^2 = -\lambda, \qquad (\text{A}.3)$$

where λ is a separation constant. Multiplying the first equation in (A.3) by v^2 and the second one by u^2 and subtracting one finds that

$$\frac{1}{2M}\left[v^2\left(\frac{df}{du}\right)^2 - u^2\left(\frac{dg}{dv}\right)^2\right] + k(u^2 - v^2) = \lambda(u^2 + v^2), \quad \text{(A.4)}$$

therefore, since $u^2 + v^2 = 2r$, $df/du = \partial S/\partial u = p_u = p_x(\partial x/\partial u) + p_y(\partial y/\partial u) = up_x + vp_y$ and $dg/dv = \partial S/\partial v = p_v = -vp_x + up_y$, from Eq. (A.4) it follows that

$$\lambda = -\frac{1}{M}\left[(xp_y - yp_x)p_y - \frac{Mkx}{r}\right] = -\frac{A_x}{M},$$

where $\mathbf{A} \equiv \mathbf{p} \times \mathbf{L} - Mk\mathbf{r}/r$ now denotes the classical HBLRL vector [*cf.* Eq. (30)]. From Eqs. (A.3) it follows that, if $E \leq 0$, then $-k \leq \lambda \leq k$ [*cf.* Eqs. (26)].

In a similar way one finds that A_y corresponds to a separation constant using the coordinate system $y = \frac{1}{2}(v^2 - u^2)$, $x = uv$.

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