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Matroid theory and supergravity

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In this work, we consider matroid theory. After presenting three different (but equivalent) definitions of matroids, we mention some of the most important theorems of such theory. In particular, we note that every matroid has a dual matroid and that a matroid is regular if and only if it is binary and includes no Fano matroid or its dual. We show a connection between this last theorem and octonions which at the same time, as it is known, are related to the Englert's solution of D = 11 supergravity. Specifically, we find a relation between the dual of Fano matroid and D = 11 supergravity. Possible applications to M-theory are speculated upon.

Keywords: Matroid theory; Fano matroid

En este trabajo, consideramos la teoría de matroides. Después de presentar tres formas diferentes (pero equivalentes) de definir los matroides, mencionamos algunos de los teoremas más importantes de tal teoría. En particular, hacemos notar que todo matroide tiene un matroide dual y que un matroide es regular si y sólo si es binario y no incluye al matroide de Fano o su dual. Mostramos cómo este último teorema está ligado a los octoniones, los cuales al mismo tiempo, como se sabe, están relacionados con la solución de Englert de supergravedad D = 11. Específicamente, mostramos una relación entre el matroide dual de Fano y supergravedad D = 11. Posibles aplicaciones a la teoría-M son especuladas.

Descriptores: Teoría de matroides; matroide de Fano

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At present, the concept of duality is widely recognized by its central role in non-perturbative dynamics of superstrings [1] and supersymmetric Yang-Mills [2]. In particular, the five known superstring theories (called Type I, Type IIA, Type IIB, Heterotic SO(32) and Heterotic $E_8 \times E_8$) may now be thought, thanks to duality, as different vacua of an underlying unique theory called M-theory [3–9]. This feature of duality in superstring theories is so relevant that lead us to believe that there must be a duality principle supporting M-theory.

M-theory is a non-pertubative theory and in addition to the five superstring theories describes supermembranes [10], 5-branes [11] and D = 11 supergravity [12]. Although the complete M-theory is unknown there are two main proposed routes to construct it. One is the N = (2,1) superstring theory [13] and the other is M(atrix)-theory [14]. Recently, Martinec [15] has suggested that these two scenarios may in fact be closely related.

In this work we propose an alternative formalism to construct M-theory. We propose that the mathematical formalism necessary to support the duality principle in M-theory is matroid theory [16]. As it is known, matroid theory can be understood as a generalization of matrix and graph theory and among its remarkable features is that every matroid has its dual. Since M(atrix)-theory and N = (2,1) superstrings have had an important success in describing some essential aspects of M-theory a natural question is to see whether matroid theory is related to these two approaches. As a first step to answer this question we may attempt to investigate if matroid theory is connected somehow to D = 11 supergravity, which is a common feature of both approaches. In this work, we find that the Fano matroid and its dual are related to Englert's compactification [17] of D = 11 supergravity. This relation is physically interesting for at least two reasons. First, since in matroid theory the Fano matroid plays a fundamental role [18] we should expect that matroids may be helpful to describe some important properties of D = 11 supergravity. Second, it turns out that such a relation allows to connect the Fano matroid with octonions (one of the division algebras [19]) which are possible related with the four forces of nature. In fact, some time ago, Blencowe and Duff [20] raised the question whether the four forces of nature correspond to the four division algebras. If this conjecture turns out to be true then, according to our results, matroids must be deeply connected with the four forces of nature.

Let us start with a brief historical review of matroid theory. It seems that the theory began in 1935 with Whitney's paper [16]. In the same year, Birkhoff [21] established the connection between simple matroids (also known as combinatorial geometries [22]) and geometric lattices. In 1936, Mac Lane [23] gave an interpretation of matroids in terms of projective geometry. And important progress was made in 1958, with two Tutte's papers [18]. At present, there is a large body of information about matroid theory and the closely related combinatorial geometries. Concrete applications may be found in circuit theory, network-flow theory, linear and integer programming and the theory (01)-matrices, for example. For further details about the history of matroid theory and related topics see, for example, the excellent books by Welsh [24], Lawler [25] and Tutte [26]; and also by Wilson [27], Kung [28] and Ribnikov [29].

An interesting feature of matroid theory is that there are many different but equivalent ways of defining a matroid. In this respect, it turns out interesting to briefly review Whitney's [16] original discovery of the matroid concept. While working with linear graphs, Whitney noticed that, for certain matrices, duality had a simple geometrical interpretation quite different that in the case of graphs. He also observed that any subset of columns of a matrix is either linearly independent or linearly dependent and that the following two statements must hold:

(a) any subset of an independent set is independent.

(b) if N_p and N_{p+1} are independent sets of p and $p+1$ columns respectively, then N_p together with some column of N_{p+1} forms an independent set of $p+1$ columns.

Moreover, Whitney discovered that if these two statements are taken as axioms then there are examples of systems satisfying these axioms but not representing any matrix or graph. Thus, he concluded that a system obeying (a) and (b) should be a new one and therefore deserved a new name: matroid.

The definition of a matroid (o pregeometry) in terms of independent sets has been refined and it is now expressed as follows: A matroid M is a pair $(E,\mathcal{I})$, where E is a non-empty finite set, and $\mathcal{I}$ is a non-empty collection of subsets of E (called independent sets) satisfying the following properties:

($\mathcal{I}$ i) any subset of an independent set is independent;

($\mathcal{I}$ ii) if I and J are independent sets with $I \subseteq J$, then there is an element e contained in J but not in I, such that $I \cup \{e\}$ is independent.

A base is defined to be any maximal independent set. By repeatedly using the property *($\mathcal{I}$ ii)* is straightforward to show that any two bases have the same number of elements. A subset of E is said to be dependent if it is not independent. A minimal dependent set is called a circuit. Contrary to the bases not all circuits of a matroid have the same number of elements.

An alternative definition of a matroid in terms of bases is as follows:

A matroid M is a pair $(E, \mathcal{B})$, where E is a non-empty finite set and $\mathcal{B}$ is a non-empty collection of subsets of E (called bases) satisfying the following properties:

($\mathcal{B}$ i) no base properly contains another base;

($\mathcal{B}$ ii) if B_1 and B_2 are bases and if h is any element of B_1, then there is an element g of B_2 with the property that $(B_1 - \{h\}) \cup \{g\}$ is also a base.

It is worth point out that if E is finite set of vectors in a vector space V, then we can define a matroid on E by taking as bases all linearly independent subsets of E which span the same subspace as E; a matroid obtained in this way is called vector matroid.

A matroid can also be defined in terms of circuits:

A matroid M is a pair $(E, \mathcal{C})$, where E is a non-empty finite set, and $\mathcal{C}$ is a collection of a non-empty subsets of E (called circuits) satisfying the following properties.

($\mathcal{C}$ i) no circuit properly contains another circuit;

($\mathcal{C}$ ii) if $\mathcal{C}_1$ and $\mathcal{C}_2$ are two distinct circuits each containing an element c, then there exists a circuit in $\mathcal{C}_1 \cup \mathcal{C}_2$ which does not contain c.

If we start with any of the three definitions $(\mathcal{I})$, $(\mathcal{B})$ and $(\mathcal{C})$ the other two follow as theorems. For instance, it is possible to prove that $(\mathcal{I})$ implies $(\mathcal{B})$ and $(\mathcal{C})$. In other words, these three definitions are equivalent. There are other definitions of a matroid also equivalent to these three, but for the purpose of this work it is not necessary to consider all of them.

Notice that even from the initial structure of a matroid theory we find relations such as independent-dependent and base-circuit which suggest duality. The dual of M, denoted by M^*, is defined as a pair $(E, \mathcal{B}^*)$, where $\mathcal{B}^*$ is a non-empty collection of subsets of E formed with the complements of the bases of M. An immediate consequence of this definition is that every matroid has a dual and this dual is a unique matroid. It also follows that the double-dual M^{**} is equal to M. Moreover, if A is a subset of E, then the size of the largest independent set contained in A is called the rank of A and is denoted by $\rho(A)$. If $M = M_1 + M_2$ and $\rho(M) = \rho(M_1) + \rho(M_2)$ we shall say that M is separable. Any maximal non-separable part of M is a component of M. An important theorem due to Whitney [16] is that if $M_1, \ldots, M_p$ and $M_1^{'}, \ldots, M_p^{'}$ are the components of the matroids M and M' respectively, and if $M_i^{'}$ is a dual of M_i (i = 1, . . . ,p), then M' is a dual of M. Conversely, let M and M' be dual matroids, and let $M_1, \ldots, M_p$ be components of M. Let $M_1^{'}, \ldots, M_p^{'}$ be the corresponding submatroids of M'. Then $M_1^{'}, \ldots, M_p^{'}$ are the components of M', and $M_i^{'}$ is a dual of M_i.

Among the most important matroids we find the binary and regular matroids. A matroid is binary if it is representable over the integers modulo two. Let us clarify this definition. An important problem in matroid theory is to see which matroids can be mapped into some set of vectors in a vector space over a given field. When such a map exists we speak of a coordinatization (or representation) of the matroid over the field. This is equivalent to represent a matroid by a matrix over a given field. (An example of a matroid that cannot be represented as a matrix is a matroid of rank 3, which has 9 elements $\{1,2,3,4,5,6,7,8,9\}$and the following 20 circuits: $\{7,1,2\}$, $\{8,1,4\}$, $\{9,2,3\}$, $\{7,3,4\}$, $\{8,3,6\}$, $\{9,4,5\}$, $\{7,5,6\}$, $\{8,2,5\}$; $\{1,6\}$, $\{1,9\}$, $\{6,9\}$, $\{1,3\}$, $\{1,5\}$, $\{2,4\}$, $\{2,6\}$, $\{3,5\}$, $\{4,6\}$, $\{7,8\}$, $\{7,9\}$, $\{8,9\}$.) Let GF(q) denote a finite field of order q. Thus, we can express the definition of a binary matroid as follows: A matroid which has a coordinatization over GF(2) is called binary. Furthermore, a matroid which has a coordinatization over every field is called regular. It turns out that regular matroids play an important role in matroid theory, among other things, because they play a similar role as planar graphs do in graph theory [27]. It is known that a graph is planar if and only if it contains no subgraph homeomorphic to K_5 or $K_{3,3}$. (Recall that K_n is a simple graph in which every pair of distinct vertices are adjacent, while $K_{r,s}$, where r and s are the number of vertices in two disjoint sets V_1 and V_2, is a complete bipartite graph in which every vertex of V_1 is joined to every vertex of V_2.) The analogue of this theorem for matroids was provided by Tutte [18]. In effect, Tutte showed that a matroid is regular if and only if it is binary and it includes no Fano matroid or its dual. In order to understand this theorem it is necessary to define the Fano matroid which is some times referred to as PG(2,2), the projective plane over FG(2). We shall see that the dual of the Fano matroid is linked with octonions which, at the same time, are connected to the Englert's compactification of D = 11 supergravity.

A Fano matroid F is the matroid defined on the set E $= \{1,2,3,4,5,6,7\}$ whose bases are all those subsets of E with three elements except $f_1 = \{1,2,4\}$, $f_2 = \{2,3,5\}$, $f_3 = \{3,4,6\}$, $f_4 = \{4,5,7\}$, $f_5 = \{5,6,1\}$, $f_6 = \{6,7,2\}$ and $f_7 = \{7,1,3\}$. The circuits of the Fano matroid are precisely these subsets and their complements. It follows that these circuits define the dual F* of the Fano matroid.

Let us write the set E in the form $\mathcal{E} = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$. Thus, the subsets used to define the Fano matroid now become $f_1 = \{e_1, e_2, e_4\}$, $f_2 = \{e_2, e_3, e_5\}$, $f_3 = \{e_3, e_4, e_6\}$, $f_4 = \{e_4, e_5, e_7\}$, $f_5 = \{e_5, e_6, e_1\}$, $f_6 = \{e_6, e_7, e_2\}$ and $f_7 = \{e_7, e_1, e_3\}$. The central idea is to identify the quantities e_i, where $i = 1,2,3,4,5,6$ and 7, with the octonionic imaginary units. Specifically, we write an octonion q in the form $q = q_0e_0 + q_1e_1 + q_2e_2 + q_3e_3 + q_4e_4 + q_5e_5 + q_6e_6 + q_7e_7$. Here, e_0 denotes the identity. The product of any two octonions is determined by the formula

$$e_ie_j = -\delta_{ij} + \psi_{ijk}e_k \tag{1}$$

Here, δ_{ij} is the Kronecker delta and ψ_{ijk} are fully antisymmetric structure constants. By taking the tensor ψ_{ijk} equals 1 for each one of the seven combinations f_i we get all the values of ψ_{ijk}.

The octonion (Cayley) algebra is not associative, but it is alternative. This means that the basic associator of any three imaginary units is

$$(e_i, e_j, e_k) = (e_ie_j)e_k - e_i(e_je_k) = \varphi_{ijkm}e_m, \tag{2}$$

where φ_{ijkl} is a fully antisymmetric tensor. It turns out that φ_{ijkl} and ψ_{ijk} are related by the expression

$$\varphi_{ijkl} = (1/3!)\epsilon_{ijklmnr}\psi_{mnr}, \tag{3}$$

where $\epsilon_{ijklmnr}$ is the completely antisymmetric Levi-Civita tensor. It is interesting to note that associating the numerical values (elements) of the subsets f_i to the indices of ψ_{mnr} and using (3) we get the other seven subsets of E (with four elements) of the dual Fano matroid F*. For instance, if we take f_1, then we have ψ_{124} and (3) gives φ_{3567} which leads to the circuit subset $\{3,5,6,7\}$.

Now, we shall relate the above mathematical structure to the Englert's octonionic solution [17] of eleven dimensional supergravity. First, let us introduce the metric

$$g_{ab} = \delta_{ij}h_a^ih_b^j, \tag{4}$$

where $h_a^i = h_a^i(x^\mu)$ is a sieben-bein. Here, x^μ are coordinates in a patch of the geometrical seven sphere S^7. The quantities ψ_{ijk} can now be related to the S^7 torsion in the form

$$T_{abc} = R_0^{-1}\psi_{ijk}h_a^ih_b^jh_c^k, \tag{5}$$

where R_0 is the S^7 radius. The quantities φ_{ijkl} can be identified with the four-indexed gauge field strength F_{abcd} through the formula

$$F_{abcd} = R_0^{-1}\varphi_{ijkl}h_a^ih_b^jh_c^kh_d^l. \tag{6}$$

Moreover, it is possible to prove that Englert's 7-dimensional covariant equations can be solved with the identification $F_{abcd} = \lambda T_{[abc;d]}$, where λ is a constant. Therefore, $\lambda T_{abc} = A_{abc}$ is the fully antisymmetric gauge field which is a fundamental object in 2-brane theory [6].

It is important to mention that in Englert's solution to D = 11 supergravity the torsion T_{abc} satisfies the Cartan-Schouten equations

$$T_{acd}T_{bcd} = 6R_0^{-2}g_{ab}, \tag{7}$$

$$T_{ead}T_{dbf}T_{fce} = 3R_0^{-2}T_{abc}. \tag{8}$$

But, as Gursey and Tze [30] noted, these equations are mere septad-dressed, *i.e.* covariant forms of the algebraic identities

$$\psi_{ikl}\psi_{jkl} = 6\delta_{ij}, \tag{9}$$

$$\psi_{lim}\psi_{mjn}\psi_{nkl} = 3\psi_{ijk} \tag{10}$$

respectively. It is worth it to mention that Englert's solution realize the riemannian curvature-less but torsion-full Cartan geometries of absolute parallelism on S^7.

Let us conclude by making some final comments. In this work, we have shown that the dual of the Fano matroid is closely related to octonions which at the same time are essential part of the Englert's solution of absolute parallelism on S^7 of D = 11 supergravity. The Fano matroid and its dual are the only minimal binary irregular matroids. We know from Hurwitz theorem (see reference [19]) that octonions is one of the alternative division algebras (the others are the reals, complex numbers and quaternions). While among the only parallelizable spheres we find S^7 (the others are the spheres S^1 and S^3 [31]). This distinctive and fundamental role played by the Fano matroid, octonions and S^7 in such a different areas in mathematics as combinatorial geometry, algebra and topology respectively lead us to believe that the relation between these three concepts must have a deep significate in nature. Of course, it is known that the parallelizability of S^1, S^3 and S^7 has to do with the existence of the complex numbers, quaternions and octonions respectively (see Ref. 32). It is also known that using an algebraic topology called K-theory [33] we find that the only dimensions for division algebras structures on Euclidean spaces are 1, 2, 4, and 8. We may now add to these remarkable results another fundamental concept in matroid theory; the Fano matroid. But besides the importance of the Fano matroid in D = 11 supergravity the matroid theory offer us the possibility to provide the basis for a duality principle in M-theory. This is because among other reasons every matroid has its unique dual matroid. It is interesting to mention that in matroid theory there is a duality principle [34], which establishes that if A is a statement in the theory of matroids that has been proved true, then also its dual A^* is true. Perhaps a duality principles such as "everything in the physical world is dual for an observer" or "the fundamental laws of physics must be dual" may constitute the fundamental principles in M-theory.

For further research, it will be interesting to find the exact relation between D = 11 supergravity and the Fano matroid. It may be also interesting to see if local supersymmetry is connected with matroids and if matroid theory may be helpful to find other solutions of D = 11 supergravity [35]. Moreover, it may be of interest to find the connection between M(atrix)-theory and M(atroid)-theory. At present, we are working in these problems and we hope to report our results elsewhere.

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