A variational approach to some properties of endoreversible heat engines

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Endoreversible heat engine models are irreversible thermal cycles where the whole entropy production is only ascribed to the couplings between the working fluid and its heat reservoirs. This kind of thermodinamic systems have been extensively studied under diverse criteria of merit. In this work we recover the so-called semisum property for an endoreversible engine working in a regime that represents a good compromise between high power output and low entropy production (the ecological regime) by means of a variational approach. We apply variational calculus over functionals corresponding to thermodynamic processes.

Keywords: Heat engine; endoreversibility; variational calculus

Los modelos de máquinas de calor endorreversibles son ciclos térmicos irreversibles donde la producción total de entropía se asigna al acoplamiento entre el fluido de trabajo y sus reservorios de calor. Esta clase de sistemas termodinámicos ha sido estudiada en forma extensiva bajo diversos criterios de mérito. En este trabajo recuperamos la llamada propiedad de la semisuma para una máquina endorreversible trabajando en un régimen que representa un buen compromiso entre alta potencia saliente y baja producción de entropía (el régimen ecológico), utilizando para ello un criterio variacional. Aplicamos el cálculo variacional a funcionales correspondientes a procesos termodinámicos.

Descriptores: Máquina térmica; endorreversibilidad; cálculo variacional

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1. Introduction

In 1975, Curzon and Ahlborn (CA) [1] found that a finite-time Carnot-like thermal engine with a working fluid exchanging heat with its heat reservoirs by means of a linear heat transfer law and working in the regime of maximum power output has an efficiency given by

$$\eta_{CA} = 1 - \sqrt{\frac{T_L}{T_H}}, \tag{1}$$

where T_L and T_H are the absolute temperatures of the cold and hot reservoirs respectively (see Fig. 1). Equation (1) has been obtained by means of different approaches [2–6]. It has been also demonstrated that the result given by Eq. (1) is highly sensitive to the transfer law used for describing the irreversible heat fluxes between the working substance and the thermal reservoirs [5, 7–10], that is, Eq. (1) is an exclusive result for a CA-engine with a Newton law of cooling. For another heat transfer laws, different results from Eq. (1) are obtained. Thus, the CA-formula for the efficiency has not the same class of universality than the Carnot efficiency. A very important ingredient in the CA-thermal engine model is the so-called endoreversibility hypothesis, which consists in assuming that the working fluid undergoes reversible transformations [4] and the whole entropy production of the irreversible engine is only ascribed to the couplings between the working fluid and its surroundings, that is, the thermal resistances (see Fig. 1). The concept of endoreversible cycle was coined by Rubin [4] based in the idea that for many cases the internal relaxation times of the working substance are very short compared with the evolution time of the whole process.

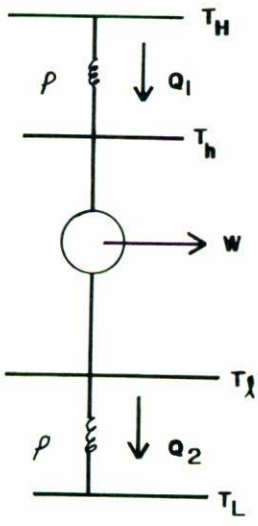

FIGURA 1. Diagram of a CA endoreversible thermal cycle.

Many criteria of merit have been proposed for the study of a CA-engine. Among others, for example minimization of entropy generation [11, 12], maximization of power output [1], minimization of cost [12], and maximization of a kind of ecological function [14]. This last criterion consists of the maximization of a function E that represents a good compromise between high power output and low entropy production. This function is given by

$$E = P - T_L \sigma, \tag{2}$$

where P is the power output of the cycle, σ is the total entropy production (system plus surroundings) per cycle, and T_L is the temperature of the cold reservoir. When function E is maximized, the CA-cycle reaches a configuration which produces around 80% of the maximum power, while entropy production is reduced down to around the 30% of the entropy produced in the maximum power regime [14]. The ecological function has also the property that when the CA-cycle works at maximum E, the thermal efficiency η_E is given by

$$\eta_E \approx \frac{1}{2}\left(\eta_c + \eta_{PM}\right), \tag{3}$$

where η_c is the Carnot efficiency and η_{MP} is the efficiency at the maximum power regime. It has been recently showed [15] that the semisum property expressed by Eq. (3) is independent of the heat transfer law used for describing the heat fluxes at the couplings of the engine. As far as we know, this is the first universal property found for endoreversible heat engine models.

In the CA-paper [1], Eq. (1) was found by the maximization of the power output as a two-variable-function $P = P(x, y)$ with $x = T_H - T_h$ and $y = T_\ell - T_L$ (see Fig. 1). Later De Vos [2, 3] also found Eq. (1) by means of an one-variable formalism, namely $P = P(\eta)$, that is, the power depending on the engine efficiency. It is usual in finite-time thermodynamics to state problems in the context of optimization criteria. Generally, the quantities to optimize are funtionals such as the work, the power and the entropy production which can be expressed as integrals over certain trajectories (the thermodinamic processes). Thus, it results natural to use the variational calculus to treat the optimization problems previously mentioned. In fact, the alternative deduction of Eq. (1) made by Rubin [4] was by maximizing a Langrangian with the power output as the objective funtion and the endoreversibility condition as the integral restriction. The semisum property [Eq. (3)] has been obtained by both procedures, the CA-xy-formalism [14] and the De Vos' η-formalism [10,15]. In this paper we now calculate the semisum property by means of variational calculus for the case of a CA-engine with a linear heat transfer law for describing the heat exchanges between the working fluid and its thermal baths.

2. CA-engine's properties and the variational calculus

2.1. The Rubin's approach

In his 1979 paper, Rubin recovered Eq. (1) by means of the application of the variational calculus to certain thermodynamic functionals. In fact, he treated a CA-engine with the operational goals of maximizing both the power output and the efficiency. For the case of the maximum power regime, equivalently he maximized the work w since he took the cycling period τ as fixed, thus, the work produced in a cycle is given by

$$w = \int_0^\tau p\,\dot{v}\,dt, \tag{4}$$

where p and v are the pressure and volume of the working fluid and $\dot{v}$ means the time derivative of v. By using a local equilibrium condition, the first law of thermodynamics is

$$\dot{q} = \dot{u} + p\dot{v}, \tag{5}$$

where $\dot{q}$ is the heat flux and $\dot{u}$ is the rate of change of the internal energy of the working fluid. Since the process is cyclic Eq. (4) may be rewritten using Eq. (5) as

$$w = \int_0^\tau \dot{q}\,dt. \tag{6}$$

The endoreversibility hypothesis means that the internal entropy change of the fluid in one period is zero, that is

$$\Delta s_w = \int_0^\tau \frac{\dot{q}}{T}\,dt = 0, \tag{7}$$

where T is the working fluid temperature. Thus, to maximize w subject to the endoreversible constraint, Rubin [4] wrote the following Lagrangian:

$$L = w - \lambda \Delta s, \tag{8}$$

where λ is Lagrange multiplyer. For the case where the heat exchanges through the thermal couplings, ρ_L and ρ_H (see Fig. 1) are linear, $\dot{q}$ is given by

$$\dot{q} = \rho\left(T_R - T\right), \tag{9}$$

where ρ is a thermal conductance and T_R is the heat reservoir temperature. Equation (9) provides both the heats entering and leaving the working fluid by means of the use of the Heaviside step function, for example Q_1 in Fig. 1 is $Q_1 = \int_0^\tau \dot{q}\theta\left(T_R - T\right) dt$. Thus, by using Eqs. (6), (7) and (9) in Eq. (8), one obtains

$$L = \int_0^\tau \left[\rho\left(T_R - T\right) - \lambda\rho\frac{\left(T_R - T\right)}{T}\right] dt. \tag{10}$$

In the present paper we only will take into account the parts of the Rubin's method necessary to obtain Eq. (1) by means of one variational approach and we will not discuss his reasonings about the general optimal configuration of a class of irreversible heat engines [4]. Thus, for solving the variations of Eq. (10), we take the same variable change of Rubin, that is

$$T_R = \frac{1}{2}\left(T_H + T_L\right) + \frac{1}{2}\left(T_H - T_L\right)\tanh\psi, \tag{11}$$

where T_H and T_L are the temperatures of the hot and cold reservoirs and ψ is an unconstrained variational parameter. Taking the first variation of L [Eq. (10)] Rubin gets

$$\delta_L = \int_0^\rho dt\rho \left[\delta T_R\left(1-\frac{\lambda}{T}\right) + \delta_T\left(-1+\lambda\frac{T_R}{T^2}\right)\right], \quad (12)$$

where

$$\delta T_R = \frac{T_H - T_L}{2cosh^2\psi}. \quad (13)$$

When $\delta L = 0$, Rubin found that for $\rho \neq 0$, there are two possibilities:

$$T = \lambda \quad (14a)$$

or

$$\psi = \pm\infty \quad (14b)$$

and in both cases,

$$T = (\lambda T_R)^{1/2}. \quad (15)$$

The case given by Eq. (14a) is discarded, because if $T = \lambda$, then from Eq. (15), $T = T_R$ and consequently $\dot{q} = 0$ [see Eq. (9)]. Furthermore, for this case the second variation of L corresponds to a saddle point [4]. Since T_R necessarily is in the interval $T_L \leq T_R \leq T_H$ this is an unilateral constraint [16], thus it is normal that Eq. (14a) will not correspond to a physical situation since $\delta T_R = 0$, *i.e.*, T_R is fixed for the hot and cold reservoirs (T_H and T_L). In the following subsection we will not take into account this variation and we will analize just the variation of the fluid temperature T. On the other hand, by using Eq. (14b) one obtains $T_R = T_H$ or T_L, that is, the temperatures of the hot and cold reservoirs respectively. Thus one has two solutions:

$$T_R = T_H \quad \text{and} \quad T = T_h = (\lambda T_H)^{1/2} \quad (16a)$$

and

$$T_R = T_L \quad \text{and} \quad T = T_\ell = (\lambda T_L)^{1/2}, \quad (16b)$$

where T_h and T_ℓ are depicted in Fig. 1.

Since the internal cycle of the working fluid operating between T_ℓ and T_h is reversible (endoreversible) (see Fig. 1), then its thermal efficiency is given by

$$\eta = 1 - \frac{T_\ell}{T_h}. \quad (17)$$

By substitution of Eqs. (16a) and (16b) in Eq. (17), one immediately obtains

$$\eta_{CA} = 1 - \sqrt{\frac{T_L}{T_H}},$$

that is, Eq. (1), the so-called CA-efficiency.

2.2. The ecological function

Now, we will obtain the semisum property of the maximum-E regime given by Eq. (3) by means a variational approach based in the Rubin method previously explained. The so-called ecological function is defined by Eq. (2). Our objective is the maximization of the function $E = P - T_L\sigma$, for a fixed cycling period τ. Thus we can replace the power output by the work per cycle w, and rewrite the ecological function as

$$E = w - T_L\sigma. \quad (18)$$

We first propose a Lagrangian function L given by

$$L = w - T_L\sigma - \lambda\Delta S_w, \quad (19)$$

where σ is the universe entropy production, λ is a Lagrange multiplier and $\Delta S_w = 0$ is again the endoreversibility constraint.

The entropy production rate can be written as

$$\sigma = \rho\frac{T - T_R}{T_R}. \quad (20)$$

By means of Eq. (20), the universe entropy change can be calculated using again the Heaviside step function. Then, substituting Eqs. (6), (7) and (20), into Eq. (19) yields

$$L = \int_0^\rho \left[\rho(T_R - T) - \frac{\lambda\rho(T_R - T)}{T} - T_L\rho\frac{(T - T_R)}{T_R}\right] dt. \quad (21)$$

Reorganizing terms, we obtain

$$L = \int_0^\rho \left[(T_R - T)\left(1 + \frac{T_L}{T_R}\right) - \lambda\left(\frac{T_R - T}{T}\right)\right] dt. \quad (22)$$

Taking the first variation of L, we get

$$\delta L = \int_0^\tau \rho\left\{\delta T_R\left[\left(1 + \frac{T_L}{T_R} - \frac{\lambda}{T}\right) - \frac{T_L}{T_R^2}(T_R - T)\right] + \delta T\left[-1 - \frac{T_L}{T_R} + \frac{\lambda T_R}{T^2}\right]\right\} dt. \quad (23)$$

That is

$$\delta L = \int_0^\tau \rho\left[\delta T_R g(T, T_R, T_L, \lambda) + \delta T f(T, T_R, T_L, \lambda)\right] dt, \quad (24)$$

where

$$g(T, T_R, T_L, \lambda) = 1 - \frac{\lambda}{T} + \frac{T_L T}{T_R^2} \tag{25}$$

and

$$f(T, T_R, T_L, \lambda) = -1 - \frac{T_L}{T_R} + \lambda \frac{T_R}{T^2}. \tag{26}$$

As in the Rubin's paper we obtain two solutions. The first one is obtained by setting $g = 0$ [Eq. (25)], because this is an unilateral constraint [16], which leads to a nonphysical solution and it is discarded. Thus,we keep our attention over the second solution which gives us

$$T_{R'} = T_H \quad \text{and} \quad T = T_h = \left(\frac{\lambda T_{R^1}}{1 + \frac{T_L}{T_{R'}}} \right)^{\frac{1}{2}} \tag{27}$$

and

$$T_R = T_L \quad \text{and} \quad T = T_\ell = \left(\frac{\lambda T_R}{1 + \frac{T_L}{T_R}} \right)^{\frac{1}{2}}. \tag{28}$$

As in the previous case (maximum power) the internal cycle is an endoreversible one, and then

$$\eta_E = 1 - \frac{T_\ell}{T_h}.$$

By substitution of Eqs. (27) and (28) into η_E, we immediately obtain

$$\eta_E = 1 - \left[\frac{T_L}{2T_H} \left(1 + \frac{T_L}{T_H} \right) \right]^{\frac{1}{2}}, \tag{29}$$

which is the same result obtained for the efficiency in the regime of maximum ecological function in Refs. 10, 14 and 15 by means of other kind of approaches.

In Fig. 2, we see that Eq. (29) leads practically to the same values than

$$\eta_E = (\eta_C + \eta_{CA})/2 \tag{30}$$

where $\eta_C = 1 - \frac{T_L}{T_H}$ and $\eta_{CA} = 1 - \sqrt{\frac{T_L}{T_H}}$. Thus, we have obtained the so-called semisum property by means of a variational approach.

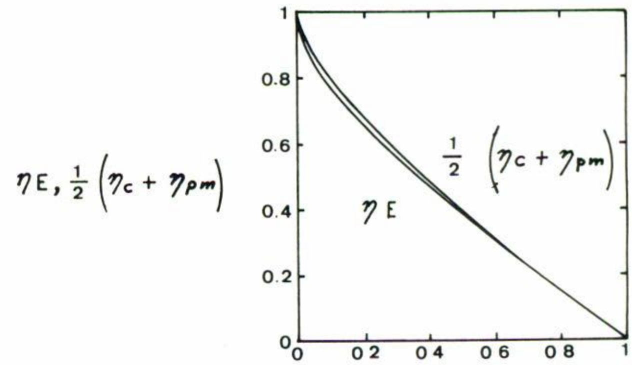

FIGURA 2. Comparison between η_E (Eq. 29) and the semisum property $\frac{1}{2}(\eta_c + \eta_{PM})$ (Eq. 3). The horizontal axis is T_ℓ/T_h

3. Conclusions

The so-called ecological function defined in the context of finite-time thermodynamics has showed to have some interesting properties. Among them the semisum property given in an approximate manner by Eq. (3). In other articles this property has been obtained by means of algebraic formalisms of both two-variable (CA-case) and one-variable (De Vos-case) treatments. In this paper, we have also obtained the semisum property by means of a variational approach for the case of a linear heat transfer law describing the exchanges of heat between the working fluid and its thermal reservoirs. Recently, it has showed that the semisum property is independent of any heat transfer law. We now are working in this general demonstration by means of a variational approach.

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