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General method for the calculation of Stokes parameters with a division of wavefront polarimeter

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We report a generalization of the derivation of the Stokes parameters measured with a division of wavefront polarimeter. This novel approach allows for arbitrary settings of three polarizers and one quarter wave retarder. The transmission coefficients of non-ideal polarizers and retarder have been considered. An experimental validation was performed with a fiber optic based polarization sensor and an analogue-to-digital receiver system.

Keywords: Polarimeter; polarized light; Stokes parameters

Reportamos una generalización de la derivación de los parámetros de Stokes medidos con un polarímetro de división de frente de onda. Este tratamiento innovador permite que los tres polarizadores y una placa retardadora de un cuarto de onda se posicionen con ángulos arbitrarios. Se consideraron los coeficientes de transmisión para polarizadores y placa retardadora no ideales. Se verificaron experimentalmente los resultados utilizando un sensor de polarización basado en fibras ópticas y un sistema de procesamiento digital.

Descriptores: Polarímetro; luz polarizada; parámetros de Stokes

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1. Introduction

In a division of wavefront polarimeter, or DOWP, the incident beam is divided at least into four segments that evenly illuminate the test sample. Then, a polarizing device, used as an analyzer, is located in each of the beam paths before their intensities are detected [1]. The polarimeters reported in the literature during the past decade [2–7] require one linear polarizer (LP) in two of the channels, one quarter wave retarder (QWR) followed by a LP in another of the channels, and one clear channel. Different designs diverge practically, although not fundamentally, in the orientations of the transmission axes of the polarizing elements used. In consequence, they also differ in the form in which the Stokes parameters are extracted and the polarimeter is calibrated. All polarimeters of this type must localize the orientation of the LPs and retarder at specific angles. Therefore, this produces inaccurate results derived from errors caused by positioning the LPs and QWR at different angles than those required.

The DOWP that we designed and developed is a generalization to the above mentioned methods (See Fig. 1). Our instrument does not require setting the LPs and QWR in the sensor head at unique and predetermined azimuth angles. In addition we have considered that the LPs and QWR are not ideal by incorporating their respective transmission coefficients into the theory.

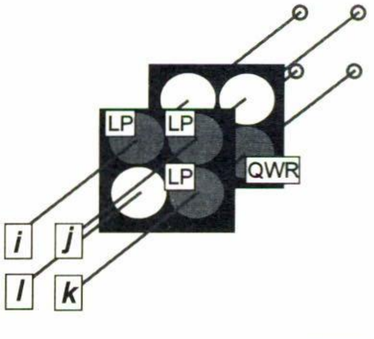

FIGURE 1. Schematic of the optics in the DOWP.

2. Theory

An arbitrary polarization state is characterized by the azimuth, ellipticity and handedness of a polarization ellipse (in two dimensions and observing only a cross section of the ellipse). The azimuth (α) is the angle that the major axis of the ellipse makes with the x axis. Angles are positive when measured anti-clockwise from the x axis. The ellipticity (e) is defined as the ratio of the length of the semi-minor axis (b) of the polarization ellipse to that of the semi-major axis (a).

According to Fig. 2, the electric field components (E_x, E_y) of an arbitrary fully polarized source are given by

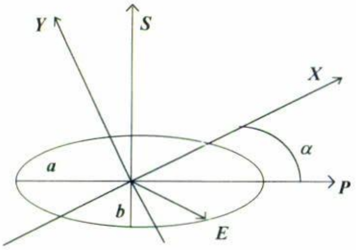

FIGURE 2. The polarization ellipse.

$$E_x = a\cos\alpha \sin(\omega t+\phi_0) - b\sin\alpha\cos(\omega t+\phi_0), \quad (1)$$

$$E_x = a\sin\alpha \sin(\omega t+\phi_0) - b\cos\alpha\cos(\omega t+\phi_0), \quad (2)$$

where ω is the wave frequency, and ϕ_0 is a phase constant. When the source is fully polarized, the corresponding average source intensity can be obtained as the average in time of the sum of both squared components of the electric field,

$$I_p(t) = \langle (E_x^2 + E_y^2) \rangle . \quad (3)$$

Intregrating Eq. (3) over a time period $T = 2\pi/\omega$ and using Eqs. (1) and (2), the average source intensity is given by

$$I_p = (a^2+b^2)/2 . \quad (4)$$

When the electric field given by Eqs. (1) and (2), traverses a LP inclined at an angle θ, its components are projected over the LP parallel and perpendicular axes (p and s respectively). The electric field components emerging from the LP are therefore given by

$$\begin{aligned} E_p &= \sqrt{\tau_p}\left(E_x\cos\theta + E_y\sin\theta\right), \\ E_s &= \sqrt{\tau_s}\left(E_x\sin\theta - E_y\cos\theta\right), \end{aligned} \quad (5)$$

where constants factors τ_p and τ_s represent the intensity transmission parameters for the LP in transmission mode and extinction mode respectively. Replacing Eqs. (1) and (2) into Eq. (5) we obtain

$$\begin{aligned} E_p &= \sqrt{\tau_p}\left\{\cos\theta\left[a\cos\alpha\sin(\omega t+\phi_0) - b\sin\alpha\cos(\omega t+\phi_0)\right] + \sin\theta\left[a\sin\alpha\sin(\omega t+\phi_0) + b\cos\alpha\cos(\omega t+\phi_0)\right]\right\}, \quad (6)\\ E_s &= \sqrt{\tau_s}\left\{\sin\theta\left[a\cos\alpha\sin(\omega t+\phi_0) - b\sin\alpha\cos(\omega t+\phi_0)\right] - \cos\theta\left[a\sin\alpha\sin(\omega t+\phi_0) + b\cos\alpha\cos(\omega t+\phi_0)\right]\right\}. \end{aligned}$$

The normalized transmission coefficient, $T(\theta)$, for the LP is obtained by averaging in time the sum of the square of the above components and dividing by the averaged intensity, so that

$$T(\theta) = \tau_s + (\tau_p - \tau_s)\left[\frac{\cos^2(\theta-\alpha) + e^2\sin^2(\theta-\alpha)}{1+e^2}\right]. \quad (7)$$

The total transmitted intensity through a channel in the polarimeter, which uses a LP labeled by the angle θ, is then given by

$$I = I_0\tau_0^{\mathrm{LP}} + I_p T(\theta) , \quad (8)$$

where I_0 is the non-polarized light and I_p is the intensity of completely polarized light. the transmission coefficient of a LP for randomly polarized light is τ_0^{LP}, given by

$$\tau_0^{\mathrm{LP}} = \left(\tau_p^{\mathrm{LP}} + \tau_s^{\mathrm{LP}}\right)/2 . \quad (9)$$

The intensities I_i and I_j admitted by two of the channels in the polarimeter (See Fig. 1) can be labeled by the orientations of the LPs θ_i and θ_j such that

$$\begin{aligned} I_{i,j} =& \tau_0^{\mathrm{LP}} I_0 + Ip\tau_0^{\mathrm{LP}} \\ &+ \frac{I_p}{2(e^2+1)}\left[(\tau_p^{\mathrm{LP}} - \tau_s^{\mathrm{LP}})\cos 2(\theta_{i,j}-\alpha)(1-e^2)\right] \end{aligned} \quad (10)$$

The total intensity k is given by

$$I_k = I_0\tau_0^{\mathrm{QWR}} + I_p T(\theta,\psi) , \quad (11)$$

where τ_0^{QWR} is the transmission coefficient for a QWR followed by a LP, when randomly polarized light is transmitted through the combination, namely

$$\begin{aligned} \tau_0^{\mathrm{QWR}} &= 1/2(\tau_s^{\mathrm{QWR}} - \tau_p^{\mathrm{QWR}})(\tau_p^{\mathrm{LP}} - \tau_s^{\mathrm{LP}})\sin^2(\theta-\psi) \\ &\quad + 1/2(\tau_s^{\mathrm{QWR}}\tau_s^{\mathrm{LP}} + \tau_p^{\mathrm{QWR}}\tau_p^{\mathrm{LP}}) , \end{aligned} \quad (12)$$

in which τ_p^{QWR} and τ_s^{QWR} are the transmission parameters along the fast and low axes of the QWR. $T(\theta,\phi)$ is the transmission coefficient of the combination QWR-LP, for completely polarized light, where θ is the azimuth angle of the LP in channel k and ψ is the azimuth angle of the QWR. This coefficient can be derived in a similar approach as Eq. (7), giving the intensity I_k as

$$\begin{aligned} I_k = I_0\tau_0^{\mathrm{QWR}} &+ \frac{I_p}{(e^2+1)}\left\{e^2\left[p_1\cos^2(\alpha-\psi) + p^2\right]\right. \\ &\left. + ep_3 + p_1\sin^2(\alpha-\psi) + p_2\right\} \end{aligned} \quad (13)$$

with p_1, p_2 and p_3 defined by the following:

$$\begin{aligned} p_1 &= \tau_s^{\mathrm{QWR}}\tau_s^{\mathrm{LP}} - \tau_p^{\mathrm{QWR}}\tau_p^{\mathrm{LP}} \\ &\quad + \left(\tau_p^{\mathrm{QWR}} + \tau_s^{\mathrm{QWR}}\right)\left(\tau_p^{\mathrm{LP}} - \tau_s^{\mathrm{LP}}\right)\sin^2(\theta_k-\psi) , \\ p_2 &= \tau_p^{\mathrm{QWR}}\tau_s^{\mathrm{LP}}\sin^2(\theta_k-\psi) + \tau_p^{\mathrm{QWR}}\tau_p^{\mathrm{LP}}\cos^2(\theta_k-\psi) , \\ p_3 &= \sqrt{\tau_s^{\mathrm{QWR}}\tau_p^{\mathrm{QWR}}}\left(\tau_p^{\mathrm{LP}} - \tau_s^{\mathrm{LP}}\right)\sin 2(\theta_k-\psi) . \end{aligned} \quad (14)$$

The clear channel I_l is assumed to measure the total partially polarized light intensity emerging from the sample

$$I_l = I_p + I_0 \tag{15}$$

By algebraic inversion of the expressions for the four intensities measured by the polarimeter I_i, I_j, I_k and I_l we can derive of the values α, ε the amount of completely polarized light (I_p) and the amount of non-polarized light (I_0). The polarization azimuth is obtained from the following expression

$$\tan 2\alpha = \frac{\left(I_i - I_l\tau_0^{\mathrm{LP}}\right)\cos 2\theta_j - \left(I_j - I_l\tau_0^{\mathrm{LP}}\right)\cos 2\theta_i}{\left(I_j - I_l\tau_0^{\mathrm{LP}}\right)\sin 2\theta_i - \left(I_i - I_l\tau_0^{\mathrm{LP}}\right)\sin 2\theta_j} \tag{16}$$

The ellipticity is given by

$$e = \frac{p_3 \pm \sqrt{p_3^2 + 4(q_3q_4 - q_1)(q_3q_4 + q_2)}}{2(q_3q_4 - q_1)} \tag{17}$$

where

$$\begin{aligned} q_1 &= p_1\cos^2(\alpha - \psi) + p_2 - \tau_0^{\mathrm{QWR}}\,, \\ q_2 &= p_1\sin^2(\alpha - \psi) + p_2 - \tau_0^{\mathrm{QWR}}\,, \\ q_3 &= \left(\tau_p^{\mathrm{LP}} - \tau_s^{\mathrm{LP}}\right)\cos 2(\theta_i - \alpha)\,, \\ q_4 &= \frac{\left(I_k - \tau_0^{\mathrm{QWR}} I_l\right)}{2\left(I_i - \tau_0^{\mathrm{LP}} I_l\right)}\,. \end{aligned} \tag{18}$$

The two solutions to Eq. (17) simply correspond to reciprocal definitions of the ellipticity $\varepsilon = b/a$ and $\varepsilon^{-1} = a/b$, the former is used in this work. The amount of completely polarized light is obtained from

$$I_p = \frac{2(I_i - I_j)(e^2 + 1)}{\left(\tau_p^{\mathrm{LP}} - \tau_s^{\mathrm{LP}}\right)(e^2 - 1)} \times \frac{1}{\left[\cos 2(\theta_i - \alpha) - \cos 2(\theta_j - \alpha)\right]}\,. \tag{19}$$

The value of I_0 for the amount of randomly polarized intensity incident on the sensor head is ,

$$I_0 = I_l - I_p\,, \tag{20}$$

and therefore the degree of polarization (DOP) is obtained from

$$DOP = \frac{I_p}{I_p + I_0} \tag{21}$$

Finally the Stokes parameters $\{I = 1,\, Q,\, U,\, V\}$ can be written in terms of the polarization azimuth and ellipticity, using the well known relationships [8] for normalized intensities given by

$$Q = \cos\left[2\arctan(e)\right]\cos 2\alpha\,, \tag{22}$$

$$U = \cos\left[2\arctan(e)\right]\sin 2\alpha\,, \tag{23}$$

$$V = \sin\left[2\arctan(e)\right]. \tag{24}$$

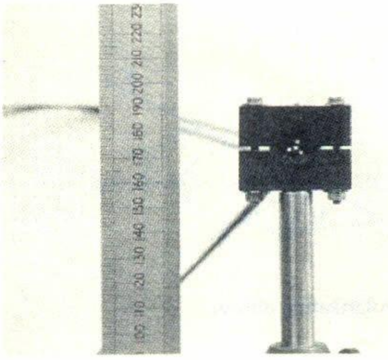

FIGURE 3. Photograph of the sensor head.

3. Prototype instrument and calibration

The sensor head of the polarimeter shown in Fig. 3 was designed according to the specifications indicated in Fig. 1. The complete instrument [9] consists of the optics integrating the sensor head, a bundleof optical fibers conducting the received light from the sensor head to the photo-diode receivers and a data acquisition system. Each of the LPs and QWR are 2mm diameter. The four apertures are arranged in a triangular geometry, within a 7 mm diameter circle, as shown in Fig. 3. The photo-diodes are multiplexed into a 12-bit analogue to digital converter. Data stored by a processor are serially communicated to a PC, in blocks, at regular intervals.

The accuracy with which the angles and transmission parameters of the LP and QWR are known is crucial for the polarimeter to yield accurate measurements. Usually the values of parameters τ_p, τ_s, τ_r and τ_l are provided by the manufacturer, but it is more convenient to obtain the parameters under operational conditions. Before performing any polarization measurement using the DOWP, the values of the angles and transmission parameters of the LP and QWR must be measured. Following this, a calibration routine integrated within the software must be followed, which includes measurement of the photo-diode offsets. The calibration procedure also compensates for errors in the readings introduced by uneven performance of each photo-diode, non-uniform illumination of all channels and losses in the fibers. When non-polarized or randomly polarized light is illuminating the sensor head, the two channels i and j should measure the same intensity (the total intensity scaled by τ_0^{LP}) which is linearly proportional to the value recorded in channel l. To restore this relationship in the presence of the effects described above, the following equalization factors are introduced

$$\begin{aligned} f_i &= \frac{\tau_0^{\mathrm{LP}} I_l}{I_i}\,, \\ f_j &= \frac{\tau_0^{\mathrm{LP}} I_l}{I_j}\,, \\ f_k &= \frac{\tau_0^{\mathrm{QWR}} I_l}{I_k}\,. \end{aligned} \tag{25}$$

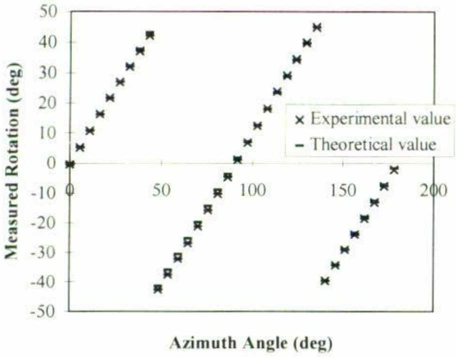

FIGURE 4. Polarization azimuth measurements: The ordinate is the angle by which the external LP has been rotated by the motor. The abscissa is the measured rotation in degrees (α). The averaged experimental error is 0.1 of a degree.

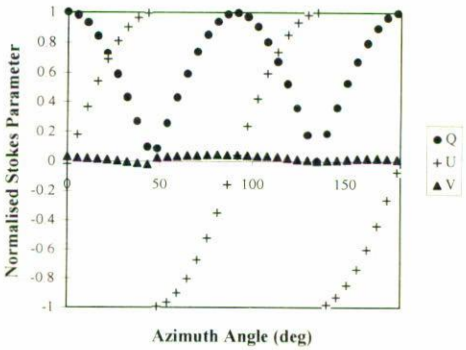

FIGURE 5. Measurement of Stokes Parameters Q, U, and V. The average experimental errors for the normalised parameters are: 0.001 for parameter Q, 0.004 for parameter U and 0.004 for parameter V. These parameters are functions of the wrapped polarization azimuth from Fig. 4.

After these calibration factors are calculated, all subsequent intensity measurements are scaled by the corresponding factor prior to subsequent calculation of the polarization parameters.

A LP external to the polarimeter was used as a test polarization modulator and was illuminated by a randomly polarized light source (670 nm light emitting diode) and then rotated of the polarimeter by a stepper motor. Measurements were obtained for the various polarization states generated for rotations from 0° to 178.2° in 33 equal steps. The motor had a resolution of 1.8° $\pm$ 5%. Each experimental result is an average of 500 measurements produced under the same conditions. The experimental error has been quantified as the standard deviation of all the measured values for a particular setting of the LP. The results are illustrated in Figs. 4 and 5. The curves for rotation, shown in Fig. 4, are wrapped at intervals of $\pi/4$, because Eq. (16) was used to extract the value of (α). The experimental results for the Stokes parameters (Fig. 5) show how they fulfill the well known relationship for completely polarized light

$$U^2 + V^2 + Q^2 = 1 . \tag{26}$$

4. Performance evaluation

The main sources of experimental errors were the incorrect determination of the angles and transmission coefficients of the LPs and QWR, followed by inaccuracies introduced by the electronics in the receivers and converters. Another probable source of error that was not compensated by the calibration procedure, was the deviation of the retardance of the QWR from the nominal retardance of 90°, the retarder was calibrated for 630 nm. For the particular experimental system that was used, additional errors were caused by unintentional partial polarization of the source. The factors most likely to affect the experimental performance of the DOWP were:

1. The error in the assessment of the transmission parameters and the azimuth angles of the polarizing optics in the sensor head.
2. The resolution of the A/D converter used (the experiments were performed with one of 12 bit). The minimum error that could be expected in the quantification of the Stokes parameters, produced by quantization effects, was 0.0001.
3. The performance of the four photo-diodes and their respective linear amplifiers.

From the three factors listed above, only the first one was intrinsic to the DOWP, while the other two greatly depended on the particular data acquisition system employed. The performance of the photo-diodes, the signal amplifiers and the accuracy of the A/D converters determined the accuracy with which the angles and parameters were measured. To ascertain the magnitude of the absolute error introduced into the Stokes parameters measured by the DOWP, due to quantization and incorrect measurement of the azimuth angles, the two effects were simulated theoretically. The computer simulation examined the situation in which the DOWP measured changes in the polarization state of a linear LP rotating from 0° to 180°. Initially the simulation routine assumed that the angles and the parameters were exactly known, but the four intensities were quantized into 4096 digital levels, simulating the accuracy of a 12 bit A/D converter. The error in each of the channels was estimated as the difference between the exact and the estimated intensities. The results are reported in Table I, second column. A deviation of one degree from the exact value was added to one of the angles, θ_i leaving all the other angles and absorbance parameters intact. The purpose was to show the dependency of the Stokes parameters on the accuracy in the measurement of the angles. The results are shown in Table I, third column. Although the A/D converter used for the experimental work had a resolution of 12 bits, a number of factors would contribute to reduce the effective precision in reality. Firstly, it was difficult to produce experimental conditions which utilize the whole dynamic range. In

TABLE I. Summary of the theoretical RMS errors and experimental standard deviation

	12-bit resolution RMS error	12-bit resolution + deviation of 1° RMS error	10-bit resolution RMS error	Experimental data Standar deviation
I_i	0.00007	0.00007	0.00027	0.0025
I_j	0.00007	0.00007	0.00026	0.0011
I_k	0.00007	0.00007	0.00027	0.0013
Ellipticity	0.0001	0.0024	0.00035	0.0017
Q	0.00013	0.0463	0.00049	0.0014
U	0.00010	0.0794	0.00045	0.0044
V	0.00033	0.0050	0.00079	0.0045

addition, there were sources of noise present from amplification and switching circuitry. We estimate that the experiments were conducted as if using a resolution smaller than that provided by a 10-bit A/D converter. This situation was simulated and reported in Table I, fourth column.

When the resolution of the A/D converter was reduced, the values that the reconstructed polarization azimuth could have were spread over a smaller number of digital levels. In Table I the standard deviation of the experimental intensities was one order of magnitude greater than the 10-bit resolution RMS error. This could mean that the experiment was performed using an even smaller dynamic range, or that incorrect values of the experimental parameters and angles affected the intensities through incorrect calibration factors. Considering that a resolution of 12 bits meant that the four intensities could use 4096 quantization levels, and that in reality only about 25% of the number of available levels were used, one could not expect to observe experimental errors smaller than those produced by a 10-bit quantization.

5. Conclusions and discussion

We have developed a DOWP whose polarizing elements can be positioned at arbitrary orientations and their characteristics may be non-ideal. The theoretical description of any polarization state measured by this polarimeter was made in terms of explicit transmission coefficients and azimuth angles. We built a system to test the concepts without intending to achieve a highly accurate instrument, however the accuracy was comparable with the most advanced polarimeters [10] reported in the literature. We therefore expect further refinements of the optical and electronic systems to yield improved accuracy, especially when the polarimeter described here is optimized for a given range of polarization state measurements.

Some of the limitations of a DOWP are that it is clearly not capable of the minimum spatial resolution and is fundamentally limited by the need to maintain coherence over the entire aperture. Nevertheless its importance is significant in the context of future polarizing imaging systems based on the pixelisation of polarizing elements. The use of polarization imaging in applications such as robotics and industrial inspection will require simple cost effective technologies and the DOWP may very well occupy an important place in those contexts.

Acknowledgment

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