Introduction to the Fock quantization of the Maxwell field

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In this article we give an introduction to the Fock quantization of the Maxwell field. At the classical level, we treat the theory in both the covariant and canonical phase space formalisms. The approach is general since we consider arbitrary (globally-hyperbolic) space-times. The Fock quantization is shown to be equivalent to the definition of a complex structure on the classical phase space. As examples, we consider stationary space-times as well as ordinary Minkowski space-time. The account is pedagogical in spirit and is tailored to beginning graduate students. The paper is self contained and is intended to fill an existing gap in the literature.

Keywords: Symplectic mechanics; Maxwell field; Fock quantization

En este artículo presentamos una introducción a la cuantización del campo de Maxwell. A nivel clásico se da un tratamiento tanto en el formalismo de espacio fase covariante como en el de espacio fase canónico. El tratamiento es general pues se consideran espacios tiempo arbitrarios (globalmente hiperbólicos). Se muestra que la cuantización de Fock es equivalente a la definición de una estructura compleja en el espacio fase de la teoría. Como ejemplos se toman espacios tiempo estacionarios así como el espacio de Minkowski ordinario. La presentación se pretende sea pedagógica, está enfocada a estudiantes de posgrado y llena un vacío existente en la literatura.

Descriptores: Mecánica simpléctica; campo de Maxwell; cuantización de Fock

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1. Introduction

The motivation to write this article comes from the author's discomfort with the usual treatment that textbooks give to the *canonical quantization* of free fields in their first chapters [1]. There seems to be a "quantum jump" from the quantization of mechanical systems with a finite number of degrees of freedom to the quantization of fields. Here, by fields we mean that the classical system to be quantized is described by (at least) one function of space-time. The best known example is precisely the electro-magnetic field, described by six quantities at each space-time point. In ordinary quantum mechanics, one starts with the phase space Γ of the system, which is normally given by pairs (q^i, p_i), $i = 1, 2, \ldots, n$ of generalized coordinates and their conjugate momenta. The quantization procedure implies a passage from the basic Poisson Brackets (PB) relations $\{q^i, p_j\} = \delta^i_j$ to the Canonical Commutation Relations (CCR): $[\hat{q}^i, \hat{p}_j] = i\hbar\delta^i_j$. This is usually called the *Dirac quantization condition*. One finally finds a Hilbert space $\mathcal{H}$ and a representation of the basic observables $\hat{q}^i$ and $\hat{p}_i$ as self-adjoint operators on $\mathcal{H}$ satisfying the CCR. More precisely, one should find at the classical level a set $\mathcal{S}$ of *elementary observables* (real functions) on Γ that are: i) large enough to generate, via linear combinations of products of them, any function on Γ and ; 2) small enough to be closed under Poisson brackets [2]. To these observables, elements of $\mathcal{S}$, there will be associated a quantum operator in a unique way, satisfying the Dirac quantization condition. For details see Sect. 2.

When the classical system to be quantized is a field theory, one is led to ask: Can we follow the same prescription? that is, can we identify the phase space of the problem and a set $\mathcal{S}$ of basic observables? How is the Poisson bracket defined? Can we implement the Dirac quantization condition and find representations of the CCR? If yes, which is the Hilbert space $\mathcal{H}$? The aim of this paper is to give answers to all this questions when the classical system to be quantized is the free Maxwell field. In the case of a Klein-Gordon field, the problem is satisfactorily addressed by Wald [3] (The reader is urged to read the first three chapters of that book). Recall that the Klein Gordon field is described by a scalar field Φ on space-time satisfying the Klein-Gordon equation: $(\Box - m^2)\Phi = 0$. The main difference between the Klein-Gordon and the Maxwell field is gauge invariance. This in turn brings some subtleties to the program of quantization. These problem are dealt with in this paper.

The particular quantization method we shall consider is the one known as Fock quantization. The intuitive idea is that the Hilbert space of the theory is constructed from "n-particle states". (In certain cases one is justified to interpret the quantum states as consisting of n-particle states. For a discussion see below.) As we shall see later, the Fock quantization is naturally constructed from *solutions to the classical equations of motion* and relies heavily on the linear structure of the space of solutions (The Klein-Gordon and Maxwell equations are linear). Thus, it can only be implemented for quantizing *linear (free) field theories*. The main steps of the quantization

are the following: Given a 4-dimensional globally hyperbolic space-time $(M, g)^{(a)}$, the first step is to consider the vector space $\bar{\Gamma}$ of solutions of the equations of motion and construct from it the vector space of *physically indistinguishable* states Γ. One then constructs the algebra $\mathcal{S}$ of fundamental observables to be quantized, which in this case consists of suitable linear functionals on Γ. The next step is to construct the so called *one-particle Hilbert space* $\mathcal{H}_0$ from the space Γ. As mentioned before, the one particle Hilbert space $\mathcal{H}_0$ receives this name since it can be interpreted as the Hilbert space of a one particle relativistic system (in the electro-magnetic case, the photon). The one-particle space is constructed by defining a complex structure on Γ compatible with the naturally defined symplectic structure thereon, in order to define a Hermitian inner product on Γ. The completion with respect to this inner product will be the one-particle Hilbert space $\mathcal{H}_0$. From the Hilbert space $\mathcal{H}_0$ one constructs its symmetric (since we are considering Bose fields) Fock space $\mathcal{F}_s(\mathcal{H})$, the Hilbert space of the theory. The final step is to represent the algebra $\mathcal{S}$ of observables in the Fock space as suitable combinations of (naturally defined) creation and annihilation operators.

We will construct in detail the quantization outlined above for the case of the Maxwell field. In our opinion, an unified treatment (although completely elementary) is not available elsewhere. The structure of the paper is as follows. In Sect. 2 we give an overview of the prerequisites to tackle the quantization program. In particular, we review the canonical quantization using symplectic language. In the Sect. 3 we consider the classical treatment of the Maxwell field. We follow two paths in the phase space description of the theory. The first one, the so called *covariant phase space* starts from the solutions to the equations of motion. The second approach, the 'standard' 3+1 formulation, is considered next and compared to the covariant framework. Section 4 addresses the quantization. We outline the quantization strategy starting from the classical analysis and show that it depends on certain extra structure (a complex structure) defined on the classical phase space. We consider then two examples of particular interest on Minkowski space-time: the standard 'positive frequency' decomposition and the self dual decomposition. We end with a discussion in Sect. 5.

Throughout the paper, we use Penrose's abstract index notation$^{(b)}$, and units in which $c = 1$, but keep $\hbar$ explicit.

2. Preliminaries

In this section we shall present some background material, both in classical and quantum mechanics. This section has two parts. In the first one we will introduce some basic notions of symplectic geometry that play a fundamental role in the Hamiltonian description of classical systems. In the second part we outline the canonical quantization starting from a classical system as described in Sect. 2.1.

2.1. Classical mechanics

A physical system is normally represented, at the classical level, by a *phase space*. This consists of a manifold Γ of dimension $\dim(\Gamma) = 2n$, where n is an integer corresponding to the dimension of the configuration space. Physical states are represented by the points on the manifold. Observables are smooth, real valued functions on Γ. There is a non-degenerate, closed *symplectic* two-form Ω defined on it. The two-form Ω_{ab} satisfies: $\nabla_{[c}\Omega_{ab]} = 0$, and if $\Omega_{ab}V^b = 0$ then $V^b = 0$. Therefore, there exists an inverse Ω^{ab} and it defines an isomorphism between the cotangent and the tangent space at each point of Γ. Here square brackets over a set of indices means antisymetrization. That is $A_{[ab]} := \frac{1}{2}(A_{ab} - A_{ba})$ [and $A_{(ab)} := \frac{1}{2}(A_{ab} + A_{ba})$]. The space Γ with the symplectic two-form Ω is called a symplectic space (or a space endowed with a symplectic structure) and denoted by (Γ, Ω).

A vector field V^a generates infinitesimal canonical transformations if it Lie drags the symplectic form, *i.e.*:

$$\mathcal{L}_V \Omega = 0. \tag{1}$$

This condition is equivalent to saying that locally the symplectic form satisfies: $V^b = \Omega^{ba}\nabla_a f := X_f^b$, for some function f. The vector X_f^a is called the *Hamiltonian vector field of* f (w.r.t. Ω). Note that the symplectic structure gives us a mapping between functions on Γ and Hamiltonian vector fields. Thus, functions on phase space (*i.e.* observables) are generators of infinitesimal canonical transformations.

The Lie Algebra of vector fields induces a Lie Algebra structure on the space of functions.

$$\{f, g\} := \Omega_{ab} X_f^a X_g^b = \Omega^{ab}\nabla_a f \nabla_b g, \tag{2}$$

such that $X^a_{\{f,g\}} = -[X_f, X_g]^a$. The 'product' $\{\cdot, \cdot\}$ is called *Poisson bracket* (PB).

Note that the Poisson bracket $\{f, g\}$ gives the change of f given by the motion generated by (the HVF of) g, *i.e.*,

$$\{f, g\} = \mathcal{L}_{X_g} f. \tag{3}$$

The PB is antisymmetric so it is also (minus) the change of g generated by f.

The role of the symplectic structure Ω in symplectic geometry is somewhat similar to the role of the metric in Riemannian geometry. It provides a one to one mapping between vectors and one-forms at each point of the manifold. There is however a very important difference: In symplectic geometry one can always find coordinates (q^i, p_j) in a finite neighborhood such that the symplectic form takes the canonical form (known as Darboux theorem),

$$\Omega_{ab} = 2\nabla_{[a} p_i \nabla_{b]} q^i \tag{4}$$

With this form, the Poisson bracket between the coordinate functions takes the form

$$\{q^i, p_j\} = \Omega^{ab}\nabla_a(q^i)\nabla_b(p_j) = \delta^i_j, \tag{5}$$

$$\{q^i, q^j\} = \Omega^{ab}\nabla_a(q^i)\nabla_b(q^j) = \{p_i, p_j\} = \Omega^{ab}\nabla_a(p_i)\nabla_b(p_j) = 0. \tag{6}$$

In such a chart, the q^i coordinates are like 'position' and p_i are like 'momenta'.

Since the symplectic form is closed, it can be obtained locally from a *symplectic potential* ω_a,

$$\Omega_{ab} = 2\nabla_{[a}\omega_{b]} \tag{7}$$

Time evolution is given by a vector field h^a whose integral curves are the dynamical trajectories of the system. On phase space there is a *preferred* function, the *Hamiltonian* H whose Hamiltonian vector field corresponds precisely with h^a, *i.e.*,

$$h^a = \Omega^{ab}\nabla_b H \tag{8}$$

Adopting the viewpoint that all observables generate canonical transformations we see that the motion generated by the Hamiltonian corresponds to 'time evolution'. The 'change' in time of the observables will be simply given by the Poisson bracket of the observable with H: $\dot{g} := h^a\nabla_a g = \Omega^{ac}\nabla_c H\nabla_a g = \{g, H\}$.

If the system has a configuration space $\mathcal{C}$, then the phase space Γ is automatically "chosen" to be the cotangent bundle of the configuration space $T^*\mathcal{C}$. There is also a preferred 1-form on $\mathcal{C}$ that can be lifted to $T^*\mathcal{C}$ and taken to be the symplectic potential which determines uniquely the symplectic structure. Therefore, the fact that there exists a configuration space picks for us the phase space and the symplectic two-form. For field theories this description is obtained when one performs a 3+1 decomposition on space-time and the phase space is defined from the initial data of the theory. An alternative is to consider the covariant variational principle, without any decomposition, and construct a naturally defined symplectic two-form. This is the covariant phase space formalism that will be seen in Sect. 3.

Let us look in detail at the simplest example: a particle in 3 dimensional Euclidean space. The state of the system is specified by the value of its configuration q^i and its momenta variables p_i. In this case, q^i are coordinates in the configuration space $\mathcal{C}$. Here $i = 1, 2, 3$ and the dimension of Γ is 6. The phase space has in this case a cotangent bundle structure $\Gamma = T^*\mathcal{C}$, and the naturally defined symplectic potential is

$$\omega_a = p_i\nabla_a q^i \tag{9}$$

from which the natural symplectic structure can be derived,

$$\Omega_{ab} = 2\nabla_{[a}p_i\nabla_{b]}q^i. \tag{10}$$

That is, in the dual basis $\{\nabla_a q^i, \nabla_a p_i\}$ for the cotangent space the 2-form (10) has a matrix representation that can be written as

$$\Omega_{\underline{ab}} = \begin{pmatrix} 0 & -I_{n\times n} \\ I_{n\times n} & 0 \end{pmatrix}.$$

In the basis of the tangent space to Γ, the inverse of the symplectic two-form is given by

$$\Omega^{ab} = 2\left(\frac{\partial}{\partial q^j}\right)^{[a}\left(\frac{\partial}{\partial p_j}\right)^{b]} \tag{11}$$

The Poisson bracket in this coordinates has the usual form

$$\{f, g\} = \frac{\partial f}{\partial q^i}\cdot\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i}\cdot\frac{\partial f}{\partial p_i}, \tag{12}$$

and the evolution equations are

$$\dot{q}^i = \{q^i, H\} = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \dot{p}_i = \{p_i, H\} = -\frac{\partial H}{\partial q^i} \tag{13}$$

In this form, we recover the usual textbook treatment of Hamiltonian mechanics.

In the case that the system exhibits come gauge freedom in the classical theory, its description in symplectic language gets modified. The details are different for the covariant and canonical phase space descriptions, but the common theme is that the phase space accessible to the system is not a true symplectic space: the two-form Ω is degenerate. In this case the space is called a pre-symplectic space and Ω is a called a pre-symplectic structure. In Sect. 3 we treat the Maxwell system and comment on the strategy to deal with gauge systems in both descriptions. Let us now look at the quantization.

2.2. Quantization

In very broad terms, by quantization one means the passage from a classical system, as described in the last part, to a quantum system. Observables on Γ are to be promoted to self-adjoint operators on a Hilbert Space. However, we know that not all observables can be promoted unambiguously to quantum operators satisfying the CCR. A well known example of such problem is factor ordering. What we *can* do is to construct a subset $\mathcal{S}$ of *elementary classical variables* for which the quantization process has no ambiguity. This set $\mathcal{S}$ should satisfy two properties:

- $\mathcal{S}$ should be a vector space large enough so that every (regular) function on Γ can be obtained by (possibly a limit of) sums of products of elements in $\mathcal{S}$. The purpose of this condition is that we want that enough observables are to be unambiguously quantized.

- The set $\mathcal{S}$ should be small enough such that it is closed under Poisson brackets.

The next step is to construct an (abstract) quantum algebra $\mathcal{A}$ of observables from the vector space $\mathcal{S}$ as the free associative algebra generated by $\mathcal{S}$ (for a definition and discussion of free associative algebras see [7]). It is in this quantum algebra $\mathcal{A}$ that we impose the Dirac quantization condition: Given A, B and $\{A, B\}$ in $\mathcal{S}$ we impose

$$[\hat{A}, \hat{B}] = i\hbar\widehat{\{A, B\}}. \tag{14}$$

It is important to note that there is no factor order ambiguity in the Dirac condition since A, B and $\{A, B\}$ are contained in $\mathcal{S}$ and they have associated a unique element of $\mathcal{A}$.

The last step is to find a Hilbert space $\mathcal{H}$ and a representation of the elements of $\mathcal{A}$ as operators on $\mathcal{H}$. For details of this approach to quantization see [2].

In the case that the phase space Γ is a linear space, there is a particular simple choice for the set $\mathcal{S}$. We can take a global chart on Γ and we can choose $\mathcal{S}$ to be the vector space generated by *linear* functions on Γ. In some sense this is the smallest choice of $\mathcal{S}$ one can take. As a concrete case, let us look at the example of $\mathcal{C} = \mathbb{R}^3$. We can take a global chart on Γ given by (q^i, p_i) and consider $\mathcal{S} = \text{Span}\{1, q^1, q^2, q^3, p_1, p_2, p_3\}$. It is a seven dimensional vector space. Notice that we have included the constant functions on Γ, generated by the unit function since we know that $\{q^1, p_1\} = 1$, and we want $\mathcal{S}$ to be closed under PB.

We can now look at linear functions on Γ. Denote by Y^a an element of Γ, and using the fact that it is linear space, Y^a also represents a vector in $T\Gamma$. Given a one form λ_a, we can define a linear function of Γ as follows: $F_\lambda(Y) := \lambda_a Y^a$. Note that λ is a label of the function with Y^a as its argument. First, note that there is a vector associated to λ_a:

$$\lambda^a := \Omega^{ab}\lambda_b,$$

so we can write

$$F_\lambda(Y) = \Omega_{ab}\lambda^a Y^b = \Omega(\lambda, Y). \tag{15}$$

If we are now given another label ν, such that $G_\nu(Y) = \nu_a Y^a$, we can compute the Poisson Bracket

$$\{F_\lambda, G_\nu\} = \Omega^{ab}\nabla_a F_\lambda(Y)\nabla_b G_\nu(Y) = \Omega^{ab}\lambda_a\nu_b. \tag{16}$$

Since the two-form is non-degenerate we can re-write it as $\{F_\lambda, G_\nu\} = \Omega_{ab}\lambda^a\nu^b$. Thus,

$$\{\Omega(\lambda, Y), \Omega(\nu, Y)\} = \Omega(\lambda, \nu). \tag{17}$$

As we shall see in Sect. 4 we can also make such a selection of linear functions for the Maxwell field.

The quantum representation is the ordinary Schrödinger representation where the Hilbert space is $\mathcal{H} = L^2(\mathbb{R}^3, d^3x)$ and the operators are represented:

$$\begin{aligned}(\hat{1}\cdot\Psi)(q) &= \Psi(q),\\ (\hat{q^i}\cdot\Psi)(q) &= q^i\Psi(q),\\ (\hat{p}_i\cdot\Psi)(q) &= -i\hbar\frac{\partial}{\partial q^i}\Psi(q)\end{aligned} \tag{18}$$

Thus, we recover the conventional quantum theory.

3. Classical description for the Maxwell Field

In the classical phase space description of the Maxwell field there are two equivalent but complementary viewpoints, namely the *covariant* and the *canonical* formalisms. In what follows we shall develop both approaches and show their equivalence.

3.1. Covariant Phase Space

In this part we shall introduce and employ the *covariant phase space formulation* [8]. Since in our opinion this formalism is not widely known, we shall outline the main steps using the Maxwell field as an example. The starting point for the construction of the covariant phase space is the identification of the symplectic vector space Γ, the phase space of the problem, starting from solutions to the equations of motion. Let us start by writing down the action for the free Maxwell theory:

$$\begin{aligned}S_{\rm M} &:= -\frac{1}{4}\int_M F^{ab}F_{ab}\,\sqrt{|g|}\,d^4x\,,\\ &= -\frac{1}{2}\int_M F^{ab}\nabla_{[a}A_{b]}\,\sqrt{|g|}\,d^4x\,,\end{aligned} \tag{19}$$

where $F_{ab} := 2\nabla_{[a}A_{b]}$. The variation of the action is given by

$$\delta S_{\rm M} = \int_M (\nabla_a F^{ab})\delta A_a\,\sqrt{|g|}\,d^4x - \int_{\partial M} F^{ab}\delta A_b\,d\Sigma_a. \tag{20}$$

The volume term tells us that the action is extremized when $\nabla_a F^{ab} = 0$. Since we are assuming that there exists a one-form A_a such that its exterior derivative is the Maxwell field $F_{ab} := 2\nabla_{[a}A_{b]}$, the equation $\nabla_{[a}F_{bc]} = 0$ is automatically satisfied (the Bianchi identity). Therefore we have the full set of Maxwell equations. The second term in Eq. (20), the boundary term, is often referred to as the *symplectic current*. It can be interpreted as a 1-form on the space $\bar{\Gamma}$ of solutions to the equations of motion (it is analog to the symplectic potential ω introduced in Sect. 2). It is acting on the vector δA_a and producing a number. We can take now another 'variation' of this term in order to get the conserved (pre)-symplectic structure $\Omega(\cdot,\cdot)$,

$$\Omega(\delta A, \widetilde{\delta A}) := \int_\Sigma (\delta F^{ab}\widetilde{\delta A_b} - \widetilde{\delta F}^{ab}\delta A_b)\,d\Sigma_a\,, \tag{21}$$

where $\sum$ is any Cauchy surface in the space-time $M^{(c)}$. We have not been very precise about functional analytic issues. We are just requiring falloff conditions (on any Σ) such that the symplectic form at spatial ∞ vanishes. If, in particular, we restrict ourselves to solutions of the Maxwell equations

that induce data of *compact support* on any Cauchy surface, that conditions will be satisfied[(d)]. This bilinear mapping defined by Ω is, however, degenerate. There are tangent vectors X_α such that $\Omega(X,Y) = 0,\ \forall\ Y \in T\bar{\Gamma}^{(e)}$. These are the degenerate directions of Ω. The fact that the two-form Ω is degenerate on $\bar{\Gamma}$ is an indication that there is some *gauge freedom* in the system. Let us now try to identify what the degenerate directions of Ω are. Since we are restricting ourselves to the space $\bar{\Gamma}$, the tangent vectors satisfy the *linearized* equation of motion, that in this case coincide with the Maxwell equations. Consider vectors of the type $X_a = \nabla_a \Lambda$ for some function Λ. Then, using the fact that it satisfies $\nabla_{[a} X_{b]} = 0$ we have,

$$\Omega(X, \delta A) = \int_\Sigma -\delta F^{ab} \nabla_b \Lambda \ d\Sigma_a$$
$$= \int_\Sigma \Lambda \nabla_b (\delta F^{ab}) \ d\Sigma_a = 0\,.$$

We can conclude that the degenerate directions of Ω are of the form $\nabla_a \Lambda$. This is the manifestation, in the covariant phase space approach, of the "gauge freedom" present in electromagnetism. In order to get a true symplectic space, we should take the quotient of $\bar{\Gamma}$ by the degenerate directions of Ω to get Γ, the (reduced) phase space of the theory. Note that Γ can be equivalently parameterized by the equivalence class of gauge potentials $[A_a]$, where $A \sim \bar{A}$ iff $A_a = \bar{A}_a + \nabla_a \Lambda$, or alternatively, by the gauge fields F_{ab}, satisfying Maxwell equations.

We can now write the (weakly non-degenerate) symplectic form on Γ:

$$\Omega(F, \tilde{F}) = \int_\Sigma (F^{ab} \tilde{A}_b - \tilde{F}^{ab} A_b) \ d\Sigma_a\,. \qquad (22)$$

Note that it is well defined on Γ since it does not depend on the representative of the equivalence class $[A]$. Note that in writing (22) we have used the fact that Γ is a linear space and therefore we can identify points in Γ with tangent vectors.

The next step is to construct observables of the theory, namely, real valued functions on Γ. A natural strategy is to use the symplectic form in order to construct such functions. Let h_a be a "test 1-form". The observable $\mathcal{O}[h] : \Gamma \to \mathbb{R}$, labeled by h, is defined in complete analogy with Sect. 2 by the expression,

$$(\mathcal{O}[h])(F) := \Omega(F, T) = \int_\Sigma (F^{ab} h_b - T^{ab} A_b) \ d\Sigma_a\,, \qquad (23)$$

where $T_{ab} := 2\nabla_{[a} h_{b]}$. We need it to be a well defined function on Γ, so $\mathcal{O}[h]$ should be invariant under gauge transformations $A_a \to A_a + \nabla_a \Lambda$. Thus, we have to require that

$$\int_\Sigma T^{ab} \nabla_b \Lambda \ d\Sigma_a = 0\,, \qquad (24)$$

which implies $\nabla_a T^{ab} = 0$. Therefore, an element h_a of Γ defines by itself a linear observable, since h_a and $h_a + \nabla_a \Lambda$ define the same function. In the quantum theory, to each of this observables there will correspond a quantum operator, making the correspondence between solutions to Maxwell equations and quantum operators precise.

Let us re-write the symplectic form (22) in terms of the familiar electric and magnetic fields. Recall that given a local observer with four velocity t^a ($t_a t^a = -1$), then the *electric field* with respect to this observer is given by $E_a := t^b F_{ba}$. It is naturally defined as a 1-form. Since we have a metric we can 'raise' the index and define the corresponding vector field. We can also define the *dual tensor* of the field F_{ab} by: ${}^*F^{ab} := \frac{1}{2}\epsilon^{abcd} F_{cd}$, where ϵ^{abcd} is the canonical volume form defined by the metric g_{ab} with all its indices raised with the metric. The magnetic field is defined by $B_a := t^b {}^*F_{ba}$. In the integrand of the symplectic form, one is contracting the tensor F^{ab} with the unit normal n_a to the surface Σ (that is the meaning of $d\Sigma_a := \epsilon_{abcd}\, d\Sigma^{bcd}$), so we get naturally the electric field E^a with respect to Σ. We can now express (22) as follows:

$$\Omega(F, \tilde{F}); = \int_\Sigma (E^a \tilde{A}_a - \tilde{E}^a A_a) \ \sqrt{h}\, d^3x\,. \qquad (25)$$

This expression can be rewritten in terms of objects defined purely on the hyper-surface $\sum$. We can write

$$F^{ab} A_b \, d\Sigma_a = \frac{1}{2} \epsilon^{abcd} {}^*F_{cd} \, A_b \, d\Sigma_a\,,$$
$$= \frac{1}{2} {}^*F_{cd} \, A_b \, \epsilon^{abcd} \epsilon_{afgh} \, d\Sigma^{fgh}\,,$$
$$= -\frac{1}{2} {}^*F_{cd} \, A_b \, d\Sigma^{cdb}\,.$$

Therefore, one can take the 3-form ${}^*F \wedge A$ and integrate it on Σ,

$$\Omega(F, \tilde{F}) = -\frac{1}{2} \int_\Sigma ({}^*F_{[ab} \tilde{A}_{c]} - {}^*\tilde{F}_{[ab} A_{c]}) \ d\Sigma^{abc}. \qquad (26)$$

Note that the pullback to Σ of the dual tensor ${}^*F_{ab}$ is, in a 3-dimensional sense, the electric field two-form: $E_{ab} := {}^*F_{ab}$. This is naturally dual to a vector density of weight one $\tilde{E}^c := \tilde{\eta}^{cab} E_{ab}$, which is, as we shall later see, the electric field arising from the canonical approach. Here, $\tilde{\eta}^{abc}$ is the naturally defined completely anti-symmetric Levi-civita density of weight one on Σ.

Finally, one can ask what the Poisson bracket of the observables defined by (23) is. Given h_a and h'_a in Γ the Poisson bracket of the observables they define is given by

$$\{\mathcal{O}[h], \mathcal{O}[h']\} := \Omega(T, T')$$
$$= \int_\Sigma (T^{ab} h'_b - T'^{ab} h_b) \ d\Sigma_a\,. \qquad (27)$$

We have seen that starting from the action, there is a naturally defined symplectic structure Ω on Γ. We constructed the lineal observables $\mathcal{O}[h]$, the generators of the algebra $\mathcal{S}$ and computed the Poisson bracket amongst them. We shall now go to the canonical approach.

3.2. Canonical Phase Space

In this part we shall present the canonical phase space description of the Maxwell Field, which is normally known as the 'Dirac analysis' [9]. However, our presentation will be 'covariant' in the sense that our analysis is coordinate free; that is, we do not assume any coordinate system on M. The action (19) can be written in a 3+1 fashion. First we write the expression for the action as follows,

$$S = -\frac{1}{4}\int_M g^{ab}g^{cd}F_{ac}F_{bd}\,\sqrt{|g|}\,d^4x \qquad (28)$$

Next, we decompose the space-time metric as follows: $g^{ab} = h^{ab} - n^a n^b$. Here h^{ab} is the (inverse of) the induced metric on the Cauchy hyper-surface Σ and n^a the unit normal to Σ. We also introduce an everywhere time-like vector field t^a and a 'time' function t such that the hyper-surfaces t =constant are diffeomorphic to Σ and such that $t^a\nabla_a t = 1$. We can write $t^a = Nn^a + N^a$. The volume element is given by $\sqrt{|g|} = N\sqrt{h}$. Using this identities in Eq. (28) we get,

$$\begin{aligned} S = -\frac{1}{4}\int_I dt \int_\Sigma N\sqrt{h}\Big\{ & h^{ac}h^{bd}F_{ab}F_{cd} \\ & - \frac{2}{N^2}h^{ac}\left[(\mathcal{L}_t A_a - \nabla_a(t\cdot A) + N^b F_{ab})\right. \\ & \left.\times\,(\mathcal{L}_t A_c - \nabla_c(t\cdot A) + N^d F_{cd})\right]\Big\}, \end{aligned} \qquad (29)$$

where $(t\cdot A) := t^b A_b$, and $I = [t_0, t_1]$ is an interval in the real line.Note that since for all the terms in the previous equation, both the one-form A_a and the field strength F_{ab} are contracted with purely "spatial" objects ($n^a N_a = n^a h_{ab} = 0$), then both A_a and F_{ab} in (29) are the pull-backs to Σ of the space-time objects. For simplicity, we shall continue to write A_a for the 3-dimensional potential.

From the 3+1 form of the action (29) we can find the momenta canonically conjugated to A_a:

$$\begin{aligned} \tilde{\Pi}^a &:= \frac{\delta S}{\delta(\mathcal{L}_t A_a)} \\ &= \frac{\sqrt{h}}{N}h^{ac}(\mathcal{L}_t A_c - \nabla_c(t\cdot A) + N^d F_{cd}). \end{aligned} \qquad (30)$$

It can be rewritten as,

$$\begin{aligned} \tilde{\Pi}^a &= \frac{\sqrt{h}}{N}h^{ac}(t^b - N^b)F_{bc} \\ &= \frac{\sqrt{h}}{N}h^{ac}Nn^b F_{bc} = \sqrt{h}\,E^a\,, \end{aligned} \qquad (31)$$

thus, the canonically conjugated momenta is just the *densitized* electric field (w.r.t. Σ). In this subsection, a 'tilde' over a tensor means that it is a density of weight one.

The Eq. (30) can be solved for the 'velocity', $\mathcal{L}_t A_a$,

$$\mathcal{L}_t A_a = \frac{N}{\sqrt{h}}h_{ac}\tilde{\Pi}^a + \nabla_c(t\cdot A) - N^d F_{cd} \qquad (32)$$

We can perform a Legendre transform of the Lagrangian density in order to find the Hamiltonian:

$$\begin{aligned} H :&= \int_\Sigma d^3x\,\left(\tilde{\Pi}^a\mathcal{L}_t A_a - \tilde{\mathcal{L}}\right) \\ &= \int_\Sigma d^3x\,\Big[-(t\cdot A)\nabla_a\tilde{\Pi}^a - N^d B_{ad}\tilde{\Pi}^a \\ &\quad + \frac{N}{2\sqrt{h}}h_{ac}\tilde{\Pi}^a\tilde{\Pi}^c + \frac{N\sqrt{h}}{4}h^{ac}h^{bd}B_{ab}B_{cd}\Big]. \end{aligned} \qquad (33)$$

We have denoted by $B_{ab} = \underline{F_{ab}}$ the field strength of the 3-dimensional potential A_a. It is related to the magnetic field in the following way: $B^a := \frac{1}{\sqrt{h}}\tilde{\eta}^{abc}B_{bc}$. The last term in (33) can be rewritten: $h^{ac}h^{bd}B_{ab}B_{cd} = B^e B^f \epsilon^{cd}{}_e\epsilon_{cdf} = 2h_{ab}B^a B^b$. In the 'Dirac analysis' of the action (28) the first step is to identify the *configuration variables*. In this case, these are pairs $[\phi := (t\cdot A), A_a]$, that is, we have four configuration degrees of freedom per point. In the action there is no term corresponding to time derivative of ϕ so we have a primary constraint $\chi_1 = \tilde{\Pi}_\phi \approx 0$. The basic Poisson brackets are,

$$\begin{aligned} \{A_a(x), \tilde{\Pi}^b(y)\} &= \delta^b_a\delta^3(x,y) \\ \{\phi(x), \tilde{\Pi}_\phi(y)\} &= \delta^3(x,y)\,. \end{aligned} \qquad (34)$$

Asking that the constraint be preserved in time with respect to the Hamiltonian (33) leads to the secondary constraint $\chi_2 := \nabla_a\tilde{\Pi}^a \approx 0$. There are no extra constraints. They form a *First Class* system[f]. One can eliminate the first one by giving the gauge condition $\chi_3 := \phi - \lambda(\bar{x}) \approx 0$, with λ an arbitrary function on Σ. We can reduce the constraints (χ_1, χ_3) since they form a second class pair. We are then left with the Gauss constraint $\chi_2 = \nabla_a\tilde{\Pi}^a \approx 0$. Now, ϕ has the role of a *Lagrange multiplier*. Therefore, the phase space Γ' is coordinatized by the pairs $(A_a, \tilde{\Pi}^b)$, having three degrees of freedom per point. The constraint surface $\hat{\Gamma}$ are the point in Γ' where the Gauss constraint is satisfied. In the canonical picture, gauge transformations are those canonical transformations generated by the (first class) constraints. The reduced phase space Γ_c is then the space of orbits generated by the gauss constraint in $\hat{\Gamma}$. The canonical transformation generated by the (smeared) Gauss constraint, $G[\lambda] = \int_\Sigma \lambda\nabla_b\tilde{\Pi}^b d^3x$, is given by,

$$A_a \longrightarrow A_a - \nabla_a\lambda\,. \qquad (35)$$

Therefore, the (reduced) phase space is given by pairs $([A], \tilde{\Pi})$ of gauge equivalence class of connections and vector densities satisfying Gauss' law. Thus, we recover the two *true degrees of freedom* the the Maxwell field has (corresponding to the two types of polarization). One alternative to the reduced phase space description is to impose a *gauge condition* in order to select one particular representative from the equivalence class. A convenient gauge choice in this case is to ask

that $\chi_4 := \nabla^a A_a = 0$. This is a good gauge condition since the pair (χ_2, χ_4) forms a second class pair[(g)]. Thus, we can coordinatize Γ_c by (A_a, E^a), a pair of divergence-less (transverse) vector fields on Σ. We have used the fact that we have a metric on Σ to de-densitize the momenta $\tilde{\Pi}$.

The Poisson brackets (34) induce a (weakly) non-degenerate symplectic form Ω on pairs of tangent vectors $(\delta A, \delta E)$ on $T^*\Gamma'$:

$$\Omega\left((\delta A, \delta E); (\delta A', \delta E')\right) = \int_\Sigma \sqrt{h}\, d^3x \left(\delta A'_a \delta E^a - \delta A_a \delta E'^a\right) . \quad (36)$$

The Poisson brackets on transverse traceless quantities (The Dirac bracket in the standard terminology) are given by,

$$\{A^{\mathrm{T}}_a(x), E^b_{\mathrm{T}}(y)\} = \delta^a_b \delta^3(x,y) - \Delta^{-1} D^b D_a \delta^3(x,y), \quad (37)$$

where Δ is the Laplacian operator compatible with the metric h_{ab}.

We can now relate the two approaches and see that the phase space Γ from last section is precisely the space Γ_c constructed via the canonical approach. The key observation is that there is a one to one correspondence between a pair of initial data of compact support on Σ, satisfying the transverse condition, and solutions to the Maxwell equations on M, modulo gauge transformations (an element of Γ) [4]. Therefore, to each element F_{ab} in Γ there is a pair (A_a, E^a) on Γ_c ($2\nabla_{[a}A_{b]} = \underline{F_{ab}}$ and $E^a = h^{ab}n^c F_{cb}$ and more importantly, for each pair, there is a solution to Maxwell's equations that induces the given initial data on Σ. Here, 'underline' denotes restriction to Σ. From now on, we shall refer to elements of the vector space Γ in-distinctively either as F_{ab} or as (A_a, E^b).

Observables for the space Γ can be constructed directly by giving smearing functions on Σ (compare to the discussion of the previous section in which the observables were constructed from *space-time* smearing objects). Given a 1-form g_a on Σ we can define,

$$E[g] := \int_\Sigma \sqrt{h}\, d^3x\, E^a g_a \,. \quad (38)$$

Similarly, given a vector field f^a we can construct,

$$A[f] := \int_\Sigma \sqrt{h}\, d^3x\, A_a f^a \,, \quad (39)$$

Asking that $E[g]$ be gauge invariant does not impose any condition on g_a, since Gauss' law does not 'move' the electric field. Note however that $E[g]$ takes the same value for g_a and $g_a + \nabla_a \lambda$. It is convenient to restrict ourselves to g_a satisfying $\nabla^a g_a = 0$. The requirement that $A[f]$ be gauge invariant tells us that $\nabla_a f^a = 0$. Therefore, in order to get well defined operators, we need the pairs (g_a, f^b) to belong to the phase space Γ. These are the precise images of the observables (23) given by the identification of phase spaces. The relation is given by $g_a = \underline{h_a}$ and $f^a = 2\nabla^{[a}h^{b]}n_b$.

Note that any pair of test fields $(g_a, f^a) \in \Gamma$ defines a linear observable, but they are 'mixed'. More precisely, a one-form g_a in Σ, that is, a pair $(g_a, 0) \in \Gamma$ gives rise to an *electric field* observable $E[g]$ and, conversely, a vector field $(0, f^a) \in \Gamma$ defines a *connection* observable $A[f]$.

As we have seen, the phase space Γ can be alternatively described by equivalence classes of solutions to the Maxwell equations in the covariant formalism or by pairs of transverse vector fields on a Cauchy surface Σ in the canonical approach. In both cases, the elements of the algebra $\mathcal{S}$ to be quantized are linear functionals of the basic fields. In the covariant case they are constructed out of space-time smearing fields and in the canonical language out of a pair of space smearing fields. In the next section we consider the construction of the quantum theory.

4. Quantization

In this section we shall construct the quantum theory. This section is divided into four parts. In the first one we construct the one-particle Hilbert space $\mathcal{H}_0$ from the phase space Γ of the classical theory. In the second part, we introduce the symmetric Fock space $\mathcal{F}$ associated with the one-particle Hilbert Space $\mathcal{H}_0$. In the third part we find representations of the CCR an the given Fock space. Finally, in the last part we give some examples.

4.1. One-particle Hilbert Space

The first step in the quantization program is to identify the 1-particle Hilbert space $\mathcal{H}_0$. The strategy is the following: start with (Γ, Ω) a symplectic vector space and define $J : \Gamma \to \Gamma$, a linear operator such that $J^2 = -1$. The *complex structure* J has to be compatible with the symplectic structure. This means that the bilinear mapping defined by $\mu(\cdot,\cdot) := \Omega(\cdot, J\cdot)$ is a positive definite metric on Γ. The Hermitian (complex) inner product is then given by

$$\langle \cdot, \cdot \rangle = \tfrac{1}{2\hbar}\mu(\cdot,\cdot) + i\tfrac{1}{2\hbar}\Omega(\cdot,\cdot). \quad (40)$$

The complex structure J defines a a natural splitting of $\Gamma_{\mathbb{C}}$, the complexification of Γ, in the following way: Define the 'positive frequency' part to consist of vectors of the form $\Phi^+ := \frac{1}{2}(\Phi - iJ\Phi)$ and the 'negative frequency' part as $\Phi^- := \frac{1}{2}(\Phi + iJ\Phi)$. Note that $\Phi^- = \overline{\Phi}^+$ and $\Phi = \Phi^+ + \Phi^-$. Since $J^2 = -1$, the eigenvalues of J are $\pm i$, so one is decomposing the vector space Γ in eigenspaces of J: $J(\Phi^\pm) = \pm i\Phi^\pm$. We have used the term 'positive frequency' since in the case of M Minkowski space-time that is the standard decomposition. The Hilbert space $\mathcal{H}$ is the completion of Γ with respect to the inner product (40).

There are two alternative but completely equivalent description of the 1-particle Hilbert space $\mathcal{H}_0$:

1. $\mathcal{H}_0$ consists of *real* valued functions (solution to the Maxwell equation for instance), equipped with the complex structure J. The inner product is given by (40).

2. $\mathcal{H}_0$ is constructed by complexifying the vector space Γ (tensoring with the complex numbers) and then decomposing it using J as described above. In this construction, the inner product is given by,

$$\langle \Phi, \tilde{\Phi} \rangle = \frac{i}{\hbar} \Omega(\Phi^-, \tilde{\Phi}^+) \tag{41}$$

Note that in this case, the 1-particle Hilbert space consists of 'positive frequency' solutions.

It is important to note that the only input we needed in order to construct $\mathcal{H}$ was the complex structure J. For a general space-time there is no preferred one. This in turn leads to the infinite ambiguity in the representation of the CCR. In the case of stationary space-times there is a preferred, canonical, complex structure given by the Killing field. This construction for the case of the Klein Gordon field is described in [10]. For Minkowski space-time there are several ways of characterizing the usual quantization. The standard textbook treatment uses a (globally inertial) time coordinate t to perform the positive-frequency decomposition. Another way of selecting this decomposition is to ask that the vacuum on the resulting theory be Poincaré invariant. A third way is to ask that the coherent states in the quantum theory have the same energy as the classical solution on which they are peaked [12].

4.2. Fock Space

Given a Hilbert space $\mathcal{H}$ there is a natural way of constructing its associated Fock Space. In this part we shall describe this universal construction of the Fock space associated to the Hilbert space $\mathcal{H}$ and then give in detail the representation for the Maxwell field in Minkowski space-time.

The *symmetric Fock space* associated to $\mathcal{H}$ is defined to be the Hilbert space

$$\mathcal{F}_s(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \left(\bigotimes^n{}_s \mathcal{H} \right), \tag{42}$$

where we define the *symmetrized tensor product* of $\mathcal{H}$, denoted by $\bigotimes^n{}_s\mathcal{H}$, to be the subspace of the n-fold tensor product $(\bigotimes^n \mathcal{H})$, consisting of totally symmetric maps $\alpha : \overline{\mathcal{H}}_1 \times \cdots \times \overline{\mathcal{H}}_n \to \mathbb{C}$ satisfying

$$\sum |\alpha(\bar{e}_{i_1}, \ldots, \bar{e}_{i_n})|^2 < \infty . \tag{43}$$

The Hilbert space $\overline{\mathcal{H}}$ is the *complex conjugate* of $\mathcal{H}$ with $\{\bar{e}_1, \cdots, \bar{e}_j, \cdots\}$ an orthonormal basis. We are also defining $\bigotimes^0 \mathcal{H} = \mathbb{C}$.

We shall introduce the abstract index notation for the Hilbert spaces since it is most convenient way of describing the Fock space. Given a space $\mathcal{H}$, we can construct the spaces $\overline{\mathcal{H}}$, the complex conjugate space; $\mathcal{H}^*$, the *dual space*; and $\overline{\mathcal{H}}^*$ the dual to the complex conjugate. In analogy with the notation used in spinors, let us denote elements of $\mathcal{H}$ by ϕ^A, elements of $\overline{\mathcal{H}}$ by $\phi^{A'}$. Similarly, elements of $\mathcal{H}^*$ are denoted by ϕ_A and elements of $\overline{\mathcal{H}}^*$ by $\phi_{A'}$. However, by using Riesz lemma, we may identify $\overline{\mathcal{H}}$ with $\mathcal{H}^*$ and $\mathcal{H}$ with $\overline{\mathcal{H}}^*$. Therefore we can eliminate the use of primed indices, so $\overline{\phi}_A$ will be used for an element in $\overline{\mathcal{H}}^*$ corresponding to the element $\phi^A \in \mathcal{H}$. An element $\phi \in \bigotimes^n{}_s\mathcal{H}$ then consists of elements satisfying

$$\phi^{A_1 \cdots A_n} = \phi^{(A_1 \cdots A_n)} \tag{44}$$

An element $\psi \in \otimes^n \bar{\mathcal{H}}$ will be denoted as $\psi_{A_1 \ldots A_n}$. In particular, the inner product of vectors $\psi, \phi \in \mathcal{H}$ is denoted by

$$(\psi, \phi) =: \bar{\psi}_A \phi^A . \tag{45}$$

A vector $\Psi \in \mathcal{F}_s(\mathcal{H})$ can be represented, in the abstract index notation as

$$\Psi = (\psi, \psi^{A_1}, \psi^{A_1 A_2}, \ldots, \psi^{A_1 \ldots A_n}, \ldots) , \tag{46}$$

where, for all n, we have $\psi^{A_1 \ldots A_n} = \psi^{(A_1 \ldots A_n)}$. The norm is given by

$$|\Psi|^2 := \overline{\psi}\psi + \overline{\psi}_A \psi^A + \overline{\psi}_{A_1 A_2} \psi^{A_1 A_2} + \cdots < \infty . \tag{47}$$

Now, let $\xi^A \in \mathcal{H}$ and let $\bar{\xi}_A$ denote the corresponding element in $\overline{\mathcal{H}}$. The *annihilation operator* $\mathcal{A}(\bar{\xi}) : \mathcal{F}_s(\mathcal{H}) \to \mathcal{F}_s(\mathcal{H})$ associated to $\bar{\xi}_A$ is denoted by

$$\mathcal{A}(\bar{\xi}) \cdot \Psi := (\bar{\xi}_A \psi^A, \sqrt{2}\, \bar{\xi}_A \psi^{AA_1}, \sqrt{3}\, \bar{\xi}_A \psi^{AA_1 A_2}, \ldots) . \tag{48}$$

Similarly, the *creation operator* $\mathcal{C}(\xi) : \mathcal{F}_s(\mathcal{H}) \to \mathcal{F}_s(\mathcal{H})$ associated with ξ^A is defined by

$$\mathcal{C}(\xi) \cdot \Psi := (0, \psi \xi^{A_1}, \sqrt{2}\, \xi^{(A_1} \psi^{A_2)}, \sqrt{3}\, \xi^{(A_1} \psi^{A_2 A_3)}, \ldots) . \tag{49}$$

If the domains of the operators are defined to be the subspaces of $\mathcal{F}_s(\mathcal{H})$ such that the norms of the right sides of Eqs. (48) and (49) are finite then it can be proven that $\mathcal{C}(\xi) = [\mathcal{A}(\bar{\xi})]^\dagger$. It may also be verified that they satisfy the commutation relations,

$$[\mathcal{A}(\bar{\xi}), \mathcal{C}(\eta)] = \bar{\xi}_A \eta^A \, \mathrm{I} . \tag{50}$$

A more detailed treatment of Fock spaces can be found in [3, 13, 14].

4.3. Representation of the CCR

In the previous section we saw that we could construct linear observables in (Γ, Ω), in either of the classical constructions. For the covariant picture the observables are given by (23) and in the canonical by (38) and (39). This is the set $\mathcal{S}$ of observables for which there will correspond a quantum operator. Thus, for $\mathcal{O}[h] \in \mathcal{S}$ there is an operator $\hat{\mathcal{O}}[h]$. We want the *canonical commutation relations* to hold,

$$\left[\hat{\mathcal{O}}[h], \hat{\mathcal{O}}[\tilde{h}]\right] = i\hbar\{\mathcal{O}[h], \mathcal{O}[\tilde{h}]\} = i\hbar\,\Omega(h, \tilde{h})\,. \qquad (51)$$

Then we should find a Hilbert space and a representation thereon of our basic operators satisfying the above conditions. We have all the structure needed at our disposal. Let us take as the Hilbert space the symmetric Fock space $\mathcal{F}_{\rm s}(\mathcal{H})$ and let the operators be represented as

$$\hat{\mathcal{O}}[h] \cdot \Psi := \hbar\left[\mathcal{C}(h) + \mathcal{A}(\overline{h})\right] \cdot \Psi\,. \qquad (52)$$

Let us denote by h^A the abstract index representation corresponding to h_a in $\mathcal{H}$. First, note that by construction the operator is self-adjoint. It is straightforward to check that the commutation relations are satisfied,

$$\begin{aligned}\left[\hat{\mathcal{O}}[h], \hat{\mathcal{O}}[h']\right] &= \hbar^2\left[\mathcal{C}[h], \mathcal{A}[\overline{h}']\right] + \hbar^2\left[\mathcal{A}[\overline{h}], \mathcal{C}[h']\right] \\ &= \hbar^2\left(\overline{h}_A h'^A - \overline{h}'_A h^A\right) \\ &= \hbar^2\left(\langle h, h'\rangle - \langle h', h\rangle\right) \\ &= 2i\hbar^2\,\mathrm{Im}(\langle h, h'\rangle) = i\hbar\,\Omega(h, h')\,, \qquad (53)\end{aligned}$$

where we have used (50) in the second line and (40) in the last line. Note that in this last calculation we only used general properties of the Hermitian inner product and therefore we would get a representation of the CCR for *any* inner product $\langle\cdot,\cdot\rangle$. Since the inner product is given in turn by a complex structure J, we see that there is a one to one correspondence between them.

4.4. Examples

As mentioned at the end of Sect. 4.1, the choice of a complex structure J is far from being a straightforward process. For a general space-time, there is no *a priori* criteria to select one. Furthermore, there are an infinite number of choices that give *inequivalent* quantum theories [3]. In the special case that there exists a time-like Killing vector field t^a on the space-time (M, g); that is, for a stationary space-time, there exists a canonical choice of complex structure given by the killing field. From the physical viewpoint, this choice is motivated because it gives to coherent states peaked at a particular solution an energy equal to the classical energy associated to that solution [12]. The complex structure is given by,

$$J := -(-\mathcal{L}_t \cdot \mathcal{L}_t)^{-1/2}\,\mathcal{L}_t. \qquad (54)$$

A particular important example of a space-time with a globally defined Killing field is Minkowski space-time (in fact it has an infinite number of such vector fields, one for each inertial reference frame). From now on, let us restrict our attention to Minkowski space-time and inertial hypersurfaces Σ. Therefore, the induced metric h_{ab} is the Euclidean flat metric. We will perform two different decompositions of Γ, for two different complex structures. First, we shall consider the ordinary 'positive frequency' decomposition. This leads to the standard quantum theory of the free Maxwell field found in textbooks. Next, we decompose Γ in self-dual and anti-self-dual fields.

4.4.1. Positive frequency decomposition

Since it is completely equivalent to use the covariant or canonical notation, we shall denote elements of Γ as pairs $(A_a^{\rm T}, E_{\rm T}^a)$, of transverse (*i.e.* divergence-free) vector fields. The first step in the quantization is the introduction of the complex structure $J : \Gamma \to \Gamma$. It is given by,

$$J \cdot \begin{pmatrix} A_a \\ E_a \end{pmatrix} := \begin{pmatrix} -\Delta^{1/2} E_a \\ \Delta^{-1/2} A_a \end{pmatrix}. \qquad (55)$$

Next, we can construct the *projector operator* $K^+ : \Gamma \to \Gamma_{\mathbb{C}}$, such that $F^+_{ab} = K^+(F_{ab})$ is the *positive frequency* part of $F_{ab} \in \Gamma$. The projector is given by the following action in terms of the pairs of initial data,

$$K^+ \cdot \begin{pmatrix} A_a \\ E_a \end{pmatrix} := \frac{1}{2}\begin{pmatrix} A_a - i\Delta^{-1/2} E_a \\ E_a + i\Delta^{1/2} A_a \end{pmatrix}. \qquad (56)$$

With this definitions, we can construct the inner product in $\mathcal{H}$. For $F, \tilde{F}$ in $\mathcal{H}$ we have,

$$\begin{aligned}\langle F, \tilde{F}\rangle &= \frac{i}{\hbar}\Omega(\overline{F}^+, \tilde{F}^+) \\ &= \frac{i}{\hbar}\int_\Sigma d^3x(\overline{E}^{+a}\tilde{A}^+_a - \tilde{E}^{+a}\overline{A^+}_a) \\ &= \frac{i}{4\hbar}\int_\Sigma d^3x\,\big[(E^a\tilde{A}_a - \Delta^{1/2}A^a\Delta^{-1/2}\tilde{E}_a - \tilde{E}^a A_a \\ &\qquad + \Delta^{1/2}\tilde{A}^a\Delta^{-1/2}E_a) \\ &\quad - i(\tilde{A}_a\Delta^{1/2}A^a + E^a\Delta^{1/2}\tilde{E}_a \\ &\qquad + A_a\Delta^{1/2}\tilde{A}^a + \tilde{E}^a\Delta^{-1/2}E_a)\big]\,. \qquad (57)\end{aligned}$$

The norm of $(g_a, f^a) \in \mathcal{H}$ is given by,

$$\langle (g, f), (g, f)\rangle = \frac{1}{2\hbar}\int_\Sigma d^3x\,\left[g_a\Delta^{1/2}g^a + f^a\Delta^{-1/2}f_a)\right]. \qquad (58)$$

One should keep in mind that all the objects (g_a, f^a) are transverse. The reason for this requirement is that the complex structure takes a very simple form (55) in terms of transverse vector fields, making also the expression for the norm look simple (58).

We are now in position of asking whether an observable generated by the pair (g_a, f^a) induces a well defined operator on $\mathcal{F}_s(\mathcal{H})$. Clearly, if the pair (g_a, f^a) belongs to the 1-particle Hilbert space $\mathcal{H}$ the answer is in the affirmative. We shall take this criteria also as necessary condition. The question is now whether the pair (g_a, f^a) defines an element of Γ, namely, whether they are 'well behaved' initial data for a solution of Maxwell equations with finite norm. This will be the case iff the norm of (g_a, f^a), given by Eq. (58), is finite. This question is of relevance when defining observables given by the fluxes of electric and magnetic field across surfaces bounded by closed loops. The Heisenberg uncertainty principle takes a particular simple form when this observables are considered [15].

4.4.2. *Self-dual Decomposition*

As we mentioned in the last section, one can define the dual tensor to the electro-magnetic field tensor F_{ab}, by ${}^*F_{ab} := \frac{1}{2}\epsilon_{abcd}F^{cd}$. Note that if we apply the duality $*$-operator again we get:

$$\begin{aligned} {}^*({}^*F_{ab}) &= \frac{1}{4}\epsilon_{abcd}\epsilon^{cdef}F_{ef} \\ &= -F_{ab}\,, \end{aligned} \tag{59}$$

since $\epsilon_{abcd}\epsilon^{cdef} = -4\delta^e_{[c}\delta^f_{d]}$. Therefore, the $*$-operator defines a complex structure J on Γ. Note that this structure is available for any 4-dimensional Lorentzian manifold (M, g_{ab}) without the need to introduce extra structure. As discussed above, the $*$-operation decomposed the complexification of Γ into eigenspaces with eigenvalues $\pm i$. The elements of $F^{\uparrow}_{ab}$ of $\Gamma_{\mathbb{C}}$ such that ${}^*F^{\uparrow}_{ab} = iF^{\uparrow}_{ab}$ are called *self-dual*; and those that satisfy ${}^*F^{\downarrow}_{ab} = -iF^{\downarrow}_{ab}$ are *anti-self-dual*. The corresponding projector is given by,

$$K^{\uparrow}_{ab}{}^{cd} = \frac{1}{2}(\delta^c_{[a}\delta^d_{b]} - i\epsilon_{ab}{}^{cd})\,. \tag{60}$$

Therefore, the self-dual electro-magnetic field is of the form: $F^{\uparrow}_{ab} = \frac{1}{2}(F_{ab} - i{}^*F_{ab})$. In terms of objects defined on the hyper-surface Σ, namely electric and magnetic fields, a self dual element is of the form $E_a - iB_a$. Let us now write the projector $K^{\uparrow}$ acting on the pairs (A_a, E^a),

$$K^{\uparrow}\cdot\begin{pmatrix} A_a \\ E^a \end{pmatrix} = \frac{1}{2}\begin{pmatrix} A_a + id_a \\ E^a - iB^a \end{pmatrix}, \tag{61}$$

where d_a is the *electric vector potential*, *i.e.*, such that $E^a = \epsilon^{abd}\partial_b d_c$.

Finally, we could follow the same steps as in the previous case and write the 'norm' in the 1-particle Hilbert space constructed from the $*$-operator decomposition as follows,

$$\langle (A,E),(A,E)\rangle = -\frac{1}{2\hbar}\int_\Sigma d^3x\,(E^a d_a + A^a B_a)\,. \tag{62}$$

Note that this norm, in contrast to the positive frequency decomposition case, is not *positive definite*, and is therefore, physically incorrect. In math jargon, one says that the complex structure defined by the $*$-operator is not compatible with the simplectic structure. If one were to quantize naively this "Hilbert space", one would get a Fock representation with negative norm states. In spite of this, it is possible to quantize the system when dealing with self-dual fields. A holomorphic quantization with a positive definite inner product was constructed in [16], and the corresponding loop representation is the subject of [17].

5. Discussion

In this paper, we have introduced the Fock quantization for the classical Maxwell field. We have seen that given a phase space point, that is, a solution to Maxwell equations on space-time [or equivalently, a pair (A, E) of initial data], we can construct a quantum state via a creation operator. There are several questions that come to mind. First, how can we make contact with the ordinary treatment of Fock spaces given in textbooks? Recall that, from the outset, the basic fields are written in a Fourier expansion. This already assumes a vector space structure for the background space-time (Minkowski) and a globally defined vector field (time coordinate) in order to perform the Fourier transform. The expression (58), when re-expressed in the Fourier components takes the familiar form of the inner product found everywhere. This proof is left as an exercise for the reader.

Second, we can ask how is that the particle interpretation of the theory arises? We have used solutions to Maxwell equations to create the 'n-particle states', but a classical electro-magnetic field certainly does not look like a particle. Let us recall how it is done in ordinary textbooks. In that case, the solution to the Maxwell equations is written in terms of a plane wave expansion (via a Fourier transform), and each plane wave with wave vector $\vec{k}$ is interpreted as (the wave function) of a photon of momentum in the $\vec{k}$ direction. Thus, the Fock space is constructed from plane waves, each with the interpretation of a 'particle'. Strictly speaking, plane waves are not normalizable and, therefore, do not belong to our phase space Γ.

Finally, we can ask how the Fock quantization compares with the standard Schrödinger representation we are used to in ordinary quantum mechanics. Recall that in this case, quantum states are given by complex-valued functions on configuration space $\psi(q^i)$. There is however, a unitarily equivalent representation where the wave functions are (analytic) functions on phase space $\phi(z^j = q^j - ip_j)$. This is the so called *Bargmann representation* of quantum mechanics. This is not usually done in ordinary quantum mechanics, but

we could in fact construct a Fock space for the harmonic oscillator, where the 'particles' would be quanta of energy [3]. In this case the basis is given by the $|n\rangle$ kets, corresponding to the eigenstates of the Hamiltonian. The most natural representation for this construction, in terms of wave-functions is the one given by Bargmann. Thus, the Fock representation is the field theory analog of the complex Bargmann representation (for details see [12]). Is there in field theory the analog of the Schrödinger representation? Can we construct it? The answer to both questions is in the affirmative. In the Schrödinger representation, quantum states are functionals of the potential A_a on Σ, $\Psi(A)$ and the basic observables (38) and (39) are represented as derivative and multiplicative operator respectively [18]. Just as in ordinary quantum mechanics, where the Schrödinger and Bargmann representations are connected by a *coherent state* transform, there is a similar transformation in field theory relating Schrödinger and Fock states. Which of this representations is more useful? The answer depends on the situation. Fock representations are very useful when considering scattering processes. In perturbation theory one considers incoming free states and outgoing free states (belonging to the Fock space) and one tries to approximate the Scattering matrix relating them using a perturbative expansion. The problem with this approach, from the mathematical viewpoint, is that this procedure is not completely justified [19]. To explain why, then, perturbation theory is so succesful is still an open problem. The natural way to construct a quantum theory for non-linear fields is then the Schrödinger representation (or its path integral variant), but progress in this direction has been slow [20].

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(a). Recall that a globally hyperbolic spacetime is one in which the entire history of the universe can be predicted from conditions at the instant of time represented by a hyper-surface Σ. In technical terms Σ is a Cauchy surface. For details see [4].

(b). In this notation, the index 'a' of a vector v^a is to be seen as a label indicating that v is a vector (very much like the arrow in $\vec{v}$), and it does not take values in any set. That is, 'a' is not the component of $\vec{v}$ on any basis. For details see [4–6]

(c). A Cauchy surface is a space-like surface Σ whose domain of dependence is the entire space-time M.

(d). A function of compact support is a function that vanishes outside a compact region of Σ.

(e). We denote by X_α the infinite dimensional tangent vector (with abstract index α) defined by $X_a(x)$.

(f). A first class system has the property that the Hamiltonian vector fields $X^\alpha_{\chi_1}$ and $X^\alpha_{\chi_2}$ are tangent to the $\chi_1 = \chi_2 = 0$ surface.

(g). A second class pair of constraints is such that the symplectic structure restricted to the surface they define is non-degenerate.

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