Einstein's equations as functional geodesics

Darío Núñez, Hernando Quevedo, and Alberto Sánchez
Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México
Apartado postal 70-543, 04510 México, D.F., Mexico

Recibido el 23 de abril de 1998; aceptado el 10 de junio de 1998

We present a method to reformulate Einstein's equations as functional geodesics defined on a metric manifold. Such reformulation is made at the level of the Einstein-Hilbert action by means of a harmonic map transformation between the spacetime manifold and a functional space. We then use canonical transformations and Maupertuis' variational principle in order to reduce the dimensionality of the functional space. To show the applicability of this approach, we analyze the well-known case of stationary axisymmetric fields. We show that the symmetries of the corresponding functional geodesic equations can be used to generate new solutions of Einstein's equation. In particular, we generate solutions describing the exterior field of a dyon and that of a slowly rotating body.

Keywords: Einstein's equations; harmonic map transformation

Presentamos un método de reformulación de las ecuaciones de Einstein como geodésicas funcionales definidas en una variedad métrica. Dicha reformulación se realiza al nivel de la acción de Einstein-Hilbert por medio de una transformación con mapeos armónicos de la variedad del espacio-tiempo a un espacio funcional. Posteriormente utilizamos transformaciones canónicas y el principio variacional de Maupertuis para reducir la dimensionalidad del espacio funcional. Para demostrar la efectividad del método, analizamos el caso, bien conocido, de campos axisimétricos estacionarios. Mostramos que las simetrías de las ecuaciones geodésicas funcionales correspondientes, pueden ser utilizadas para generar nuevas soluciones a las ecuaciones de Einstein. En particular, generamos una solución que describen el campo exterior de un *dyon* y otra que describe el campo exterior de un cuerpo rotando lentamente.

Descriptores: Ecuaciones de Einstein; transformación con mapeos armónicos

PACS: 04.20.-q; 04.20.Fy

1. Introduction

To simplify the structure of Einstein's equations, it is usual to postulate the existence of one or more Killing vector fields in the spacetime under consideration or, in less technical terms, the independence of certain coordinates. In a more general sense, the omission of the coordinates can be regarded as a special case of the Kaluza-Klein approach. Indeed, to investigate solutions with two Killing vectors in a systematic fashion, we can consider a Kaluza-Klein type reduction of Einstein's theory to two dimensions [1]. The dimensional reduction just amounts to dropping, for all the fields in the spacetime, the dependence on the coordinates that can be associated with the Killing vectors.

In this work, we are concerned with a different type of dimensional reduction in which the number of fields—in our case, the metric components—is reduced to the minimum necessary for describing the spacetime. This reduction occurs at the level of the Einstein-Hilbert Lagrangian and consists in dropping the terms that can be represented as total divergences, and applying canonical transformations such that the number of "dynamical variables" decreases. The "dynamical variables" are in fact coordinates on a differential manifold which we introduce by means of a harmonic map transformation which acts on the spacetime manifold [1]. A scketch of this approach has been presented in a previous work [2].

A possible application of this reformulation is to use the symmetries of the functional geodesic equations, which are solutions of the geodesic deviation equation, to generate new solutions from known ones. Several solution generating techniques have been developed during the last years. In fact, Geroch [3] proposed a method to generate solutions when the seed solutions have at least one Killing vector field. Kramer [4] introduced the concept of potential space to investigate symmetry properties of the Einstein-Hilbert Lagrangian, and studied a technique, which can be applied when the spacetime admits a non-null Killing vector field. In order to show the validity of the functional geodesic method, we apply it to stationary axisymmetric spacetimes, which have been extensively investigated in previous works [3, 4].

In Sect. 2 we define the concept of functional geodesic by means of a harmonic map, and outline a method to reduce the dimensionality of the Einstein-Hilbert Lagrangian coupled to an arbitrary matter Lagrangian.

In Sect. 3, we apply our method to stationary axisymmetric fields in vacuum. In Sect. 4, we analyze the geodesic deviation equation in the functional space and investigate the existence of affine collination vectors. It is shown that there are only three Killing vectors, which are then used to generate new solutions.

Section 5 contains an approximate solution generated by applying on the Chazy-Curzon metric the symmetries associated with the Killing vectors. We study the properties of this solution and show that it may be interpreted as describing the exterior field of a gravitational dyon. Section 6 is devoted to derivation and study of a linearized solution which contains the necessary parameters to describe the field of a slowly rotating mass. Finally, we present the conclusions and outline some possible lines of further research.

2. General approach

Let us consider a differentiable manifold M with metric $g_{ij}(x^k)$ and line element $ds^2 = g_{ij}dx^i dx^j$ defined on M. Further, we consider a Lagrangian (density) of the form $\mathcal{L} = \mathcal{L}_{EH} + \mathcal{L}_M$, where $\mathcal{L}_{EH} = \sqrt{-g}R$ is the Einstein-Hilbert Lagrangian and $\mathcal{L}_M$ represents an arbitrary matter Lagrangian that determines an action $I_{\rm grav} = \int \mathcal{L}\, d^4x$ on M. If we neglect the divergence terms of $\mathcal{L}_{EH}$, which do not contribute to the field equations, the total Lagrangian can be written as

$$\mathcal{L} = \frac{1}{2}\sqrt{-g}\, g^{ij}(\Gamma^k{}_{il}\Gamma^l{}_{jk} - \Gamma^l{}_{ij}\Gamma^k{}_{lk}) + \mathcal{L}_M\,, \qquad (1)$$

where $\Gamma^l{}_{ij}$ are the Christoffel symbols associated with the metric g_{ij}, and $i, j, \ldots = 0, 1, 2, 3$. The Einstein-Hilbert Lagrangian is then a function of the metric g_{ij} and its first order derivatives $g_{ij,k}$. Let $\mathcal{L}_M$ be a function of the matter potentials η^A, $A = 1, 2, \ldots, p$, and their first order derivatives $\eta^A_{,i}$. The variation of the action $I_{\rm grav}$ with respect to the metric g_{ij}, $\delta I_{\rm grav} = 0$, leads to the usual Einstein equations

$$R_{ij} - \frac{1}{2}R\, g_{ij} = 8\pi T_{ij}, \qquad (2)$$

where

$$T_{ij} = -\frac{\sqrt{-g}}{8\pi}\frac{\delta\mathcal{L}_M}{\delta g^{ij}}. \qquad (3)$$

We now define new variables $X^\alpha = \{g_{ij}, \eta^A\}$, where $\alpha = 1, 2, \ldots n$. The number n may have any values in the interval $p + 1 \leq n \leq p + 10$, depending on the number of independent components of g_{ij}. In fact, a transformation of the space time coordinates with four free parameters may be used to reduce to six the number of independent functions g_{ij}. However, if we use this freedom to fix *a priori* the number of dependent components of the metric, we arrive to a variational problem which in general is not equivalent to the original one. This approach is allowed only in special cases which will be treated below.

Accordingly, the Lagrangian Eq. (1) has the functional dependence $\mathcal{L} = \mathcal{L}(X^\alpha, X^\alpha_{,i})$ and is, therefore, defined in the configuration space determined by the new variables X^α and their first order derivatives $X^\alpha_{,i}$.

The introduction of the new variables X^α may be performed in a more formal way by means of a harmonic map [5, 6]. Consider a differentiable manifold N with metric $G_{\alpha\beta}(X^\gamma)$ and line element $dS^2 = G_{\alpha\beta}\, dX^\alpha\, dX^\beta$ defined on N. The map $X : M \to N$, or equivalently $X^\alpha = X^\alpha(x^i)$, is a harmonic map if the action

$$I_{hm} = \int \sqrt{-g}\, G_{\alpha\beta}\, X^\alpha_{,i}\, X^\beta_{,j} g^{ij}\, d^m x \qquad (4)$$

is extremal, *i.e.*, $\delta I_{hm} = 0$. Here g is the determinant of the metric g_{ij} and m is the dimension of M. Accordingly, the Euler-Lagrange equations following from the action Eq. (4) can be written as

$$\frac{1}{\sqrt{-g}}\left(\sqrt{-g}\, g^{ij} X^\alpha_{,i}\right)_{,j} + \Gamma^\alpha_{\beta\gamma} X^\beta_{,i} X^\gamma_{,j} g^{ij} = 0, \qquad (5)$$

where $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols associated with the metric $G_{\alpha\beta}$. If $\dim(M) = 1$, then $x^i \to \lambda$, where λ is a parameter, and Eq. (5) reduces to the geodesic equations $X^\alpha = X^\alpha(\lambda)$ for the metric $G_{\alpha\beta}$

$$\ddot{X}^\alpha + \Gamma^\alpha_{\beta\gamma}\dot{X}^\beta \dot{X}^\gamma = 0 \qquad (6)$$

on N, where a dot represents the derivative with respect to λ. A detailed analysis of harmonic maps can be found in [7, 8]. By increasing the dimensionality of M we obtain from Eq. (5) a generalization of the geodesic equation: a functional geodesic (see below).

Our main goal is to relate Einstein's equations (2) with the harmonic map Eq. (5). To this end, we introduce the concept of functional geodesic in the following manner:

A harmonic map $X : M \to N$, where M is a 4-dimensional pseudoriemannian manifold with metric g_{ij} satisfying Einstein's equations (2), will be called a *functional geodesic* if there exists a metric $G_{\alpha\beta}$ on N such that the harmonic map Eq. (5) is equivalent to Einstein's equations (2).

Clearly, our definition of functional geodesic is conditioned to the existence of a very specific metric $G_{\alpha\beta}$ which might not exist. In fact, a comparison of the actions $I_{\rm grav}$ and I_{hm} shows that for $G_{\alpha\beta}$ to exist, the gravitational action must be represented as

$$\frac{1}{2}\sqrt{-g}g^{ij}(\Gamma^k{}_{il}\Gamma^l{}_{jk} - \Gamma^l{}_{ij}\Gamma^k{}_{lk}) + \mathcal{L}_M = \sqrt{-g}G_{\alpha\beta}\, X^\alpha_{,i}\, X^\beta_{,j} g^{ij}\,, \qquad (7)$$

a relationship which, obviously, is very restrictive and cannot always be satisfied. This condition might be relaxed by allowing the two Lagrangians to differ by a null Lagrangian (*i.e.*, a Lagrangian which leads to identically vanishing Euler-Lagrange equations) or, in particular, by a total derivative. It would be interesting to find the most general form of the Lagrangian $\mathcal{L}$ satisfying Eq. (7); however, our aim in this work is to show an explicit example (stationary axisymmetric fields) in which the representation (7) exists, and to investigate in this example the advantages of representing Einstein's equations as a set of equations for a functional geodesic.

Moreover, we will present a method which can be used in order to attack the problem of finding the necessary representation (7). The method consists in treating the Lagrangian $\mathcal{L}$ as in field theory and applying to it canonical transformations which simplify the field equations. To this end, let us consider the general Lagrangian $\mathcal{L}$ in terms of the new variables X^α with the functional dependence $\mathcal{L} = \mathcal{L}(X^\alpha, DX^\alpha, x^i)$. For the sake of simplicity, we introduce the notation DX^α which represents all partial derivatives $X^\alpha_{,i}$ so that, for instance,

$$G_{\alpha\beta}\, DX^\alpha\, DX^\beta \equiv G_{\alpha\beta}\, X^\alpha_{,i} X^\beta_{,j} g^{ij} . \tag{8}$$

Starting from the general Lagrangian $\mathcal{L}$ we can construct the corresponding "Hamiltonian" $\mathcal{H}$ as

$$\mathcal{H} = \mathcal{H}(P_\alpha, X^\alpha, x^i) = P_\alpha DX^\alpha - \mathcal{L} , \tag{9}$$

where $P_\alpha = \partial\mathcal{L}/\partial DX^\alpha$ is the canonical conjugate "momentum". Notice that our construction of the Hamiltonian (9) does not follow the usual procedure of introducing a foliation with respect to which the evolution of the system may be analyzed. Our Hamiltonian is only an auxiliary function leading to a set of "Hamiltonian equations" which are equivalent to the original Euler-Lagrange equations. The advantange of using this type of Hamiltonian function is that one can introduce canonical transformations in the usual way, *i.e.* as such transformations which preserve Hamilton equations. As we will see below, this fact can be used in order to reduce the "dimensionality" of the problem.

A canonical transformation

$$P_\alpha = P_\alpha(P^{'}_\alpha, X^{'\alpha}) , \; X^\alpha = X^\alpha(P^{'}_\alpha, X^{'\alpha}) \tag{10}$$

is applied to $\mathcal{H}$ such that one of the new coordinates, say $X^{'n}$, becomes cyclic, *i.e.*, $\partial\mathcal{H}^{'}/\partial X^{'n} = 0$, where $\mathcal{H}^{'}$ is the Hamiltonian obtained from $\mathcal{H}$ by applying the canonical transformation (10). The advantage of this type of canonical transformation is that the corresponding conjugate momentum becomes a constant of motion. Accordingly, the action of $n-2$ canonical transformations of this type leads to a Hamiltonian of the form

$$\mathcal{H}^{(n-2)} = \mathcal{H}^{(n-2)}(P^{(n-2)}_\alpha, X^1_{(n-2)}, X^2_{(n-2)}, x^i), \tag{11}$$

where the index $(n-2)$ refers to quantities obtained by means of a canonical transformation. It can be seen that the canonical transformations have reduced by $n-2$ the number of independent canonical coordinates of the Hamiltonian. In principle, one could continue this procedure until one gets a system in which all the conjugate momenta are constants of motion and hence the field equations reduce to a simple system of equations which can be solved by quadratures. However, in each step one has to solve also the equations for the canonical transformations, which can become as difficult as the original field equations [9]. Therefore, when applying this procedure to concrete Hamiltonians one has to analyze the resulting equations carefully in order to decide which is the most suitable number of canonical transformations one should apply. In the general case analyzed here, we apply $n-2$ canonical transformations, *i.e.*, we keep only two independent variables $X^1_{(n-2)}$ and $X^2_{(n-2)}$, because in the explicit example we will analyze in the subsecuent sections this is exactly the number of arising independent variables.

If it happens that the conjugate momenta $P^\alpha_{(n-2)}$ (or some of them) associated with the cyclic variables enter the Hamiltonian (11) only linearly, then it is possible to eliminate them by applying a method proposed by Routh [9]. To this end, we construct the Lagrangian $\mathcal{L}^{(n-2)} = \mathcal{L}^{(n-2)}(X^1_{(n-2)}, X^2_{(n-2)}, DX^\alpha_{(n-2)}, x^i)$ corresponding to the Hamiltonian (11). Routh's method essentially consists in performing a Legendre transformations for the cyclic coordinates only, *i.e.*,

$$\mathcal{R} = \frac{\partial\mathcal{L}^{(n-2)}}{\partial DX^s_{(n-2)}} DX^s_{(n-2)} - \mathcal{L}^{(n-2)} , \tag{12}$$

where the cyclic coordinates are labeled by $X^s_{(n-2)}$, $s = 3, 4, \ldots n$. The resulting Routhian is then a function of the noncyclic coordinates and their associated "velocities", and a set of constants λ^s related to the momenta conjugate to the cyclic coordinates, *i.e.*,

$$\mathcal{R} = \mathcal{R}(X^a_{(n-2)}, DX^a_{(n-2)}, \lambda^s, x^i), \qquad a = 1, 2. \tag{13}$$

The variation of $\mathcal{R}$ with respect to X^a leads to a set of second order differential equations (Euler-Lagrange equations) which are the main field equations. When solving these equations, we can ignore the cyclic coordinates, and consider the Routhian as a Lagrangian. Additionally, the field equations for the cyclic coordinates are obtained (variation with respect to λ^s) in the Hamiltonian form.

In the procedure described above we have used the Hamiltonian (9). This implies that the general Lagrangian does not contain terms linear in the "velocities". Therefore, the Routhian (13) will contain only quadratic "velocity" terms. Hence, after applying the above procedure, we can consider a general Routhian of the form

$$\mathcal{R} = h_{ab}(X^a, \lambda^s, x^i) DX^a DX^b - V(X^a, \lambda^s, x^i),$$
$$a, b = 1, 2 \qquad s = 3, 4, \ldots n, \tag{14}$$

where V is a "potential" term that does not contain derivatives of X^a. It turns out that the further investigation of the Routhian (Lagrangian) can be divided in two different cases.

2.1. Pure kinetic Lagrangians

Consider a Lagrangian with vanishing potential $\mathcal{L} = h_{ab}(X^a, x^i) DX^a DX^b$. We have dropped the constants λ^s since we are interested in the Euler-Lagrange equations only, which in this case take the form

$$D^2X^a + \Gamma^a_{bc} DX^b DX^c + h^{ab} D(h_{bc}) DX^c = 0 , \tag{15}$$

where Γ^a_{bc} are the Christoffel symbols associated with h_{ab}. Equation (15) corresponds to the functional geodesic equations in a two-dimensional space. The last term in the left-hand side of Eq. (15) is due to the fact that the metric h_{ab} explicitly depends on the spacetime coordinates x^i which now are being used to parametrize the coordinates X^a. Thus, a solution of Eq. (15) is a two-dimensional functional geodesic that represents a solution of Einstein's equations. This reduction of the problem is of especial importance since we can now make use of all the symmetries of two-dimensional functional geodesics in order to investigate solutions of Einstein's equations. For instance, it is possible to introduce affine parameters in Eq. (15) such that the last term dissapears. Additionally, one can always perform a coordinate transformation that brings the Lagrangian into its conformally flat form, *i.e.*, $\mathcal{L} = \Sigma(X^1, X^2)[(DX^1)^2 + (DX^2)^2]$, where Σ is a conformal factor. Furthermore, one can generate solutions using the following approach. Let X^a be a solution of Eq. (15). The infinitesimal transformation

$$X^a \to X'^a = X^a + \epsilon\eta^a, \qquad (16)$$

generates a new solution of Eq. (15), to first order in ϵ, if η^a satisfies the equation

$$\nabla^2\eta^a + R^a_{\ bcd}DX^bDX^c\eta^d - D(\Gamma^a_{bc})DX^b\eta^c = 0\,, \quad (17)$$

where $R^a_{\ bcd}$ is the Riemann tensor of the metric h_{ab}, and ∇ is the total derivative "on shell"

$$\nabla = D + DX^a\frac{\partial}{\partial X^a} - [\Gamma^a_{bc}DX^bDX^c + g^{ab}D(h_{bc})DX^c]\frac{\partial}{\partial DX^a}\,. \quad (18)$$

If the metric h_{ab} does not depend explicitly on the "parameters" x^i, Eq. (17) reduces to the equation of geodesic deviation for the connecting vector η^a. Consequently, one can use Killing vectors or affine collineations of the metric h_{ab} in order to generate solutions. Also, we have used this method to study the posibility of relating seemingly unrelated problems by means of a hypersymmetry [11].

2.2. Lagrangians with potential

Let us consider the Lagrangian

$$\mathcal{L} = h_{ab}(X^a)\frac{DX^a}{D\tau}\frac{DX^b}{D\tau} - V(X^a). \qquad (19)$$

For the sake of simplicity, we assume here that the coordinates X^a are parametrized by τ. The field equations for the Lagrangian (19) can be obtained from the variation

$$\delta\int\mathcal{L}D\tau = \delta\int\frac{\partial\mathcal{L}}{\partial\frac{DX^a}{D\tau}}DX^a - \delta\int\mathcal{H}D\tau \qquad (20)$$

Since the Lagrangian (19) does not depend explicitly on the parameter τ, the corresponding Euler-Lagrange equation implies $D\mathcal{H} = 0$, *i.e.*, the associated Hamiltonian is a conserved quantity. Consequently, the last term of the right-hand side of Eq. (20) vanishes, and we can use Maupertuis' principle [9] to derive the field equations. From the Hamiltonian associated with the Lagrangian (19), we obtain

$$D\tau = \sqrt{\frac{h_{ab}DX^aDX^b}{\mathcal{H} - V}}. \qquad (21)$$

Introducing Eqs. (19) and (21) into Eq. (20), we get

$$\delta\int\mathcal{L}D\tau = 2\delta\int\sqrt{(\mathcal{H} - V)h_{ab}DX^aDX^b}\,. \qquad (22)$$

This corresponds to the variation of the two-dimensional line element

$$ds^2 = (\mathcal{H} - V)h_{ab}DX^aDX^b \qquad (23)$$

This result shows that the case of a Lagrangian with non-vanishing potential $V \neq 0$ can be reduced to the case of a pure kinetic Lagrangian with a conformally transformed metric, when the Hamiltonian is conserved, $D\mathcal{H} = 0$. The field equations are, therefore, given by Eq. (15) with h_{ab} replaced by $h'_{ab} = (\mathcal{H} - V)h_{ab}$. Accordingly, solutions of the Einstein equations are equivalent to functional geodesics of a two-dimensional space described by the metric h'_{ab}.

In the following sections we apply the method developed here to the case of stationary axisymmetric fields, for which it is well known that different solution generating techniques apply and lead to real new solutions.

3. Stationary axisymmetric fields

We begin considering the stationary axisymmetric line element in Weyl canonical coordinates

$$ds^2 = e^{2\psi}(dt - \omega d\phi)^2 - e^{-2\psi}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\phi^2], \quad (24)$$

where ψ, ω, and γ are functions of ρ and z only. If ω = const., Eq. (24) leads to the special case of static axisymmetric fields. The calculation of the corresponding scalar curvature leads to the Lagrangian density

$$\mathcal{L} = \frac{e^{4\psi}}{2\rho}(\omega_\rho^2 + \omega_z^2) + 2\rho(\psi_{\rho\rho} + \psi_{zz} - \gamma_{\rho\rho} - \gamma_{zz} - \psi_\rho^2 - \psi_z^2) + 2\psi_\rho\,, \quad (25)$$

which generates the usual Einstein equations. We proceed to construct from Eq. (25) another Lagrangian which leads to a set of functional geodesic equations as described above.

Following the description given in the previous section, Eq. (25) can be written as in Eq. (8) in the following form: [we made the substitution $\rho D^2B = D(\rho DB) - D\rho DB$ and neglected the total divergence terms],

$$\mathcal{L} = 2D\rho D\gamma + \frac{e^{4\psi}}{2\rho}(D\omega)^2 - 2\rho(D\psi)^2\,, \qquad (26)$$

where $D = (\partial_\rho, \partial_z)$, and $X^\alpha = (\psi, \omega, \gamma, \rho)$. Now, since γ and ω are already cyclic coordinates of the Lagrangian (26), we directly use the Routhian density $\mathcal{R}$, Eq. (12) for those coordinates:

$$\mathcal{R} = \frac{\partial \mathcal{L}}{\partial(D\gamma)} D\gamma + \frac{\partial \mathcal{L}}{\partial(D\omega)} D\omega - \mathcal{L}$$
$$= \frac{1}{2}\rho e^{-4\psi}\Pi_\omega^2 + 2\rho(D\psi)^2 . \tag{27}$$

Here Π_ω is the canonically conjugate "momentum" associated with the generalized "coordinate" ω. Note that the conjugate momentum Π_γ (as well as γ) does not enter the Routhian (27) at all. As a consecuence, it can be shown that the metric function γ is determined by two first order partial differential equations that can be integrated by quadratures once ψ and ω are known [12].

It follows from Eq. (27) that Π_ω is a "constant of motion" (*i.e.*, $D\Pi_\omega = 0$) in the manifold N. Using this fact, we can define an additional differential operator $\widetilde{D} = (-\partial_z, \partial_\rho)$ such that $D\widetilde{D} \equiv 0$. Introducing a function Ω by means of the relationship

$$\Pi_\omega = \rho^{-1} e^{4\psi} D\omega = \widetilde{D}\Omega , \tag{28}$$

the Routhian (27) becomes

$$\mathcal{R} = \frac{1}{2}\rho f^{-2}[(Df)^2 + (D\Omega)^2] , \tag{29}$$

where $f = \exp(2\psi)$. The variation of $\mathcal{R}$ with respect to f and Ω leads to the Euler-Lagrange equations:

$$D^2 f - f^{-1}(Df^2 - D\Omega^2) + \rho^{-1} D\rho Df = 0,$$
$$D^2\Omega - 2f^{-1} Df D\Omega + \rho^{-1} D\rho D\Omega = 0 . \tag{30}$$

It is straightforward to show that Eqs. (30) are equivalent to the principal vacuum equations ($R_{\mu\nu} = 0$) which follow from the stationary axisymmetric line element (24). For completeness, we mention that taking $E = f + i\Omega$, the Routhian density (29) can be rewritten as

$$\mathcal{R} = \frac{2\rho}{(E + E^*)^2} DE\, DE^* , \tag{31}$$

where an asterisk represents complex conjugation. The variation of Eq. (31) with respect to E or E^* leads to the Ernst equation [13]

$$(\mathrm{Re}\, E)\Delta E = (DE)^2,$$

with

$$\Delta E = D^2 E + \rho^{-1} D\rho DE . \tag{32}$$

As expressed in Eq. (29), the final Routhian may formally be interpreted as a "line element" of the form given by Eq. (14) with no potential term, so in this case it is a pure kinetic one, where h_{ab} $(a, b = 1, 2)$ is a 2×2 symmetric matrix

$$h_{ab} = \frac{1}{2}\rho f^{-2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{33}$$

and $X^a = (f, \Omega)$.

4. Symmetries of the functional geodesics

Starting from Eq. (29), we can perform an infinitesimal transformation to generate new solutions from known ones, which are called "seed" solutions. Taken the symmetry vector η^a as a function of the parameter s and the coordinates only, the symmetry equation (17), (the geodesic deviation equation), can be rewritten as

$$\eta^a_{,ss} + 2(\eta^a_{,s})_{;b} DX^b + (\eta^a_{;bc} + R^a{}_{bcd}\,\eta^d) DX^b\, DX^c = 0, \tag{34}$$

where a semicolon represents the covariant derivative associated with the metric h_{ab} given in Eq. (33). Notice that in the case that η^a is just a function of X^a, even for a metric depending on the parameter s, the symmetry equation reduces to that of affine collineations. Consider this last case, $\eta^a = \eta^a(X^b)$. Introducing the metric (33) into the symmetry equation (17), we get

$$D^2\eta^1 - 2f^{-1}(DfD\eta^1 - D\Omega D\eta^2)$$
$$+ \eta^1 f^{-2}[(Df)^2 - (D\Omega)^2] + \rho^{-1} D\rho D\eta^1 = 0,$$
$$D^2\eta^2 - 2f^{-1}(DfD\eta^2 + D\Omega D\eta^1)$$
$$+ 2\eta^1 f^{-2} Df D\Omega + \rho^{-1} D\rho D\eta^2 = 0 . \tag{35}$$

A detailed investigation of Eq. (35) shows that it possesses three independent solutions:

$$\eta_1^a = (0, 1) , \tag{36}$$

$$\eta_2^a = (f, \Omega) , \tag{37}$$

$$\eta_3^a = (f\,\Omega, \frac{\Omega^2 - f^2}{2}) . \tag{38}$$

Moreover, it can be shown that there are no affine eigencollineations, that is, the solutions (36–38) coincide with the Killing vectors of the metric (33). To find more general symmetry vectors, it is necessary to consider the most general *ansatz* $\eta^a = \eta^a(s, X^b, DX^b)$. In this work, however, we want to focus attention on the symmetry vectors (36–38) and to show that even these simple vectors can be used to connect classes of solutions with different physical properties.

We will now consider the type of solutions which can be generated by means of the vectors (36–38). Let ϵ_1, ϵ_2, and ϵ_3 be the parameters introduced by the symmetry vectors η_1^a, η_2^a, and η_3^a, respectively, according to Eq. (16). Acting on a seed solution $\{f, \Omega\}$, the vector η_1^a leads to the new functional geodesic $f' = f$ and $\Omega' = \Omega + \epsilon_1$. According to Eq. (28), this is equivalent to adding a constant ω_0 to the metric function ω. Obviously, this symmetry transformation is trivial since a coordinate transformation of the form $t' = t - \omega_0\phi$ in the line element (24) absorbes the new term. Physically, this is equivalent to the introduction of a rotating frame for the line element (24). Similarly, it is possible to show that the parameter ϵ_2 associated with the symmetry vector η_2^a can be absorbed by means of a rescaling of coordinates. The only non-trivial symmetry vector is η_3^a and it can be used to generate new solutions of the form

$$f' = f(1 + \epsilon_3\Omega) ,$$
$$\Omega' = \Omega + \frac{\epsilon_3}{2}(\Omega^2 - f^2) . \tag{39}$$

Although, when acting alone, the symmetry vectors η_1^a and η_2^a are trivial, we will see below that they are helpful when used together with η_3^a to generate non-trivial solutions. Note, moreover, that the corresponding parameters ϵ_1 and ϵ_2 can take any real value because they do not enter in the symmetry equations at all. That is, putting the infinitesimal transformation, Eq. (16), with η_1^a and η_2^a given by Eqs. (36) and (37) respectively, one sees that the resulting geodesic deviation equations, Eq. (35), are identically satisfied regardless of the values of the parameters ϵ_1 and ϵ_2, respectively. Consequently, the symmetry vectors η_1^a and η_2^a define finite symmetry transformations of the Routhian (29). In order to find the finite transformation associated with η_3^a, we consider the corresponding infinitesimal generator $\hat{\eta}_3$ defined as

$$\hat{\eta}_3 = f\Omega\frac{\partial}{\partial f} + \frac{1}{2}(\Omega^2 - f^2)\frac{\partial}{\partial\Omega} . \tag{40}$$

The repeated application of $\hat{\eta}_3$ leads to the finite transformation which corresponds to the integral curves of $\hat{\eta}_3$ and, according to Eq. (16), satisfies the differential equations (see, for instance, Ref. 14)

$$\frac{\partial f'}{\partial\epsilon_3} = f'\Omega' , \qquad \frac{\partial\Omega'}{\partial\epsilon_3} = \frac{1}{2}(\Omega'^2 - f'^2) , \tag{41}$$

with initial values $f'(\epsilon_3 = 0) = f$ and $\Omega'(\epsilon_3 = 0) = \Omega$. The integration of Eq. (41) leads to the finite transformation law

$$f' = \frac{4\,f\,(f^2 + \Omega^2)}{[\epsilon_3\,(f^2 + \Omega^2) - 2\,\Omega]^2 + 4\,f^2} ,$$
$$\Omega' = \frac{-2\,(f^2 + \Omega^2)[\epsilon_3\,(f^2 + \Omega^2) - 2\,\Omega]}{[\epsilon_3\,(f^2 + \Omega^2) - 2\,\Omega]^2 + 4\,f^2} , \tag{42}$$

which satisfies the initial value conditions. As expressed in Eq. (42), there is no obvious relationship between the transformation generated by $\hat{\eta}_3$ and other known solution generating transformations [15]. However, the existence of such a relationship cannot be excluded at this level because it is necessary to use a different representation of (42), namely as a Bäcklund transformation. Even if the transformation $\hat{\eta}_3$ turns out to be expressable in terms of known transformations, the interpretation of solutions of Einstein's equations as functional geodesics allows us to investigate more general symmetries such as affine, curvature or Ricci collineations for the metric g_{ab}. Moreover, it is possible to analyze the symmetries mentioned above also as contact transformations, *i.e.* depending on the derivatives of the seed solutions. This task will be treated in further investigations.

The finite symmetry transformation (42) can be used to generate *exact* solutions from known ones. Nevertheless, in the next sections we will use only the infinitesimal generator (40) to derive approximate solutions, since the latter are sufficient to investigate the physical significance of the solutions generated by this method, and this is the main task we are interested in the present work.

5. Exterior field of a gravitational dyon

The interest in monopole structures has rapidly increased during the past few years due to their discovery in generalizations of the standard model of particle physics. Magnetic monopoles were first introduced by Dirac [16] in electrodynamics to symmetrize Maxwell's equation in a direct way. Certainly, the most important consequence of the existence of magnetic monopoles is the quantization of electric charge. Most grand unified theories possess t'Hooft-Polyakov monopoles [17]. In general relativity there exist two different sorts of monopole structures: a magnetically charged black hole and a gravitational dyon. In fact, the magnetic black hole is the magnetic counterpart of the electrically charged black hole described by the Reissner-Nordstrom metric, and is related to it by a duality rotation. A magnetic black hole can also be interpreted as a magnetic monopole with mass greater than a determined critical value [18].

A gravitational dyon is a hypothetical object the existence of which follows from the *relativistic* character of gravitation. In Newtonian theory, the only source of gravitation is the mass. In contrast, general relativity predicts that mass as well as rotation are stationary sources of gravitational interaction. This leads to the well-known analogy between relativistic gravity and electromagnetism. The gravitational field generated by a distribution of mass turns out to be analogous to the electric field, and the field of an angular momentum current presents characteristics similar to those of a pure magnetic field. For this reason, the field generated by an angular momentum current is called "gravitomagnetic" field. For this analogy to be complete, it is necessary to require the existence of a "gravitomagnetic monopole" as the counterpart of the magnetic Dirac monopole of electrodynamics. A gravitational dyon is thus a mass endowed with a gravitomagnetic monopole. In this section, we will investigate solutions that can be generated from a static seed metric by means of a combination of symmetry transformations, and may be used to describe the exterior field of a gravitational dyon.

To give a correct interpretation of the solutions presented in this work, we will use a coordinate-invariant method based upon the investigation of the relativistic multipole moments for asymptotically flat solutions, according to the definition proposed by Geroch and Hansen [19]. We now proceed to derive the solution for a gravitational dyon. If we consider a static asymptotically flat solution $(f, \Omega = 0)$ as seed metric and apply to it the symmetry transformation associated with the vector η_3^a, we obtain a stationary solution with $f' = f$ and $\Omega' = -\epsilon_3 f^2/2$. It can be shown that for any given asymptotically flat f the new solution does not satisfy the condition of asymptotic flatness *á la* Geroch-Hansen [20]. Consequently, it is not possible to covariantly interpret the solutions generated by this type of transformation. To avoid this difficulty, we use a combination of three different symmetry transformations (36-38). To the seed static solution f we first apply the symmetry vector η_1^a with parameter ϵ_1. The resulting solution is then used as seed solution for a transformation with the vector η_2^a and parameter ϵ_2, and, finally, we apply the symmetry vector η_3^a. The new solution can be written as

$$f' = (1+\epsilon_2) f[1 + \epsilon_1\epsilon_3(1+\epsilon_2)] \,, \tag{43}$$

and

$$\Omega' = (1+\epsilon_2)\left[\epsilon_1 - \frac{\epsilon_3}{2}(1+\epsilon_2)(f^2 - \epsilon_1^2)\right] . \tag{44}$$

It is now necessary to choose the parameters introduced by the symmetry transformations such that the new solution becomes asymptotically flat. This condition leads to the relationships

$$\epsilon_1^2 = -\frac{\epsilon_2}{2+\epsilon_2} \,, \quad \text{and} \quad \epsilon_3 = -\frac{\epsilon_2}{\epsilon_1(1+\epsilon_2)^2} \,, \tag{45}$$

where ϵ_2 is a negative constant defined in the interval $\epsilon_2 \in (-2, 0)\backslash\{-1\}$. As we mentioned at the end of Sect. 2, the parameters ϵ_1 and ϵ_2 do not need to be infinitesimally small. Consequently, they can be chosen such that Eq. (45) is satisfied and ϵ_3 becomes infinitesimally small as required by the transformation law (16). In fact, even for very large values of ϵ_1, ϵ_3 remains infinitesimal and ϵ_2 remains in its domain of definition.

To analyze a concrete solution, we have to specify the asymptotically flat seed metric. Consider the Chazy-Curzon metric [21]

$$f = \exp(-2m/r) \,, \qquad r^2 = \rho^2 + z^2 \,, \tag{46}$$

where m is a positive constant. The new solution is then given by substituting Eq. (46) in Eqs. (43) and (44). Choosing the new parameters according to Eq. (45), we calculate the corresponding Geroch-Hansen multipole moments and obtain

$$M_0 = m \,, \qquad J_0 = -m\epsilon_3 \,. \tag{47}$$

There are higher mass multipole moments M_n which corresponds to the axisymmetric mass distribution of the source, and higher moments for the angular momentum current J_n which, however, can be neglected since they are proportional to ϵ_3^2. Equation (47) shows that this solution represents the gravitational field of a body with mass m and gravitomagnetic monopole $-m\epsilon_3$. Hence, the new parameter ϵ_3 may be interpreted as the specific "gravitomagnetic" mass which may be positive as well as negative. The total "gravitoelectric" mass of the seed solution has not been affected by the action of symmetry transformations. For the sake of completeness, we present the metric functions of the new solu tion:

$$\begin{aligned} f' &= \exp(-2m/r) \,, \\ \Omega' &= -2m\epsilon_3(1+\epsilon_2)^2 z/r \,, \\ \gamma' &= -m^2\rho^2/r^4 \,. \end{aligned} \tag{48}$$

Finally, we would like to mention that using the Schwarzschild metric as starting solution, it is possible to generate the linearized Taub-NUT (Newman-Unti-Tamburino) solution which is also a candidate for describing the exterior field of a gravitational dyon. In general, it should be possible to find other solutions which, being different from the Taub-NUT metric or the one presented here, present similar properties and hence might be used to describe a dyon. They all could differ only in the set of multipole moments higher than the monopole one; that is, there may exist different distributions of mass possessing the same gravitomagnetic monopole structure.

6. Field of a slowly rotating mass

For the study of the gravitational field of astrophysical bodies like stars and planets, it is necessary to investigate solutions which possess a set of mass multipole moments as well as a set of gravitomagnetic moments representing the rotation of the source. In contrast to the solution presented in the last section, a solution with realistic rotational properties may have only gravitomagnetic multipoles higher than or equal to the dipole one. In this section we derive a solution wich satisfies this condition.

Consider any stationary seed solution (f, Ω) satisfying the conditions of asymptotic flatness. As we have done in Sect. 3, we apply three consecutive symmetry transformations according to Eqs. (16) and (36)–(38). The new solution is then given by

$$f' = (1+\epsilon_2) f[1 + \epsilon_3(1+\epsilon_2)(\Omega+\epsilon_1)] \,, \tag{49}$$

$$\begin{aligned} \Omega' = (1+\epsilon_2)\Big[&\Omega + \epsilon_1 \\ &- \frac{\epsilon_3}{2}(1+\epsilon_2)(f^2 - \epsilon_1^2 - 2\epsilon_1\Omega - \Omega^2)\Big] . \end{aligned} \tag{50}$$

In general, this new solution is not asymptotically flat. However, if we demand that the parameters ϵ_1 and ϵ_2 satisfy the relationships (45), asymptotic flatness is conserved and the resulting solution can be written as

$$f' = f[1 + \epsilon_3(1 + \epsilon_2)^2\Omega] , \quad (51)$$

$$\Omega' = \Omega + \frac{\epsilon_3}{2}(1 + \epsilon_2)^2 (1 + \Omega^2 - f^2) . \quad (52)$$

The calculation of new solutions does not present any difficulties. We will present here only one solution which illustrates our approach and can easily be interpreted. Consider the seed solution [22]

$$f = \frac{x^2 - 1 + \alpha_1^2(y^2 - 1)}{(x + 1)^2 + \alpha_1^2(y - 1)^2} ,$$

$$\Omega = \frac{2\alpha_1(x + y)}{(x + 1)^2 + \alpha_1^2(y - 1)^2} , \quad (53)$$

with

$$x = \frac{1}{2m}(r_+ + r_-) ,$$

$$y = \frac{1}{2m}(r_+ - r_-) , \quad (54)$$

$$r_\pm^2 = \rho^2 + (z \pm m)^2 ,$$

where m and α_1 are constants. To illustrate the effect of symmetry transformations, we first analyze the seed solution (53). An investigation of the corresponding multipoles show that there are gravitoelectric as well as gravitomagnetic monopole and dipole moments. Due to the presence of the gravitomagnetic monopole and gravitoelectric dipole, this solution cannot be considered as a candidate for the description of the gravitational field of any astrophysical object. Hence solution (53) is of no interest from a physical point of view. However, if we apply three different symmetry transformations to solution (53), its physical meaning can totally be changed. In fact, putting Eq. (53) into Eqs. (51) and (52), and calculating the relativistic multipole moments of the resulting solution, we see that all undesirable multipole moments vanish if α_1 is assumed to take the value

$$\alpha_1 = -\epsilon_3(1 + \epsilon_2)^2 . \quad (55)$$

Then, the only nonvanishing multipoles are

$$M_0 = m , \qquad \text{and} \qquad J_1 = \epsilon_3(1 + \epsilon_2)^2 m . \quad (56)$$

The last equation shows that the total mass of the body is given by m and that only the gravitomagnetic dipole moment survives in accordance with the dipole character of rotation. The angular momentum per unit mass is given by $\epsilon_3(1 + \epsilon_2)^2$ and can be positive as well as negative, corresponding to the two possible directions of rotation of the source with respect to the symmetry axis. Consequently, the new solution may be interpreted as describing the exterior field of a slowly rotating mass. Using Eqs. (51)–(55) and (28), the calculation of the metric components leads to

$$f' = \frac{x - 1}{x + 1} ,$$

$$\omega' = 2m\epsilon_3(1 + \epsilon_2)^2 \frac{1 - y^2}{x - 1} ,$$

$$\gamma' = \frac{1}{2} \ln \frac{x^2 - 1}{x^2 - y^2} . \quad (57)$$

This is equivalent to the Lense-Thirring metric [23], the physical meaning of which has been investigated by using other approaches and coincides with that we have obtained above by just analyzing the corresponding multipole moments.

7. Conclusions

We have presented a method to reformulate the Einstein-Hilbert Lagrangian coupled to an arbitrary matter Lagrangian, in such a way that the resulting field equations can be interpreted as functional geodesics in a different manifold. This reformulation is made by means of a harmonic map. We then proceed to perform a dimensional reduction of the new Lagrangian by applying canonical transformations.

As an application of this method, and in order to prove its validity, we studied the symmetries of the functional geodesics of the space associated with axisymmetric stationary gravitational fields, and we were able to generate some solutions and to analyze their physical significance. It is also possible to show [24] that the method presented here apply to other vacuum gravitational fields like Einstein-Rosen waves, rotating gravitational waves, plane symmetric fields, rotating cylindrically symmetric gravitational waves, spherically symmetric fields, as well as to non vacuum fields like an additional pure scalar field or a perfect fluid which preserve the symmetries of the spacetime.

The reformulation of the Einstein-Hilbert action as in Eq. (7), indicates the possibility of analyzing it in the framework of other theories, like the non-linear σ models, and string theory. The investigation of these topics could be of interest. Moreover, we want to stress the fact that the idea presented here is not only a method to generate solutions, but also a different point of view to work with Einstein's equations.

Acknowledgments

This work has been supported by DGAPA-UNAM, project IN105496, and CONACyT, Mexico, project 3567-E.

1. B. Julia, in *Superspace and supergravity*, edited by S.W. Hawking and M. Rocek, (Cambridge University Press, 1980).
2. D. Núñez and H. Quevedo, *SILARG VIII. Proceedings of the* 8^{th} *Latin American Symposium on Relativity and Gravitation*, (Águas de Lindóia, Brazil, 1993), edited by P.S. Letelier and W.A. Rodriguez, Jr., (World Scientific, 1994), p. 162.
3. R.Geroch, *J. Math. Phys.* **12** (1971) 918; *ibid.* **13** (1972) 394.
4. D. Kramer, H.Stephani, E. Herlt, and M. MacCallum, *Exact solutions of Einstein's Field Equations*, (Cambridge University Press, Cambridge, 1980).
5. D. Fuller, *Proc. Roy. Soc.* **3** (1954) 234.
6. T. Matos, *Phys. Rev. D* **49** (1994) 4296.
7. T. Matos, *J. Math. Phys.* **33** (1992) 3521.
8. T. Matos, *J. Math. Phys.* **35** (1994) 1302.
9. L.D. Landau and E.M. Lifshits, *Theoretical Physics*, Vol. 1, Mechanics, (Ed. Nauka, Moscow, 1988); H. Goldstein, *Classical Mechanics*, (Addison-Wesley Publishing Company, Reading, 1980).
10. G. Neugebauer and D. Kramer, *Ann. Phys. (Leipzig)* **24** (1969) 62.
11. S. Hojman, S. Chayet, D. Núñez, and M. Roque, *J. Math. Phys.* **32** (1991) 1491.
12. In Weyl coordinates, these equations take the form $4\gamma_\rho = \rho f^{-2}(f_\rho^2 - f_z^2) - \rho^{-1} f^2(\omega_\rho^2 - \omega_z^2)$ and $2\gamma_z = \rho f^{-2} f_\rho f_z - \rho^{-1} f^2 \omega_\rho \omega_z$, with $f = \exp(2\psi)$.
13. F.J. Ernst, *Phys. Rev.* **167** (1968) 1175.
14. H. Stephani, *Differential Equations: Their solution using symmetries*, (Cambridge University Press, Cambridge, 1989).
15. See, for instance, *Solutions of Einstein's equations: Techniques and Results*, edited by C. Hoenselaers and W. Dietz (Springer, Berlin, 1984). For an introductory review, see H. Quevedo, *Fortschr. Phys.* **38** (1990) 733.
16. P.A. Dirac, *Proc. Roy. Soc. A* **133** (1931) 133; *Phys. Rev.* **74** (1948) 817.
17. G. t'Hooft, *Nucl. Phys. B* **79** (1974) 276; A.M. Polyakov, *JETP Lett.* **20** (1974) 194.
18. W.A. Hiscock, *Phys. Rev. Lett.* **50** (1983) 1734.
19. R. Geroch, *J. Math. Phys.* **11** (1970) 1955; *ibid.* **11** (1970) 2580; R.O. Hansen, *ibid.* **15** (1974) 46. For other equivalent definitions of multipole moments see: K.S. Thorne, *Rev. Mod. Phys.* **52** (1980) 299; R. Beig and W. Simon, *Comm. Math. Phys.* **78** (1980) 75; *Proc. Roy. Soc. London* **376A** (1981) 333; *Acta Phys. Austriaca* **53** (1981) 249. For an introductory review see H. Quevedo, *Fortschr. Phys.* **38** (1990) 733.
20. It can be shown that a stationary axisymmetric solution is asymptotically flat (see Ref. 19) if for $\rho = 0$ and $z \to \infty$ the metric functions behave like $f \to 1 + O(z^{-1})$ and $\Omega \to O(z^{-1})$.
21. J. Chazy, *Bull. Soc. Math. France* **52** (1924) 17; H.E.J. Curzon, *Proc. Math. Soc. London* **23** (1924) 477.
22. This is a special case of a more general solution presented in: H. Quevedo, *Phys. Rev. D* **39** (1989) 2904.
23. H. Thirring and J. Lense, *Phys. Z.* **19** (1918) 156; see also B. Mashhoon, F.W. Hehl, and D.S. Theiss, *Gen. Rel. Grav.* **16** (1984) 711.
24. A. Sánchez, *M.S. Thesis*, UNAM-Mexico, (1996).