Inhibition of two-stream plasma instabilities due to thermal pressure gradient

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We present a study of the plasma instabilities in a system consisting of two mutually interpenetrating plasma beams, taking into account the effects of the thermal pressure gradient in one of them. We discuss how this gradient may destroy these instabilities. Also we include a derivation of the dispersion relation for this system and a commonly used criteria to classify instabilities.

Keywords: Plasma waves; two-stream instabilities

Presentamos un estudio de las inestabilidades de plasma en un sistema que consiste de dos plasmas mutuamente interpenetrantes, tomando en cuenta los efectos del gradiente de presión térmica en uno de ellos. Discutimos cómo este gradiente puede destruir estas inestabilidades. Incluimos una derivación de la relación de dispersión para este sistema así como un criterio comúnmente usado para clasificar las inestabilidades.

Descriptores: ondas de plasma; inestabilidades de dos corrientes

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1. Introduction

The propagation of electromagnetic and plasma waves in complex media has been a topic of interest in many branches of physics. The wide variety of possible technological applications and also the rich amount of basic physics involved in these phenomena has lead to study them deeply and constantly through the years [1]. It is common to deal with waves whose amplitudes decrease or remain constant as they propagate. However, this is not always the case since there are systems in which waves may grow as they propagate. For example, one of the most studied systems which may exhibit this kind of behavior are plasma streams and particularly the case of two mutually interpenetrating plasma streams [2–13].

A thorough understanding of the electromagnetic wave propagation in plasma systems requires a dynamical description of the charge carriers, such as the simple hydrodynamic model. In this model one includes, besides the electromagnetic and damping forces acting on a carrier, the force due to the carrier thermal pressure gradient. This pressure has been shown to have an important role in semiconductor heterostructures [1, 14]. Since the thermal pressure gradient has not been discussed in detail for the two-stream system, the purpose of this paper is to investigate in detail its effects on the system. For this purpose, we proceed in the following way. First, we introduce the basic framework by presenting a short derivation of the dispersion relation for longitudinal normal modes of the electric field. Then, we present wave instabilities by classifying waves according to their spatial and temporal behavior and we choose one of the simplest criteria used to decide whether a solution of the dispersion relation corresponds to an instability or not. Finally, we derive the two-stream plasma system dispersion relation including the thermal pressure gradient term and discuss quantitatively how the instabilities may be suppressed by choosing appropriate values for this term.

2. Dispersion relations

Let us discuss the problem of electromagnetic wave propagation and the corresponding dispersion relations. The variables in our problem (as for example the field intensities) may be expressed as functions of the spatial coordinates $\mathbf{r}$ and time t, that is, as $\phi(\mathbf{r},t)$. If the medium is homogeneous in $\mathbf{r}$ and t, one proceeds to look for solutions of the form

$$\phi(\mathbf{r},t) = e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\,, \tag{1}$$

which are plane monochromatic waves.

One normally finds that solutions of type (1) may exist only if ω and k are appropriately related. The equation which determines permissible relations is known as the "dispersion relation". In the case of longitudinal electrostatic waves propagating in a medium, the dispersion relation can be found from Gauss law,

$$\nabla\cdot\mathbf{D} = 4\pi\rho^{\text{ext}}\,, \tag{2}$$

where $\mathbf{D}$ is the electric displacement vector and ρ^{ext} is the external charge density. The most general linear relation among the electric displacement field and the total electric field is a relation of the form

$$D_\alpha(\mathbf{r},t) = \int_{-\infty}^{t} dt' \int dV' \varepsilon_{\alpha\beta}(\mathbf{r},t;\mathbf{r}',t')E_\beta(\mathbf{r}',t')\,, \tag{3}$$

where $\varepsilon_{\alpha\beta}$ is the dielectric tensor. The physical meaning of this relation is that the response of the media (that is, its polarization or its magnetization) is non-local. In other words, the response at a point $\mathbf{r}$ and at a time t depends on the values of the electric field at other points and at previous times. The tensorial character of the response function $\varepsilon_{\alpha\beta}$ is due to the fact that in anisotropic media the direction of the $\mathbf{D}$ and $\mathbf{E}$ vectors may not coincide. If the medium is homogeneous and its properties do not depend on time, then $\varepsilon_{\alpha\beta}$ is function of $\mathbf{r}-\mathbf{r}'$ and $t-t'$. In this case, we can take the Fourier transform of Eq. (3) to get the algebraic relation

$$D_\alpha(\mathbf{k},\omega) = \varepsilon_{\alpha\beta}(\mathbf{k},\omega)E_\beta(\mathbf{k},\omega)\,. \tag{4}$$

Thus the non-locality of the relation between $\mathbf{E}$ and $\mathbf{D}$ has the result that the dielectric tensor depends on the wave vector as well as on the frequency. This is referred to as *spatial dispersion* and *time dispersion*, respectively.

The dependence on the wave vector introduce a distinctive direction in the dielectric function $\varepsilon_{\alpha\beta}(\mathbf{k},\omega)$, namely that of $\mathbf{k}$. Therefore, when spatial dispersion is present, the dielectric function is a tensor even in an isotropic medium. The general form of such a tensor may be written as

$$\varepsilon_{\alpha\beta}(\mathbf{k},\omega) = \varepsilon_t(\mathbf{k},\omega)\left(\delta_{\alpha\beta} - k_\alpha k_\beta/k^2\right) + \varepsilon_l(\mathbf{k},\omega)k_\alpha k_\beta/k^2\,. \tag{5}$$

In this equation the scalar functions ε_t and ε_l represent the response of the medium to the transverse electric field ($\mathbf{E} \perp \mathbf{k}$) and to the longitudinal electric field ($\mathbf{E} \parallel \mathbf{k}$), respectively. Assuming plane wave fields of the form (1) and if we suppose there are no free charges, then Eq. (2) takes on the form

$$i\mathbf{k}\cdot\mathbf{E}(\mathbf{k},\omega)\varepsilon_l(\mathbf{k},\omega) = 0\,. \tag{6}$$

Thus, if we look for solutions of the electric field that are parallel to the direction of propagation we require that

$$\varepsilon_l(\mathbf{k},\omega) = 0\,. \tag{7}$$

Equation (7) is the dispersion relation for the longitudinal normal modes of the electric field (or bulk plasmons) in a medium.

3. Classification of growing waves

When dealing with propagation through a medium it is sometimes convenient to consider a wave packet $U(x,t)$ rather than a monochromatic wave. This packet can be written as a superposition of plane waves of the form given by Eq. (1), where k and ω are related to each other by means of the dispersion relation as pointed out before. In order to solve Eq. (7) we may give real values for k finding generally complex values for ω; or we can give real values for ω getting possibly complex values for k. When both, k and ω are real, the wave propagates without growth or attenuation. If k is real but ω is complex it may represent two different situations: If the imaginary part of ω is negative, the perturbation is damped in the course of time. On the other hand, if the imaginary part of ω is positive for some range of k, such perturbation grows and we say that the medium is unstable with respect to oscillations in that range of wavelengths. Hence, we say that a propagating system is unstable if any small perturbation of the system grows exponentially *in time*. We say that a disturbance that is being propagated through an unstable system is a "convective instability" if the disturbance grows, but is propagated away form the point of origin, that is, at any given point x the disturbance may initially grow in time, but eventually decays away after the disturbance has passed:

$$\lim_{t\to\infty} U(x,t) = 0 \qquad \text{for } x = \text{const.} \tag{8}$$

This is illustrated in Fig. 1, where we have plotted a wave packet contours $U(x,t)$ for the one dimensional case.

On the other hand, if the disturbance grows in amplitude and in extent, but always embraces the original point of origin it is called "absolute instability". If we see any fixed point in space, the disturbance grows as time passes:

$$\lim_{t\to\infty} U(x,t) = \infty \qquad \text{for } x = \text{const.} \tag{9}$$

This is shown in Fig. 2. In any case, the wave packet must satisfy the condition:

$$\lim_{|x|\to\infty} U(x,t) = 0 \qquad \text{for } t = \text{const.} \tag{10}$$

which means that a disturbance created at a given instant and propagating at a finite speed cannot reach an infinite distant region.

It should be noted that the distinction between absolute and convective instabilities is a relative one, in the sense that it depends on the reference frame of the observer. In the case of an absolute instability, for an observer that is moving faster than the instability can spread, the response will appear as convective. In the same way, for an observer that is moving with the wave packet of a convective disturbance, it will appear absolute. This, however, does not mean that the difference between the two types of instabilities has no physical importance. In actual problems there is always a preferred experimental reference frame to which instabilities are referred.

Now, if we are concerned with the propagation of a wave packet that *grows in space*, we have to consider the situation in which the packet has real frequencies ω and the values of k, found by solving the dispersion relation, are complex. In this case we may distinguish between amplifying and evanescent waves. Amplifying waves grow in space at an exponential rate as they propagate through the medium. On the other hand, evanescent waves decay as they propagate.

A connection between convective instabilities and amplifying waves is evident from Fig. 1. In that figure, an original impulse localized in space and time grows in amplitude as it

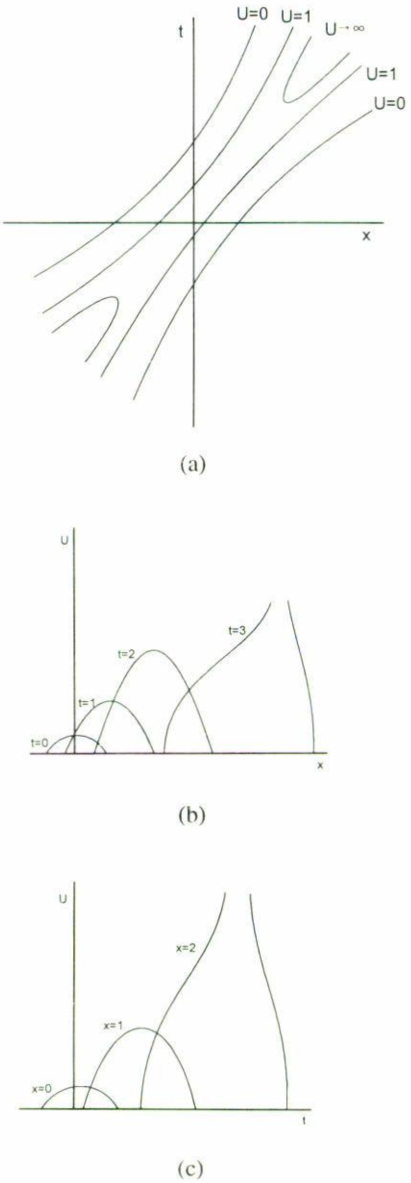

FIGURE 1. Wave packet contours $U(x,t)$ for a convective instability in one dimension. U, t, and x are the constant parameters in panels (a), (b) and (c) respectively (after Pozhela [16]).

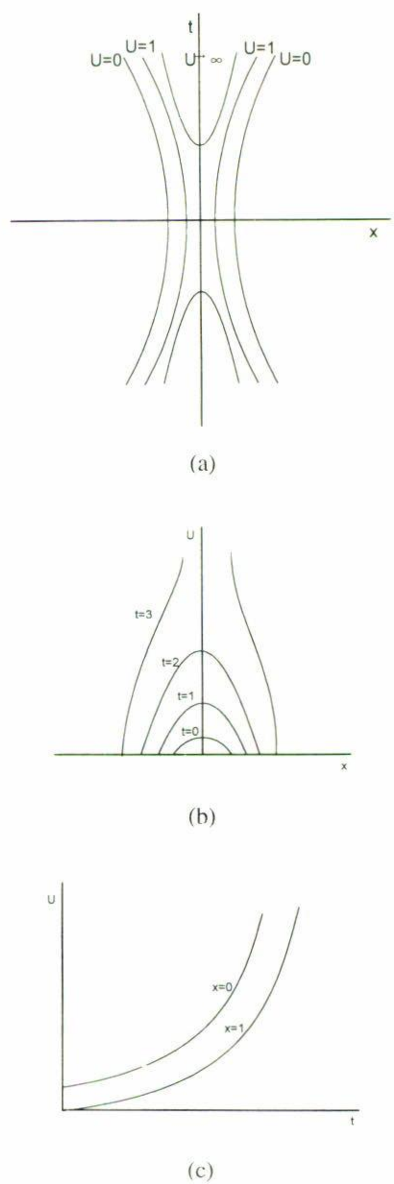

FIGURE 2. Wave packet contours $U(x,t)$ for an absolute instability in one dimension. U, t, and x are the constant parameters in panels (a), (b) and (c) respectively (after Pozhela [16]).

moves through the medium. But if instead of this impulse we perturb the same medium with a steady-state oscillating source which is spatially localized, then we expect to see a steady-state oscillation of the medium at any fixed point but the wave form shows an exponential growth in amplitude as the wave propagates away from the source as shown in Fig. 3a. This oscillation can be described as a wave with real frequency ω but complex k which is precisely an amplifying wave. On the other hand, to construct the convective wave package shown in Fig. 1, it is necessary to have a superposition of waves with some of them being amplified. In both cases the medium is the same but the initial conditions are different. Therefore, a medium that supports amplifying waves may exhibit convective instabilities. In other words, the statement that the system supports amplifying waves is synonymous with the statement that the system exhibits convective instabilities.

At this point, it is necessary to emphasize that the sign of the imaginary part of k cannot be the criterion for distinguishing between amplifying and evanescent waves. This is due, in part, because a change in the sense of propagation for a given direction is equivalent to a change in the sign of k. Nonetheless, there is a more subtle reason that arises when both, ω and k are complex. In order to see this point let us consider the wave behavior of a localized oscillatory source whose amplitude grows exponentially in time. If this growth is not very rapid, the form of the curve is similar to that shown in Fig. 3a. But if the amplitude of the oscillatory source grows

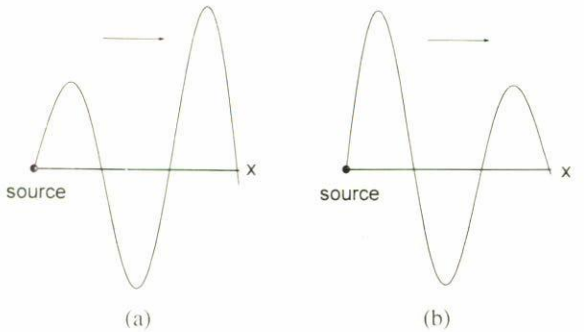

FIGURE 3. Spatially localized oscillatory source with (a) constant amplitude, (b) rapidly growing amplitude.

in time sufficiently rapid, the amplitude close to the source can be made larger than the amplitude far from the source—the latter having grown because of the convective instability at a much lower starting level. In this way the spatial gradient of the amplitude can be reversed, corresponding to the reversal of the imaginary part of k as shown in Fig. 3b. With this facts in mind, we can think on a procedure to decide whether a solution of the dispersion relation with real ω and complex k corresponds to an instability or to an evanescent wave; first we take ω with a large positive imaginary part, corresponding to growth in time of the wave and then let this imaginary part tend to zero. If in the process the imaginary part of k changes sign, then the system is convectively unstable. There are other procedures to find out if a solution of the dispersion relations correspond or not to an instability, for example, Sturrock describes a criteria [2] based on a certain diagram on complex k plane that can be constructed from knowledge of the dispersion relation. Since this criteria can be difficult to apply we have chosen the criteria described above due to its simplicity.

4. Waves on plasma streams

We proceed to obtain the dispersion relations for a system consisting of two mutually interpenetrating plasma beams. We do this by considering each plasma beam as a charged fluid formed of an specific type of charged particles designated by $s(=1,2)$. Particles of type s have mass m_s, and charge q_s. We will assume that in the unperturbed state, the density and velocities of species s are n_s^0 and v_s^0. In presence of a perturbation, the density and velocity of carrier s are

$$n_s = n_s^0 + \delta\, n_s\,, \tag{11}$$

and

$$\mathbf{v}_s = \mathbf{v}_s^0 + \delta\, \mathbf{v}_s\,. \tag{12}$$

The motion of an average carrier in stream s is given by the transport equation

$$\frac{\partial \mathbf{v}_s}{\partial t} + (\mathbf{v}_s \cdot \nabla)\mathbf{v}_s = \frac{q_s}{m_s}\left(\mathbf{E} + \frac{1}{c}\mathbf{v}_s \times \mathbf{B}\right) - \frac{\nabla P_s}{m_s n_s^0} - \upsilon_s \mathbf{v}_s\,, \tag{13}$$

where υ_s is the collision frequency, $\mathbf{E}$ and $\mathbf{B}$ are the total electric and magnetic fields in the position of the carrier, and ∇P_s is the electron thermal pressure gradient.

If we are considering two plasma beams with charges of the same sign and also that the system is electrically neutral, so that there is no electric field in the unperturbed state, then we should assume a uniform charged background with charge of opposite sign of that of the beams:

$$n_1^0 q_1 + n_2^0 q_2 + n_b^0 q_b = 0\,, \tag{14}$$

and n_b^0 and q_b are the (constant) density and charge of the background, respectively. Therefore, the only electric field is $\delta\mathbf{E}$ and it satisfies Gauss law [15]:

$$\nabla \cdot \delta\mathbf{E} = 4\pi(q_1 \delta n_1 + q_2 \delta n_2)\,. \tag{15}$$

It is also supposed that there is no magnetic field in the unperturbed state and that the magnetic fields produced by the carriers can be neglected. Finally, if we disregard collisions ($\upsilon_s = 0$), then the transport equation of an average carrier in the stream s can be written as

$$\frac{\partial \mathbf{v}_s}{\partial t} + (\mathbf{v}_s^0 \cdot \nabla)\mathbf{v}_s = \frac{q_s}{m_s}\mathbf{E} - \frac{\nabla P_s}{m_s n_s^0}\,. \tag{16}$$

Linearizing this equation we can write the equation of motion for the perturbation $\delta\mathbf{v}_s$:

$$\frac{\partial \delta\mathbf{v}_s}{\partial t} + (\mathbf{v}_s \cdot \nabla)\delta\mathbf{v}_s = \frac{q_s}{m_s}\delta\mathbf{E} - \frac{\nabla \delta P_s}{m_s n_s^0}\,, \tag{17}$$

where we wrote $P_s = P_s^0 + \delta P_s$, with P_s^0 the unperturbed electron pressure.

To complete the system of equations it is necessary to include the continuity equations

$$\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \mathbf{v}_s) = 0\,. \tag{18}$$

These can also be linearized and written for the fluctuations of carrier density

$$\frac{\partial \delta n_s}{\partial t} + \mathbf{v}_s^0 \cdot \nabla \delta n_s + n_s^0 \nabla \cdot \delta\mathbf{v}_s = 0\,. \tag{19}$$

Carrying out a Fourier analysis of the equations of motion, that is, looking for plane-wave solutions so that the first-order quantities vary as $\exp[i(\mathbf{k}\cdot\mathbf{r} - wt)]$, the linearized fluid equations [Eqs. (15), (17) and (19)] take the form

$$i\mathbf{k} \cdot \mathbf{E} = 4\pi(q_1 \delta n_1 + q_2 \delta n_2)\,, \tag{20}$$

$$-i\omega \delta\mathbf{v}_s + i(\mathbf{k} \cdot \mathbf{v}_s^0)\delta\mathbf{v}_s = \frac{q_s}{m_s}\mathbf{E} - \beta_s^2 \frac{i\mathbf{k}\delta n_s}{n_s^0}\,, \tag{21}$$

and

$$-i\omega \delta n_s + i(\mathbf{k} \cdot \mathbf{v}_s^0)\delta n_s + i n_s^0 \mathbf{k} \cdot \delta\mathbf{v}_s = 0\,. \tag{22}$$

In writing Eq. (21) it was necessary to simplify the pressure term, that appears in Eq. (13), which was written as

$$P_s = P_s^0\left(\frac{n_s}{n_s^0}\right)^\gamma = P_s^0 + \frac{\gamma P_s^0}{n_s^0}\delta n_s + \ldots \approx P_s^0 + \delta P_s, \quad (23)$$

where

$$\delta P_s = \frac{\gamma P_s^0}{n_s^0}\delta n_s\,. \quad (24)$$

Then,

$$\nabla P_s = \nabla\delta P_s = \frac{\gamma P_s^0}{n_s^0}\nabla\delta n_s = i\mathbf{k}\frac{\gamma P_s^0}{n_s^0}\delta n_s\,. \quad (25)$$

If we define the constant β_s^2 so that $\beta_s^2 = \gamma P_s^0/n_s^0 m_s$ we arrive, finally, to the simplification needed to write Eq. (21). β_s^2 depends on the properties of the electron gas, for example, if we are considering a semiconductor at zero Kelvin degrees then the electron gas is highly degenerate, the exponent $\gamma = 5/3$ and

$$\beta_s^2 = \frac{5}{3n_s^0 m_s}\frac{\hbar^2}{5m_s}(3\pi^2)^{2/3}(n_s^0)^{5/3} = \frac{2}{3}\frac{\varepsilon_{sF}^0}{m_s} = \frac{1}{3}v_{sF}^2$$

with ε_{sF}^0 the unperturbed Fermi energy, and v_{sF} the Fermi velocity. On the other hand, if we are dealing with a classical Boltzmann electron gas, then

$$\beta_s^2 = \frac{5}{3}\frac{k_B T_s}{m_s} = \frac{5}{9}v_{sT}^2$$

with k_B the Boltzmann constant, T_s the temperature of the gas, and v_{sT}^2 the root mean square velocity. These expressions change slightly when we consider a frequency regime much higher than the collision frequency. In this case, the one dimensional nature of the oscillation is maintained and the acoustical value $\gamma = 5/3$ for a gas of particles with 3 external, but not internal, degrees of freedom is not valid. Since $\gamma = (m+2)/m$, where m is the number of degrees of freedom, we have $\gamma = 3$ and $\beta_s^2 = 5v_F^2/3$ for the degenerate electron gas and $\beta_s^2 = v_T^2$ for the Boltzmann gas.

Equations (21) and (22) lead to the following expressions for the velocity and density perturbations:

$$\delta\mathbf{v}_s = \frac{i\left(q_s\mathbf{E} - \dfrac{m_s\beta_s^2}{n_s^0}\delta n_s i\mathbf{k}\right)}{m_s\left(\omega - \mathbf{k}\cdot\mathbf{v}_s^0\right)}\,, \quad (26)$$

$$\delta n_s = \frac{i n_s^0 q_s \mathbf{k}\cdot\mathbf{E}}{m_s\left[\left(\omega - \mathbf{k}\cdot\mathbf{v}_s^0\right)^2 - \beta_s^2 k^2\right]}\,. \quad (27)$$

Substituting Eq. (27) into Eq. (20) and ordering terms we get

$$i\mathbf{k}\cdot\mathbf{E}\left(1 - 4\pi\sum_{s=1}^{2}\frac{n_s^0 q_s^2}{m_s\left[\left(\omega - \mathbf{k}\cdot\mathbf{v}_s^0\right)^2 - \beta_s^2 k^2\right]}\right) = 0\,, \quad (28)$$

Comparing this equation with Eq. (6) we can identify the longitudinal dielectric function

$$\varepsilon_l(\mathbf{k},\omega) = 1 - \frac{\omega_{p1}^2}{\left[\left(\omega - \mathbf{k}\cdot\mathbf{v}_1^0\right)^2 - \beta_1^2 k^2\right]} - \frac{\omega_{p2}^2}{\left[\left(\omega - \mathbf{k}\cdot\mathbf{v}_2^0\right)^2 - \beta_2^2 k^2\right]}\,. \quad (29)$$

Therefore, if we are looking for longitudinal modes of the system ($\mathbf{E}$ parallel to $\mathbf{k}$), we must have

$$\frac{\omega_{p1}^2}{\left[\left(\omega - \mathbf{k}\cdot\mathbf{v}_1^0\right)^2 - \dfrac{2}{3}\dfrac{\varepsilon_{1F}^0}{m_1}k^2\right]} + \frac{\omega_{p2}^2}{\left[\left(\omega - \mathbf{k}\cdot\mathbf{v}_2^0\right)^2 - \dfrac{2}{3}\dfrac{\varepsilon_{2F}^0}{m_2}k^2\right]} = 1\,, \quad (30)$$

where $\omega_{ps} = \sqrt{4\pi q_s^2 n_s^0/m_s}$ is the plasma frequency of stream s.

Dispersion relation given by Eq. (30) is a very general one in the sense that we may have arbitrary velocities $\mathbf{v}_1^0$ and $\mathbf{v}_2^0$ (but of course in the non-relativistic limit). As pointed out before, we can choose a reference system in which one of the velocities becomes zero, that is, we consider a plasma beam moving through a plasma at rest. Also, for the sake of simplicity, we are going to consider only waves with wave vectors parallel to the velocity of the stream and that the plasma at rest is "cold", that is, the pressure term for this plasma is zero. Therefore, the dispersion relation for this simple case becomes

$$\frac{\omega_{p1}^2}{\omega^2} + \frac{\omega_{p2}^2}{\left(\omega - v_2^0 k\right)^2 - \beta_s^2 k^2} = 1\,. \quad (31)$$

In order to simplify the notation, it is convenient to introduce the dimensionless variables $\tilde{\alpha}^2 \equiv \omega_{p2}^2/\omega_{p1}^2$, $\tilde{\beta}^2 \equiv \left(\beta/v_2^0\right)^2$, $\tilde{\omega} \equiv \omega/\omega_{p1}$, and $\tilde{k} \equiv v_2 k/\omega_{p1}$. Then, the dispersion relation can be simply written as

$$\frac{1}{\tilde{\omega}^2} + \frac{\tilde{\alpha}^2}{\left(\tilde{\omega} - \tilde{k}\right)^2 - \tilde{\beta}^2\tilde{k}^2} = 1\,. \quad (32)$$

Introducing a "normalized phase velocity" defined as

$$\tilde{v}_\phi \equiv \frac{\tilde{\omega}}{\tilde{k}} = \frac{1}{v_2^0}\frac{\omega}{k} = \frac{v_\phi}{v_2^0}\,, \quad (33)$$

where v_ϕ is the phase velocity we can write an alternative form of Eq.(32):

$$\frac{1}{\tilde{v}_\phi^2} + \frac{\tilde{\alpha}^2}{\left(\tilde{v}_\phi - 1\right)^2 - \tilde{\beta}^2} = \tilde{k}^2\,. \quad (34)$$

5. Results and discussion

5.1. Cold beam

First, we are going to exhibit the possibility of instabilities contained in Eq. (34) ignoring the effects of thermal pressure gradient ($\tilde{\beta} = 0$). We can do this by means of a simple graph. Fig. 4a shows the left hand side (LHS) of this equation. As we can see, the curve presents two poles (zeros of the denominators) localized at the points $\tilde{v}_\phi = 0$, and $\tilde{v}_\phi = 1$. The right hand side (RHS) of Eq. (34) is plotted as horizontal lines. These lines intersect the curve at different points: when $\tilde{k}^2$ takes values above certain critical value labeled $\tilde{k}_c^2$, the horizontal lines meet the curve in four points. This represents four waves that propagate without growth or decay, but when $\tilde{k}^2$ takes values below $\tilde{k}_c^2$, that is, when the horizontal line lies in the shaded region, it crosses the curve in just two points. Now, Eq. (34) is a fourth order equation in $\tilde{v}_\phi$ and it has real coefficients, then there must be four solutions for $\tilde{v}_\phi$, therefore, two of the solutions must be a complex conjugate pair corresponding to complex frequencies, one of them represents a mode that grows exponentially in time, and hence instability.

5.2. Warm beam

Now we are going to investigate the effects of taking into account the thermal pressure gradient term ($\tilde{\beta} \neq 0$). The poles of the curve in this case are localized at the points $\tilde{v}_\phi = 0$, and $\tilde{v}_\phi = 1 \pm \tilde{\beta}$ as can be seen from Eq. (34). Several possibilities can occur as shown in Fig. 4b–4c: if $\tilde{\beta}$ is small, the general behavior of the zero thermal pressure gradient case is preserved and it is similar to that of the cold beam shown in panel (a). In panel (b) we use a small value of $\tilde{\beta}$. For the region of interest [positive values of LHS of Eq. (34)] there is not qualitative change, that is, there is still a critical value of $\tilde{k}^2$ below of which there is a pair of complex conjugated solutions one of them representing an amplifying wave and therefore, from the results of Sect. 3, the system may support convective instabilities. If we increase the value of $\tilde{\beta}$, two critical values for $\tilde{k}^2$, named $\tilde{k}_{c1}^2$ and $\tilde{k}_{c2}^2$, appear due to the three-pole structure of the curve. This structure is also present in Fig. 4b but it is not shown there because it is located in the negative region of the vertical axis. The shaded region of panel (c) ($\tilde{k}_{c1}^2 < \tilde{k}_c^2 < \tilde{k}_{c2}^2$) indicates that there are two real roots and two complex roots—one of these representing an instability. Finally, if we increase the value of $\tilde{\beta}$ even more [see panel (d)], all roots are real, and therefore no instabilities may propagate in this case. In other words, we have arrived to the important result that instabilities can be annihilated by increasing the value of $\tilde{\beta}$. Since $\tilde{\beta}^2 \equiv \left(\beta/v_2^0\right)^2$, in order to have instabilities, we need small values of β or large values of the velocity v_2^0.

It is custom to plot the dispersion relation in a diagram of ω vs. $\tilde{k}$. In order to do these plots, we find $\tilde{k}$ from Eq. (32)

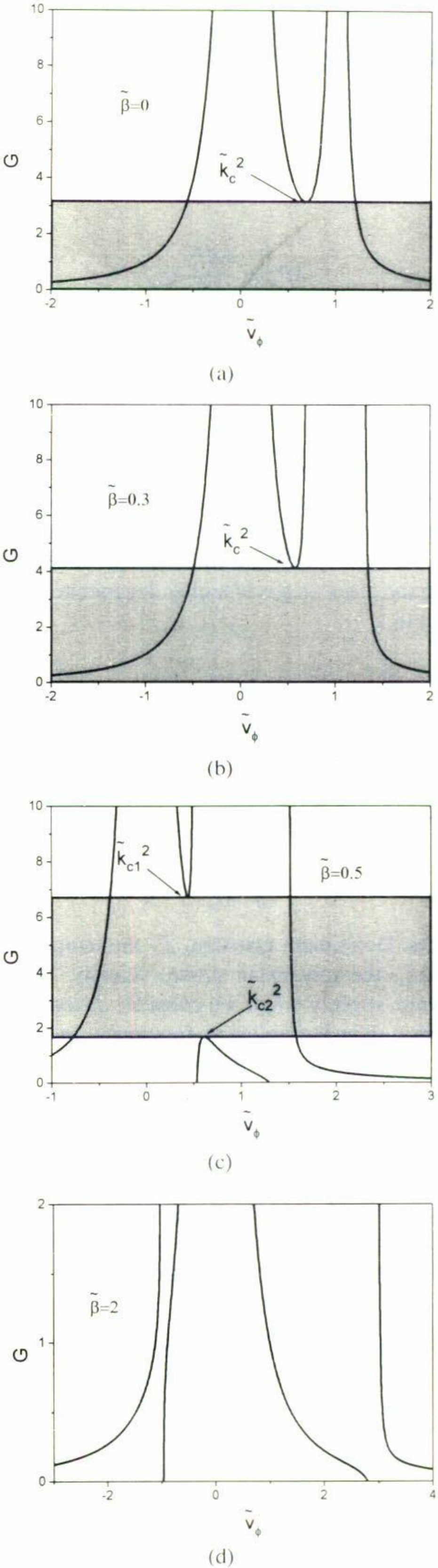

FIGURE 4. Plots of the LHS of Eq.(34) (labeled as G) for four different values of $\tilde{\beta}$. The shaded areas represent the regions where instabilities occur.

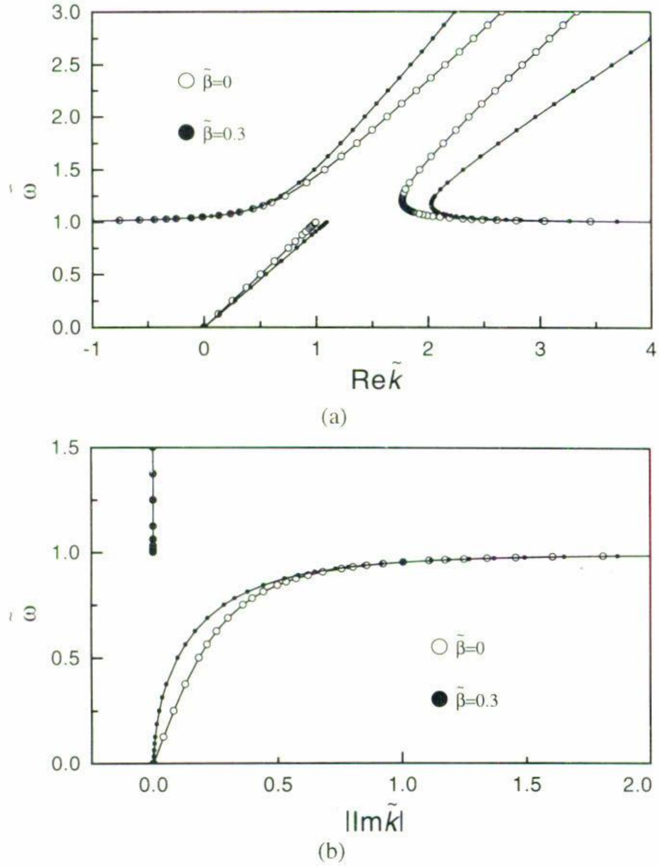

FIGURE 5. Dispersion relation [Eq.(35)] for $\tilde{\beta} = 0$ (open circles) and $\tilde{\beta} = 0.3$ (solid circles); (a) $\tilde{\omega}$ *vs.* Re $\tilde{k}$ and (b) $\tilde{\omega}$ *vs.* $|\text{Im}\,\tilde{k}|$. In these plots we used $\tilde{\alpha}^2 = 0.1$.

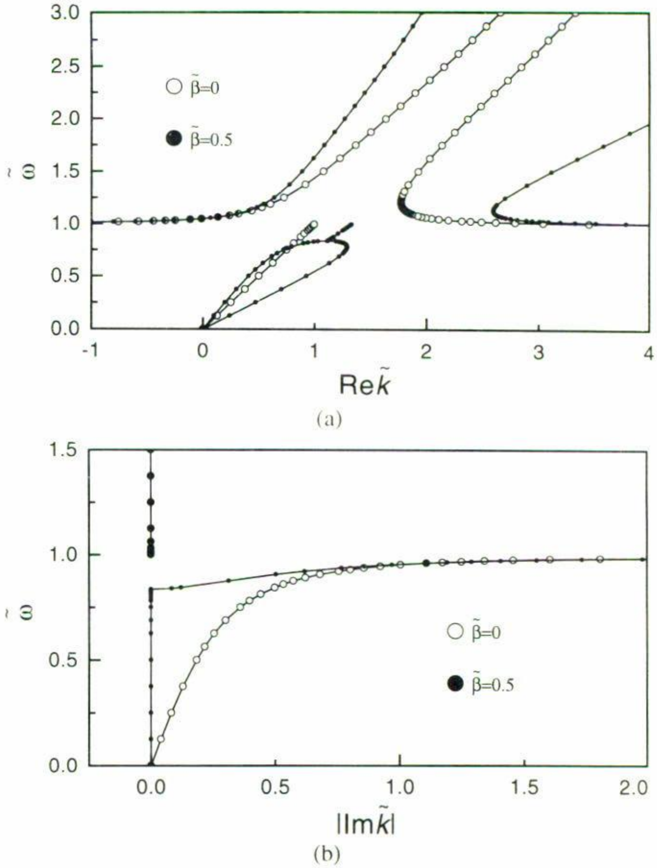

FIGURE 6. Dispersion relations [Eq.(35)] for $\tilde{\beta} = 0$ (open circles) and $\tilde{\beta} = 0.5$ (solid circles); (a) $\tilde{\omega}$ *vs.* Re $\tilde{k}$ and (b) $\tilde{\omega}$ *vs.* $|\text{Im}\,\tilde{k}|$. In these plots we used $\tilde{\alpha}^2 = 0.1$.

which is a second order equation in $\tilde{k}$ with solutions:

$$\tilde{k} = \frac{\tilde{\omega}}{1-\tilde{\beta}^2}\left(1 \pm \text{sgn}(\tilde{\omega}^2-1)\sqrt{\tilde{\beta}^2 + \frac{\tilde{\alpha}^2(1-\tilde{\beta}^2)}{\tilde{\omega}^2-1}}\right), \quad (35)$$

where the function sgn(z) is defined as $z/|z|$ with z an arbitrary complex number different from zero and we have assumed that $\tilde{\omega}^2 \neq 1$ and $\tilde{\beta}^2 \neq 1$. In Figs. 5–7 panels (a) [(b)] we show the solutions of Eq. (35) when we input real values of $\tilde{\omega}$ to get possible complex values of $\tilde{k}$. Since the solutions can be either real or complex conjugated pairs, we just show in panels (b) $|\text{Im}\,\tilde{k}|$. In all these plots we compare the case $\tilde{\beta} = 0$ with $\tilde{\beta} \neq 0$. In Fig. 5 we plot solutions of Eq. (35) for $\tilde{\beta} = 0$ and $\tilde{\beta} = 0.3$. It is clear from this figure that for each value of $\tilde{\beta}$ considered, there are four different branches; two of them have a vanishing imaginary part of $\tilde{k}$ and represent waves that propagate without growth or decay and are located in the range $\tilde{\omega} > 1$. The remaining two branches are complex conjugated of each other and correspond to an instability and an evanescent wave. Figure 6 is the same as Fig. 5 but for $\tilde{\beta} = 0.5$. It is seen that there is a shift in the region where convective instabilities occur and the range for which the waves propagate without decay (vanishing Im $\tilde{k}$) is now extended to $\tilde{\omega} <\cong 0.8$. Finally, in Fig. 7 we choose a still larger value of $\tilde{\beta}$. We observe a very rich structure of the dispersion relation where we can see a small region in which Im $\tilde{k}$ is non-zero indicating the existence of two evanescent waves. This is shown in the inset of Fig. 7a.

In order to establish the character of the instabilities we apply the criteria described at the end of Sect. 3 to Eq. (35). Figure 8 shows a plot of the imaginary part of $\tilde{k}$ as a function of the imaginary part of $\tilde{\omega}(\text{Im}\,\omega)$ for a given value of the real part of $\tilde{\omega}(\text{Re}\,\omega)$. This value of Re $\tilde{\omega}$ is chosen so that in the limit where Im $\tilde{\omega}$ is zero, we fall in a region where Im $\tilde{k}$ is non-zero, for example, if $\tilde{\beta} = 0$ we can chose for Re $\tilde{\omega}$ any value between 0 and 1. As can be seen from the Fig. 8a–8c, one of the two solutions of Eq. (35), the one that changes sign as Im $\tilde{\omega}$ goes to zero, corresponds to a convective instability. The other one corresponds to an evanescent wave. Finally, in Fig. 8d we show that the curves shown in the inset of Fig. 7a correspond to evanescent waves. This result is consistent with the one of Fig. 4d which shows that no instabilities are possible.

It is interesting to discuss the origin of instabilities in the two-stream system. Again, we consider first the situation in which there is no thermal pressure gradient term. Starting with an initial perturbation, one has to investigate the effect

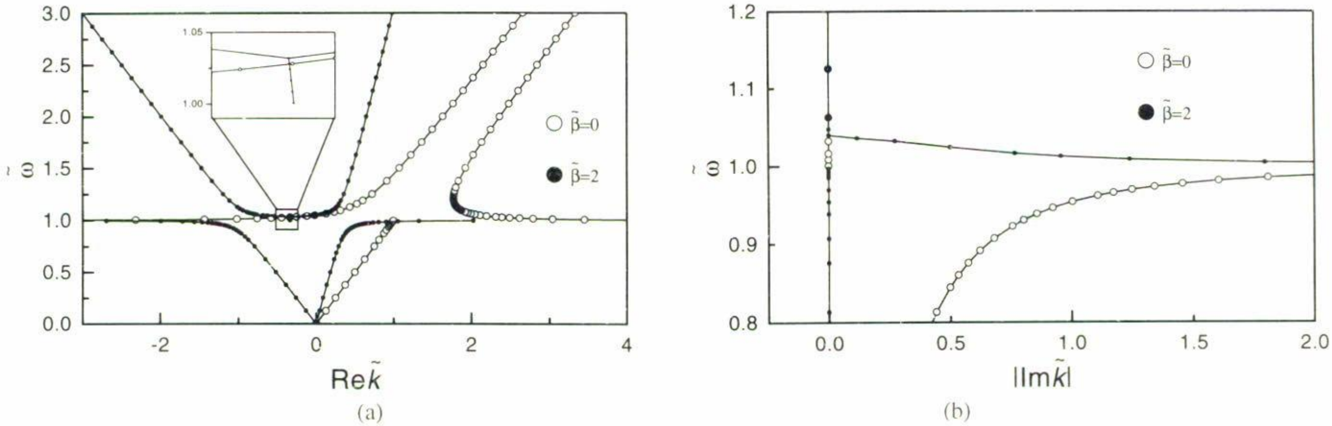

FIGURE 7. Dispersion relation [Eq.(35)] for $\tilde{\beta} = 0$ (open circles) and $\tilde{\beta} = 2$ (solid circles); (a) $\tilde{\omega}$ *vs.* Re $\tilde{k}$ and (b) $\tilde{\omega}$ *vs.* $|\text{Im}\,\tilde{k}|$. In these plots we used $\tilde{\alpha}^2 = 0.1$.

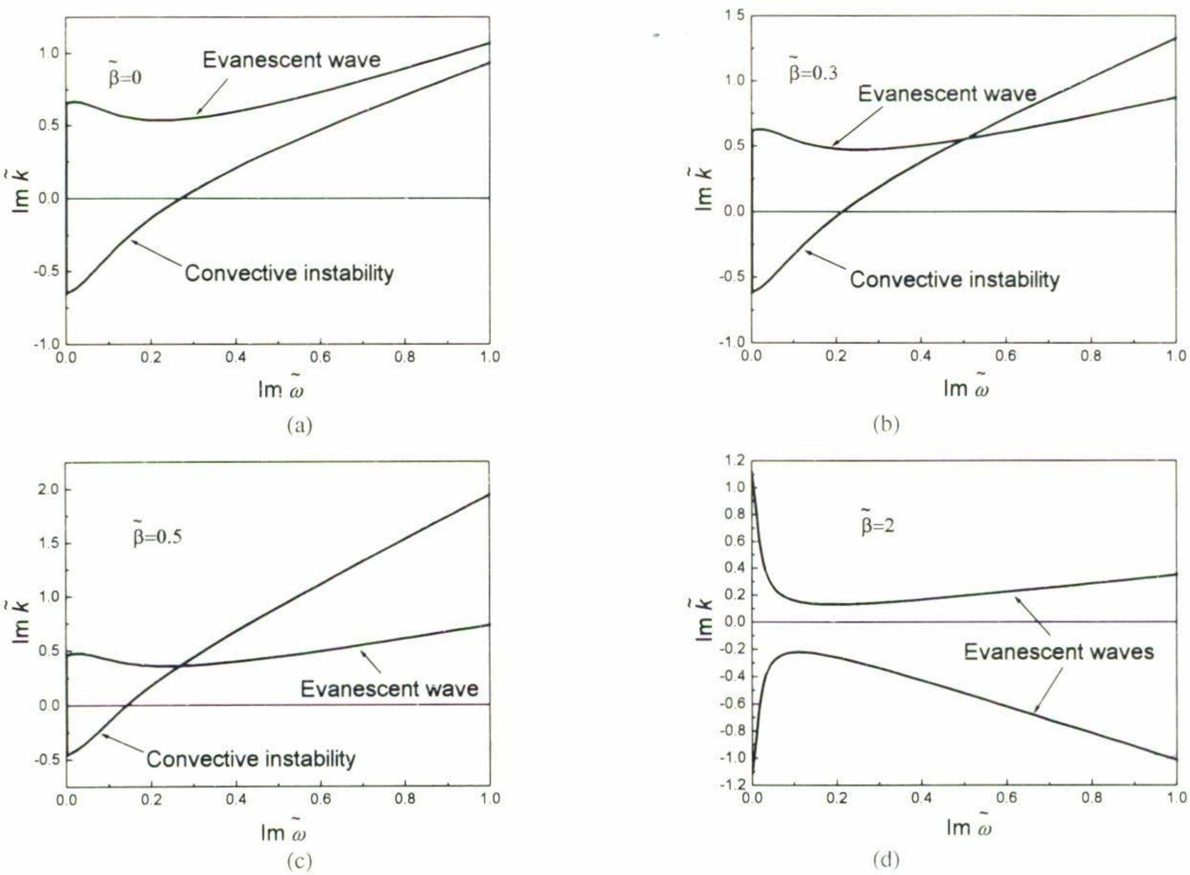

FIGURE 8. Plots of the imaginary part of $\tilde{k}$ as a function of the imaginary part of $\tilde{\omega}$ for a given value of the real part of $\tilde{\omega}$. (a)–(c) Re $\tilde{\omega} = 0.9$ and (d) Re $\tilde{\omega} = 1.01$. The curves that change sign correspond to convective instabilities and the ones that do not change sign correspond to evanescent waves.

produced by the two streams on the electrostatic potential of the system. Since particles moving in either stream are slowed down at the top of the potential hill, their density there will exceed the particle density at the bottom, where they move faster. The larger electron density, on the other hand, enhances the electrostatic potential. This feedback mechanism leads to the exponentially growing potential wave. In other words, the electric field of the propagating wave accelerates the beam particles causing a fluctuation in their density. At the same time, some of the particles are carried back towards the original point of disturbance by the zero-order streaming motion of the beams producing a bunching of the particles. This bunching leads to potential wells which further enhance the bunching. Therefore, the free streaming energy is rapidly converted to field energy, and particles become trapped in the potential wells. If the thermal pressure is included then it is harder for the particles to increase their density at the top of the potential hill and therefore we observe a tendency to reduce the two-stream instability. When thermal pressure gradient is large enough, the instability is finally destroyed.

6. Conclusions

In this work, we presented an introduction to the interesting subject of plasma instabilities, and in particular, we studied a two-stream plasma system consisting of a warm electron beam and a cold plasma at rest, mutually interpenetrating. Starting from hydrodynamic equations, we obtained the dispersion relations for this system. Using a simple intuitive criteria, the convective character of the instabilities present on this system was established and we discussed the physical origin of them. We investigated in detail the effects, on the dispersion relations, of the electron thermal pressure gradient. Even for the simple two-stream system, we found a very rich structure for these relations that show a reduction and finally an inhibition of the instabilities as the thermal pressure gradient term is increased.

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