Equivalent Hamiltonians by means of the Hamilton-Jacobi equation

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Making use of the Hamilton-Jacobi equation, two Hamiltonians in three-dimensional space that determine the paths of a light ray in a conformally stationary gravitational field are obtained; one of these Hamiltonians coincides with that of a non-relativistic charged particle in a magnetic field. Similarly, it is shown that the orbits of a non-zero rest mass particle in a stationary gravitational field correspond to those of a non-relativistic charged particle in an electromagnetic field. It is also shown that the Hamiltonian of the Kepler problem for bounded motion is equivalent to that of a free particle in a sphere in four dimensions and a simplified proof of the Jacobi principle is given.

Keywords: Hamilton-Jacobi theory; geodesics; stationary space-times; Kepler problem

Usando la ecuación de Hamilton-Jacobi se obtienen dos hamiltonianas en el espacio tridimensional que determinan las trayectorias de un rayo de luz en un campo gravitacional conformemente estacionario; una de estas hamiltonianas coincide con la de una partícula cargada no relativista en un campo magnético. Similarmente, se muestra que las órbitas de una partícula con masa en reposo distinta de cero en un campo gravitacional estacionario corresponden a las de una partícula cargada no relativista en un campo electromagnético. Se muestra también que la hamiltoniana del problema de Kepler para movimiento acotado equivale a la de una partícula libre en una esfera en cuatro dimensiones y se da una prueba simplificada del principio de Jacobi.

Descriptores: Teoría de Hamilton-Jacobi; geodésicas; espacio-tiempo estacionarios; problema de Kepler

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1. Introduction

As is well known, the Hamilton-Jacobi (HJ) equation constitutes an alternative to Hamilton's and Lagrange's equations to solve the equations of motion of a mechanical system. Instead of the system of ordinary differential equations given by the Lagrange or the Hamilton equations, the HJ equation is a single first-order partial differential equation and a complete solution of this equation gives the solution of the equations of motion. An advantage of the HJ equation is that, in the case of a system with a time-independent Hamiltonian, the time dependence can be eliminated from the very beginning (looking for Hamilton's characteristic function) and the solution of the resulting equation yields the orbits of the mechanical system.

In the case of a system with a time-independent Hamiltonian, the HJ equation also allows us to find alternative (equivalent) Hamiltonians that lead to the same orbits determined by the original Hamiltonian. In this paper we consider three nice examples of this application of the HJ equation. The possibility of relating different problems of classical or quantum mechanics by means of coordinate transformations has been known for a long time. Some systematic procedures employed to obtain such transformations are based on the HJ equation (see, *e.g.*, Ref. 1), the Jacobi principle (see, *e.g.*, Ref. 2) and the generalized canonical transformations [3]. However, two of the examples given below do not involve coordinate transformations; the alternative Hamiltonians correspond to different parametrizations of the orbits.

In Sect. 2, it is shown that the light rays in a conformally stationary space-time coincide with the orbits of a charged particle in a certain magnetic field in a possibly curved three-dimensional space. This result was previously obtained in Ref. 4 by a different procedure. We also show that the path of a test particle with a non-zero rest mass in a stationary space-time coincides with that of a charged particle in a certain combination of electric and magnetic fields in a possibly curved three-dimensional space. In Sect. 3, it is shown that the Kepler problem with negative energy is related to the problem of a free particle on the unit sphere in four-dimensional space. This result shows that the Kepler problem for bounded motion is invariant under the group of rotations in four dimensions and is analogous to Fock's transformation of the Schrödinger equation for the bound states of the hydrogen atom into an integral equation on a sphere in four dimensions [5]. Finally, in Sect. 4, a simplified proof of Jacobi's principle is given.

2. Test particles in gravitational fields

If $H(q^i, p_i, t)$ is the Hamiltonian of a system with n degrees of freedom, the corresponding HJ equation is given by

$$H\left(q^i, \frac{\partial S}{\partial q^i}, t\right) + \frac{\partial S}{\partial t} = 0, \qquad (1)$$

which is a partial differential equation for the principal function S. If $S(q^i, \alpha_i, t)$ is a complete solution of the HJ equation (1), *i.e.*, a solution of Eq. (1) containing n non-additive independent constants α_i, then the equations $\partial S/\partial \alpha_i = \beta^i$, where the β^i are also constants, yield the solution to the equations of motion. If H does not depend explicitly on t, one may look for complete solutions of Eq. (1) of the form $S = W(q^i, \alpha_i) - Et$, where W is the characteristic function which contains $n-1$ non-additive independent constants and obeys the condition

$$H\left(q^i, \frac{\partial W}{\partial q^i}\right) = E. \tag{2}$$

Then, the equations $\partial W/\partial \alpha_i = \beta^i$ $(i = 1, 2, \ldots, n-1)$, which do not contain the time, yield the equations of the orbit.

The world line of a particle subject only to the gravitational field can be obtained from the Hamiltonian

$$H = \tfrac{1}{2} g^{\alpha\beta} p_\alpha p_\beta \tag{3}$$

(Greek lower case indices $\alpha, \beta, \ldots$, run from 0 to 3), where the matrix $(g^{\alpha\beta})$ is the inverse of $(g_{\alpha\beta})$, and the $g_{\alpha\beta}$ are the components of the metric tensor of the space-time in some coordinate system. In other words, the Hamilton equations, $dx^\alpha/d\lambda = \partial H/\partial p_\alpha$, $dp_\alpha/d\lambda = -\partial H/\partial x^\alpha$, lead to the geodesic equations corresponding to the metric $ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$. If the particle has a non-vanishing rest mass, the parameter λ can be chosen as the particle's proper time, which amounts to the condition $H = -c^2/2$, assuming that the signature of the metric is $(-+++)$. On the other hand, for zero rest mass particles, $g^{\alpha\beta} p_\alpha p_\beta = 0$ and therefore $H = 0$.

The HJ equation for the Hamiltonian (3), in the case of zero rest mass particles (or light rays) leads to $\frac{1}{2} g^{\alpha\beta} (\partial S/\partial x^\alpha)(\partial S/\partial x^\beta) = 0$ or, equivalently,

$$\frac{1}{2} g^{00} \left(\frac{\partial S}{\partial x^0}\right)^2 + g^{0i} \left(\frac{\partial S}{\partial x^0}\right)\left(\frac{\partial S}{\partial x^i}\right) + \frac{1}{2} g^{ij} \left(\frac{\partial S}{\partial x^i}\right)\left(\frac{\partial S}{\partial x^j}\right) = 0, \tag{4}$$

where the lower case Latin indices $i, j, \ldots$, run from 1 to 3. Hence, writing $x^0 = ct$, from Eq. (4) we have

$$\frac{\partial S}{\partial t} = \frac{c}{g^{00}} \left\{ -g^{0i} \frac{\partial S}{\partial x^i} + \sqrt{(g^{0i} g^{0j} - g^{00} g^{ij}) \frac{\partial S}{\partial x^i} \frac{\partial S}{\partial x^j}} \right\}, \tag{5}$$

which has the form of the HJ equation (1) for the Hamiltonian

$$h = \frac{c}{g^{00}} \left\{ g^{0i} p_i - \sqrt{(g^{0i} g^{0j} - g^{00} g^{ij}) p_i p_j} \right\}. \tag{6}$$

If the space-time is *stationary*, there exists a coordinate system in which the metric components, $g_{\alpha\beta}$, are independent of x^0; then, in such a coordinate system, the Hamiltonian (6) is independent of t and, hence, conserved. Since the Hamiltonian (6) is a homogeneous function of degree one of the p_i, Hamilton's equations are invariant under the transformation $p_i \mapsto \mu p_i$, where μ is any constant greater than zero; therefore, we can normalize the p_i by requiring that the constant value of h be c (with this choice we recover the normalization employed in the Hamiltonian formulation of geometrical optics in flat space-time, see, *e.g.*, Ref. 6).

Before discussing the general case of a stationary space-time, it is convenient to consider the simpler subcase of the static space-times. A space-time is *static* if there exists a coordinate system in which the metric components are independent of x^0 and furthermore $g_{0i} = 0$, which implies that $g^{0i} = 0$, $g^{00} = (g_{00})^{-1}$ and the 3×3 matrix (g_{ij}) is the inverse of (g^{ij}) [see Eqs. (9) below]. Thus, assuming that the space-time is static, in an appropriate coordinate system, the orbits of the light rays coincide with those corresponding to the Hamiltonian [see Eq. (6)]

$$h = c\sqrt{(-g_{00}) g^{ij} p_i p_j}. \tag{7}$$

The orbits of the Hamiltonian (7) are the geodesics corresponding to the metric $d\sigma^2 \equiv (-g_{00})^{-1} g_{ij}\, dx^i\, dx^j = (-g_{00})^{-1}\, dl^2$, where $dl^2 = g_{ij}\, dx^i\, dx^j$ is the spatial metric (induced by the space-time metric ds^2 on the hypersurfaces t = const), *i.e.*, the curves that locally minimize the integral $\int d\sigma = \int (-g_{00})^{-1/2}\, dl$. In fact, the HJ equation for the Hamiltonian (7), with $h = c$, amounts to $(-g_{00}) g^{ij} (\partial W/\partial x^i)(\partial W/\partial x^j) = 1$, which is the HJ equation for the Hamiltonian $(-g_{00}) g^{ij} p_i p_j$, whose orbits are the geodesics corresponding to the metric $d\sigma^2$ [*cf.* Eq. (3)]. The metric $d\sigma^2$ is sometimes called the optical metric and has several remarkable properties (see, *e.g.*, Ref. 7 and the references cited therein). Since for a light ray $ds^2 = 0$, under the present assumptions we have $0 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta = g_{00} (dx^0)^2 + g_{ij}\, dx^i\, dx^j = g_{00}(dx^0)^2 + dl^2$; therefore, $\int (-g_{00})^{-1/2}\, dl = \int dx^0 = c \int dt$, which proves that in a static space-time the Fermat principle holds. Note that $n \equiv (-g_{00})^{-1/2}$ plays the role of a refractive index and the normalization $h = c$ amounts to $\sqrt{g^{ij} p_i p_j} = n$. The Fermat principle also holds in conformally stationary space-times; as we shall show below. (The Fermat principle is considered in many textbooks on general relativity; for a recent treatment see Ref. 4 and the references cited therein.)

Going back to the case of stationary space-times, from Hamilton's equations $dx^i/dt = \partial h/\partial p_i$ and Eq. (6) we find that

$$\dot{x}^i \equiv \frac{dx^i}{dt} = c \frac{g^{ij} p_j - g^{0i}}{g^{0k} p_k - g^{00}} \tag{8}$$

on the hypersurface $h = c$. Then, it can be readily verified, making use of the formulas

$$\begin{aligned} 1 &= g_{0\alpha}g^{0\alpha} = g_{00}g^{00} + g_{0i}g^{0i}, \\ 0 &= g_{0\alpha}g^{i\alpha} = g_{00}g^{i0} + g_{0j}g^{ij}, \\ 0 &= g_{i\alpha}g^{0\alpha} = g_{i0}g^{00} + g_{ij}g^{0j}, \\ \delta_i^j &= g_{i\alpha}g^{j\alpha} = g_{i0}g^{j0} + g_{ik}g^{jk}, \end{aligned} \tag{9}$$

that the inverse of Eq. (8) is given by

$$p_i = \frac{g_{ij}(\dot{x}^i/c) + g_{0i}}{-g_{00} - g_{0k}(\dot{x}^k/c)}, \tag{10}$$

which differs from the commonly encountered relations between the velocities and the momenta.

Alternatively, assuming that the space-time is stationary, Eq. (4) admits separable solutions of the form $S(x^\alpha) = W(x^i) - \nu x^0$, where ν is a constant. Since $p_0 = \partial S/\partial x^0 = -\nu$ and the signature of the metric is $(- + + +)$, for a future directed world line ν must be positive and without loss of generality we can set $\nu = 1$ (which is equivalent to making $h = c$ as above). Substituting $S = W - x^0$ into Eq. (4) one obtains

$$\frac{1}{2}g^{00} - g^{0i}\frac{\partial W}{\partial x^i} + \frac{1}{2}g^{ij}\frac{\partial W}{\partial x^i}\frac{\partial W}{\partial x^j} = 0$$

and making use of Eqs. (9) one finds that this equation can be written as

$$\frac{1}{2}(-g_{00})g^{ij}\left(\frac{\partial W}{\partial x^i} + \frac{g_{0i}}{g_{00}}\right)\left(\frac{\partial W}{\partial x^j} + \frac{g_{0j}}{g_{00}}\right) = \frac{1}{2}, \tag{11}$$

which is the HJ equation for the Hamiltonian

$$\tilde{h} \equiv \frac{1}{2}(-g_{00})g^{ij}\left(p_i + \frac{g_{0i}}{g_{00}}\right)\left(p_j + \frac{g_{0j}}{g_{00}}\right) \tag{12}$$

if $\tilde{h} = 1/2$. The Hamiltonian (12) coincides with that of a *non-relativistic* charged particle in a magnetic field with vector potential proportional to g_{0i}/g_{00} in a three-dimensional space with metric tensor $d\sigma^2 = (g_{00})^{-2}(g_{0i}g_{0j} - g_{00}g_{ij})\,dx^i\,dx^j$, whose components form the inverse matrix of $[(-g_{00})g^{ij}]$; however, $\tilde{h}$ is not conjugate to the "true" time (or a constant multiple of it) in all cases. In fact, denoting by τ the fictitious "time" conjugate to $\tilde{h}$, from Eqs. (12) and (9) we obtain

$$\frac{dx^i}{d\tau} = \frac{\partial\tilde{h}}{\partial p_i} = (-g_{00})g^{ij}\left(p_j + \frac{g_{0j}}{g_{00}}\right) = (-g_{00})(g^{ij}p_j - g^{0i}). \tag{13}$$

Comparison with Eq. (8) shows that

$$\frac{dx^i}{d\tau} = (-g_{00})(g^{0k}p_k - g^{00})\frac{1}{c}\frac{dx^i}{dt}, \tag{14}$$

i.e., $c\,dt = (-g_{00})(g^{0k}p_k - g^{00})d\tau$; in particular, if the space-time is static, we can take $\tau = ct$. Making use of Eqs. (9) and (13) one can verify that the $dx^i/d\tau$ are the components of a unit vector with respect to the metric $d\sigma^2$

$$\frac{(g_{0i}g_{0j} - g_{00}g_{ij})}{(-g_{00})^2}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} = (-g_{00})g^{ij}\left(p_i + \frac{g_{0i}}{g_{00}}\right)\left(p_j + \frac{g_{0j}}{g_{00}}\right) = 1. \tag{15}$$

Thus, the *orbits* determined by the Hamiltonian (12) are those of a non-relativistic charged particle in a magnetic field with vector potential proportional to g_{0i}/g_{00} in a three-dimensional space with metric $d\sigma^2$. On the other hand, if there exists a function $\xi(x^i)$ such that $g_{0i}/g_{00} = \partial\xi/\partial x^i$, then the magnetic field mentioned above would vanish, which means that the components g_{0i} can be eliminated by a suitable coordinate transformation and the space-time is actually static (in fact, if $g_{0i}/g_{00} = \partial\xi/\partial x^i$, then $ds^2 = g_{00}\,(dx^0)^2 + 2g_{0i}\,dx^0\,dx^i + g_{ij}\,dx^i\,dx^j = g_{00}(dx^0 + d\xi)^2 + (g_{ij} - g_{0i}g_{0j}/g_{00})\,dx^i\,dx^j$; hence, by replacing x^0 by $x^0 + \xi$ one shows that the space-time is static). It can be shown that $g_{ij} - g_{0i}g_{0j}/g_{00}$ is a positive-definite metric, which is induced by the space-time metric $g_{\alpha\beta}$ on the three-dimensional space orthogonal to the timelike direction $\partial/\partial x^0$ [8] (see also Ref. 9).

Clearly, the Hamiltonian (12) can be derived from the Lagrangian

$$L_2 \equiv \frac{1}{2}\frac{g_{0i}g_{0j} - g_{00}g_{ij}}{(-g_{00})^2}x^{i\prime}x^{j\prime} - \frac{g_{0i}}{g_{00}}x^{i\prime},$$

where $x^{i\prime} = dx^i/d\tau$, and the Euler–Lagrange equations for L_2 coincide with those for the Lagrangian [see Eq. (15)]

$$L_1 \equiv \frac{[(g_{0i}g_{0j} - g_{00}g_{ij})x^{i\prime}x^{j\prime}]^{1/2}}{(-g_{00})} - \frac{g_{0i}}{g_{00}}x^{i\prime}$$

which, in turn, are the conditions for

$$\int_{(x_1^i)}^{(x_2^i)} \frac{\left[(g_{0i}g_{0j} - g_{00}g_{ij})(dx^i/d\lambda)(dx^j/d\lambda)\right]^{1/2} + g_{0i}(dx^i/d\lambda)}{(-g_{00})}\,d\lambda \tag{16}$$

to have a stationary value among the curves joining the points with coordinates (x_1^i) and (x_2^i). Since $0 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta = g_{00}\,(dx^0)^2 + 2g_{0i}\, dx^i\, dx^0 + g_{ij}\, dx^i\, dx^j$ for a light ray, the integral (16) is just $c\int_{(x_1^i)}^{(x_2^i)} dt$, thus showing that Fermat's principle also holds in this case.

It is easy to see that, actually, the preceding results are also valid if the space-time is *conformally stationary* [4], *i.e.*, if there exists a coordinate system such that the metric is of the form $\phi^2 g_{\alpha\beta}$, where ϕ is some non-vanishing function and $\partial g_{\alpha\beta}/\partial x^0 = 0$, since the factors ϕ^{-2} and ϕ^2 drop out from Eq. (4) and from the condition $\phi^2 g_{\alpha\beta} dx^\alpha dx^\beta = 0$. For instance, the metric of the Friedmann-Robertson-Walker cosmological models can be written in the form

$$ds^2 = R_0^2 e^{a(t)}[-dt^2 + dr^2 + \xi^2(r)(d\theta^2 + \sin^2\theta\, d\varphi^2)], \quad (17)$$

where R_0 is a constant, a depends only on t and $\xi(r)$ is equal to r, $\sin r$ or $\sinh r$. Clearly, the metric (17) is conformally stationary (in fact, conformally static). Taking $\phi^2 = R_0^2 e^{a(t)}$ and $(g_{\alpha\beta}) = \mathrm{diag}(-1, 1, \xi^2, \xi^2\sin^2\theta)$, Eq. (12) corresponds to the Hamiltonian of a non-relativistic free particle in a three-dimensional space of constant curvature.

As in Ref. 4, we can go in the opposite direction, giving a magnetic field and the metric of a three-dimensional space and then finding a space-time metric for which the light rays follow the orbits of a charged particle in the given magnetic field and the given three-dimensional space. For example, the Hamiltonian of a particle of unit mass and unit electric charge in flat three-dimensional space, subject to the magnetic field of a monopole, written in cartesian coordinates, has the form (12) with $[(-g_{00})g^{ij}] = \mathrm{diag}(1, 1, 1)$ and $g_{0i}/g_{00} = -A_i/c$, where A_i is a vector potential for the field of a magnetic monopole, *e.g.*,

$$A_i = k\frac{(y, -x, 0)}{r(\pm r - z)}, \quad (18)$$

k is the magnetic charge of the monopole and $r = \sqrt{x^2 + y^2 + z^2}$. Hence, $[(g_{00})^{-2}(g_{0i}g_{0j} - g_{00}g_{ij})] = \mathrm{diag}(1, 1, 1)$; and

$$\begin{aligned} ds^2 &= (-g_{00})\left[-\left(dx^0 + \frac{g_{0i}}{g_{00}}\, dx^i\right)^2 + (g_{00})^{-2}(g_{0i}g_{0j} - g_{00}g_{ij})\, dx^i\, dx^j\right] \\ &= (-g_{00})\left[-\left(dx^0 - \frac{1}{c}A_i\, dx^i\right)^2 + dx^2 + dy^2 + dz^2\right] \\ &= (-g_{00})\left[-\left(c\,dt + \frac{k}{c}(\pm 1 + \cos\theta)\, d\varphi\right)^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right], \quad (19) \end{aligned}$$

with the factor $(-g_{00})$ unspecified, in accordance with the conformal invariance of the null geodesics. Thus, as in the case of the orbits of a charged particle in the field of a magnetic monopole, a light ray in a space-time with metric (19) lies on a cone with its vertex at $r = 0$ in such a way that the ray becomes a straight line when the cone is unfolded. Choosing $-g_{00} = 1$, one can readily verify that the metric (19) is equivalent to the Taub–NUT solution of the Einstein vacuum field equations (see, *e.g.*, Ref. 10)

$$\begin{aligned} ds^2 = &-U^{-1}\, dr^2 + (2l)^2 U(d\psi + \cos\theta\, d\varphi)^2 \\ &+ (r^2 + l^2)(d\theta^2 + \sin^2\theta\, d\varphi^2), \end{aligned}$$

where

$$U \equiv -1 + \frac{2(mr + l^2)}{r^2 + l^2},$$

to first order in l, when $m = 0$, making the identifications $c\,dt \pm (k/c)\, d\varphi = 2l\, d\psi$ and $2l = k/c$. Further examples are given in Ref. 4.

In the case of particles with non-zero rest mass we have some similar results. In place of Eq. (4) we now have

$$\frac{1}{2}g^{00}\left(\frac{\partial S}{\partial x^0}\right)^2 + g^{0i}\left(\frac{\partial S}{\partial x^0}\right)\left(\frac{\partial S}{\partial x^i}\right) + \frac{1}{2}g^{ij}\left(\frac{\partial S}{\partial x^i}\right)\left(\frac{\partial S}{\partial x^j}\right) = -\frac{c^2}{2}, \quad (20)$$

which can be written in the form $\partial S/\partial t + h(x^i, \partial S/\partial x^i) = 0$, with

$$h(x^i, p_i) = \frac{c}{g^{00}}\left\{g^{0i}p_i - \sqrt{(g^{0i}g^{0j} - g^{00}g^{ij})p_i p_j - g^{00}c^2}\right\}. \quad (21)$$

On the other hand, in a stationary space-time, in coordinates such that $\partial g_{\alpha\beta}/\partial x^0 = 0$, Eq. (20) admits solutions of the form $S(x^0, x^i) = W(x^i) - \varepsilon x^0/c$, where ε is a positive constant (assuming that the world line is directed to the future) and

$$\frac{1}{2}g^{00}\frac{\varepsilon^2}{c^2} - g^{0i}\frac{\partial W}{\partial x^i}\frac{\varepsilon}{c} + \frac{1}{2}g^{ij}\frac{\partial W}{\partial x^i}\frac{\partial W}{\partial x^j} = -\frac{c^2}{2}$$

or, equivalently,

$$\frac{1}{2}(-g_{00})g^{ij}\left(\frac{\partial W}{\partial x^i} + \frac{\varepsilon}{c}\frac{g_{0i}}{g_{00}}\right)\left(\frac{\partial W}{\partial x^j} + \frac{\varepsilon}{c}\frac{g_{0j}}{g_{00}}\right) + \frac{c^2}{2}(-g_{00}) = \frac{\varepsilon^2}{2c^2} \quad (22)$$

which is the HJ equation for the Hamiltonian

$$\tilde{h} \equiv \frac{1}{2}(-g_{00})g^{ij}\left(p_i + \frac{\varepsilon}{c}\frac{g_{0i}}{g_{00}}\right)\left(p_j + \frac{\varepsilon}{c}\frac{g_{0j}}{g_{00}}\right) + \frac{c^2}{2}(-g_{00}) \quad (23)$$

with $\tilde{h} = \varepsilon^2/(2c^2)$ [*cf.* Eq. (12)]. This Hamiltonian can be interpreted as that of a non-relativistic charged particle in a magnetic field with vector potential proportional to g_{0i}/g_{00} and an electric field with scalar potential proportional to $(-g_{00})$ in a three-dimensional space with metric tensor $d\sigma^2 = (g_{00})^{-2}(g_{0i}g_{0j} - g_{00}g_{ij})\,dx^i\,dx^j$. In the present case the "time" parameter, τ, conjugate to $\tilde{h}$ is related to the proper time of the particle, λ, by means of $d\tau = d\lambda/(-g_{00})$. By contrast with the Hamiltonian (12), a Hamiltonian of the form (23) determines the space-time metric $g_{\alpha\beta}$ up to a constant factor. Comparison of Eqs. (12) and (23) shows that when g_{00} is constant, the orbits of test particles with zero or non-zero rest mass differ only in the values of the "electric charge" and of the "energy".

3. Kepler problem

An alternative to the usual form of the HJ equation is given by

$$H\left(-\frac{\partial S}{\partial p_i}, p_i, t\right) + \frac{\partial S}{\partial t} = 0, \quad (24)$$

where the coordinates q^i appearing in the Hamiltonian $H(q^i, p_i, t)$ are replaced by $-\partial S/\partial p_i$. Equation (24) can be derived from the usual HJ equation (1) taking into account that the coordinate transformation $q^i \mapsto -p_i$, $p_i \mapsto q^i$ leaves Hamilton's equations invariant (*i.e.*, is a canonical transformation). In the case of the Kepler problem, the Hamiltonian can be expressed as

$$H = \frac{p_x^2 + p_y^2 + p_z^2}{2M} - \frac{k}{\sqrt{x^2 + y^2 + z^2}}, \quad (25)$$

where k is a positive constant and from Eq. (24), writing $S = W(p_i) - Et$, one obtains

$$\frac{p_x^2 + p_y^2 + p_z^2}{2M} - k\left[\left(\frac{\partial W}{\partial p_x}\right)^2 + \left(\frac{\partial W}{\partial p_y}\right)^2 + \left(\frac{\partial W}{\partial p_z}\right)^2\right]^{-1/2} = E. \quad (26)$$

Now, following Refs. 5 and 11 and assuming $E < 0$ (bounded motion), the vector $\mathbf{p} = (p_x, p_y, p_z)$ will be replaced by a point (u_x, u_y, u_z, u_w) on the unit sphere in four dimensions by means of the stereographic projection, *i.e.*,

$$\mathbf{p} = p_0 \frac{(u_x, u_y, u_z)}{1 - u_w}, \quad (27)$$

where

$$p_0 \equiv \sqrt{-2ME}. \quad (28)$$

By expressing the unit vector (u_x, u_y, u_z, u_w) in terms of spherical coordinates in the form $(\sin\chi\sin\theta\cos\phi$, $\sin\chi\sin\theta\sin\phi$, $\sin\chi\cos\theta$, $\cos\chi)$, one finds that

$$(dp_x)^2 + (dp_y)^2 + (dp_z)^2 = \left[\frac{p_0}{2\sin^2(\chi/2)}\right]^2 \times \left[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\,d\phi^2)\right],$$

which corresponds to the well-known fact that the stereographic projection is a conformal transformation (note that $d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\,d\phi^2)$ is the standard metric of the unit sphere). Hence

$$\left(\frac{\partial W}{\partial p_x}\right)^2 + \left(\frac{\partial W}{\partial p_y}\right)^2 + \left(\frac{\partial W}{\partial p_z}\right)^2 = \left[\frac{2\sin^2(\chi/2)}{p_0}\right]^2 \left(\left(\frac{\partial W}{\partial \chi}\right)^2 + \frac{1}{\sin^2\chi}\left[\left(\frac{\partial W}{\partial \theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial W}{\partial \phi}\right)^2\right]\right) \quad (29)$$

and making use of Eqs. (27) and (28) it follows that Eq. (26) amounts to

$$\frac{1}{2}\left(\left(\frac{\partial W}{\partial \chi}\right)^2 + \frac{1}{\sin^2\chi}\left[\left(\frac{\partial W}{\partial \theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial W}{\partial \phi}\right)^2\right]\right) = \frac{1}{2}\left(\frac{Mk}{p_0}\right)^2, \quad (30)$$

which is the HJ equation corresponding to the Hamiltonian

$$h = \frac{1}{2}\left[p_\chi^2 + \frac{1}{\sin^2\chi}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right)\right], \quad (31)$$

if $h = \frac{1}{2}(Mk/p_0)^2$. Equation (31) is the Hamiltonian of a free particle of unit mass on the unit sphere in four dimensions, S^3 (or of a free spherical top; note, however, that the configuration space of the top, which can be identified with the group SO(3), corresponds to S^3 with antipodal points identified). The orbits of the Hamiltonian (31), which is of the form $\frac{1}{2}g^{ij}p_ip_j$, where (g^{ij}) corresponds to the standard metric of S^3, are the geodesics of S^3 [*cf.* Eq. (3)], *i.e.*, great circles, which, under the stereographic projection (27) are mapped onto circles lying on planes passing through the origin in $\mathbf{p}$ space [12].

The explicit invariance of the Hamiltonian (31) under the group of rotations in four dimensions, SO(4), shows that the Hamiltonian (25) for $E < 0$ is invariant under this group. In a similar way, making use of the coordinate transformation $\mathbf{p} = \mathbf{u}/|\mathbf{u}|^2$ one finds that Eq. (26) with $E = 0$ is equivalent to the HJ equation for a free particle in three-dimensional space and, in the case where $E > 0$, taking

$$\mathbf{p} = \sqrt{2ME}\,\frac{(u_x, u_y, u_z)}{1 - u_w} \tag{32}$$

with

$$u_x^2 + u_y^2 + u_z^2 - u_w^2 = 1, \tag{33}$$

one finds that Eq. (26) is equivalent to the HJ equation for a free particle on the hyperboloid (33) in Minkowski space (*cf.* Ref. 12).

4. Jacobi's principle

The result of the preceding section is analogous to Jacobi's principle, according to which the orbits of a conservative system, with forces derivable from a velocity-independent potential energy, are the geodesics of a suitably defined metric. Jacobi's principle can also be easily derived making use of the HJ equation. The HJ equation for a system described by the time-independent Hamiltonian

$$H = \frac{1}{2} g^{ij} p_i p_j + V(q^k) \tag{34}$$

(here $i, j, \ldots = 1, \ldots, n$) leads to

$$\frac{1}{2} g^{ij} \frac{\partial W}{\partial q^i} \frac{\partial W}{\partial q^j} + V = E \tag{35}$$

[see Eq. (2)] hence,

$$\frac{1}{2(E - V)} g^{ij} \frac{\partial W}{\partial q^i} \frac{\partial W}{\partial q^j} = 1. \tag{36}$$

Equation (36) is the HJ equation for the Hamiltonian

$$h = \frac{1}{2(E - V)} g^{ij} p_i p_j \tag{37}$$

with $h = 1$, whose orbits, and, hence, those of (34), are the geodesics of the metric

$$(E - V) g_{ij}\, dq^i\, dq^j . \tag{38}$$

5. Concluding remarks

The examples considered here show that, even without a change of coordinates, the orbits of a given time-independent dynamical system can be determined by a huge variety of distinct Hamiltonians, which allows us to find relationships between different dynamical systems.

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