Connection between the Kepler problem and the Maxwell fish-eye

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Making use of Hamilton's equations, the formal equivalence between the Maxwell fish-eye and the Kepler problem is established. The analogs of some properties of the Maxwell fish-eye in the Kepler problem are also discussed.

Keywords: Geometrical optics; Kepler problem

Utilizando las ecuaciones de Hamilton, se establece la equivalencia formal entre el ojo de pescado de Maxwell y el problema de Kepler. Se discuten también los análogos de algunas propiedades del ojo de pescado de Maxwell en el problema de Kepler.

Descriptores: Óptica geométrica; Problema de Kepler

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1. Introducción

The Kepler problem in classical mechanics and the Maxwell fish-eye in geometrical optics are systems thoroughly studied (see, *e.g.*, Refs. 1–7 and the references cited therein). On one side, the Maxwell fish-eye is a medium with a refractive index

$$n = \frac{a}{b + r^2}, \tag{1}$$

where a and b are constants and r denotes the distance to the origin. All light rays crossing this medium are circles or arcs of circles; furthermore, if $b > 0$, then, given a point P in the medium, there exists another point, P′, opposed to P through the origin (the conjugate of P), such that every ray passing through P, passes through P′. Only when $b > 0$, the ray is a complete circle. For instance, when $b = 0$, the ray is a circle passing through the origin, with the origin removed; the light approaches the origin, but never reaches it (this behavior is somewhat similar to what an observer outside a black hole would see when a light ray is entering the black hole; the speed of light tends to zero as the light approaches the black hole horizon, so that it takes an infinite time to cross it).

On the other side, in the Kepler problem, a particle of mass m moves under the action of a central force given by

$$\mathbf{F} = -\frac{k}{r^3}\mathbf{r}, \qquad (k > 0).$$

The particle's trajectory can be, depending on the energy, a hyperbola (for $E > 0$), a parabola (for $E = 0$) or an ellipse (for $E < 0$). Furthermore, in momentum space, the hodograph is a circle [3, 4].

A proof of the formal equivalence between these two systems was given by Buchdahl in 1978 [6]. In this paper this equivalence is demonstrated by means of a simpler and elementary procedure. We also prove the existence of conjugate points in the momentum space of the Kepler problem.

In Sect. 2 the light rays in the Maxwell fish-eye are considered showing that for all values of b in Eq. (1), the light rays are (arcs of) circles. The Kepler problem is dealt with in Sect. 3 and Sect. 4 contains two proofs of the existence of conjugate points in the Kepler problem, one is based on Hamilton's vector and the other employs the stereographic projection.

2. The Maxwell fish-eye

The Hamiltonian for the light rays in a medium with refractive index n can be taken as [8, 9]

$$H = \frac{c}{2n^2} g^{ij} p_i p_j - \frac{c}{2}, \tag{2}$$

where c is the speed of light in vacuum, p_i are the components of the momentum $\mathbf{p}$ (which is tangent to the light ray and $|\mathbf{p}| = n$ at each point of the ray), (g^{ij}) is the inverse of the matrix formed by the components of the metric tensor and there is summation on repeated indices. In the case of the Maxwell fish-eye, n is given by Eq. (1) and owing to the symmetry of the problem, it is convenient to employ spherical coordinates. Moreover, since the rays must lie on planes containing the origin, we shall consider a ray on the plane

$\theta = \pi/2$, therefore $p_\theta = 0$ and Eq. (2) reduces to

$$H = \frac{c}{2n^2}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) - \frac{c}{2}. \tag{3}$$

The trajectory followed by the light is determined by the Hamilton equations

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}. \tag{4}$$

Substituting Eq. (3) into Eqs. (4) we obtain

$$\frac{dr}{dt} = \frac{\partial H}{\partial p_r} = \frac{cp_r}{n^2}, \tag{5a}$$

$$\frac{d\varphi}{dt} = \frac{\partial H}{\partial p_\varphi} = \frac{cp_\varphi}{r^2 n^2}, \tag{5b}$$

$$\frac{dp_\varphi}{dt} = -\frac{\partial H}{\partial \varphi} = 0. \tag{5c}$$

Equation (5c) implies that p_φ is a constant, whose value will be denoted by L. In order to find the trajectory of the light ray, we make use of Eqs. (5a) and (5b), then, according to the chain rule,

$$\frac{d\varphi}{dr} = \frac{d\varphi/dt}{dr/dt} = \frac{L}{r^2 p_r}. \tag{6}$$

Since $|\mathbf{p}| = n$, from Eq. (1) we have

$$p_r^2 + \frac{L^2}{r^2} = \left(\frac{a}{b + r^2}\right)^2. \tag{7}$$

Eliminating p_r from Eqs. (6) and (7) we obtain the differential equation

$$\frac{d\varphi}{dr} = \frac{b + r^2}{r\sqrt{a^2 r^2/L^2 - (r^2 + b)^2}}. \tag{8}$$

It is easy to see that the solution of this equation corresponds to a circle of radius

$$R = \frac{a}{2|L|}, \tag{9a}$$

whose center is at a distance

$$d = \sqrt{R^2 - b} \tag{9b}$$

from the origin. Hence, for $b > 0$, the origin lies inside the circle, for $b = 0$, the origin lies on the circle and, for $b < 0$, the origin is outside the circle.

3. The Kepler problem

The Hamiltonian of the Kepler problem is

$$H = \frac{p^2}{2m} - \frac{k}{r}.$$

As in the previous section, the orbit is on a plane passing through the origin and we can assume that the plane of the orbit is the xy-plane. Thus, $p_z = 0$ and

$$H = \frac{p_x^2 + p_y^2}{2m} - \frac{k}{\sqrt{x^2 + y^2}}. \tag{10}$$

Now, we shall introduce polar coordinates, ρ, χ, in momentum space,

$$p_x = \rho\cos\chi, \qquad p_y = \rho\sin\chi. \tag{11}$$

Then,

$$\begin{aligned} p_x\,dx + p_y\,dy &= \rho\cos\chi\,dx + \rho\sin\chi\,dy \\ &= d(x\rho\cos\chi + y\rho\sin\chi) \\ &\quad - (x\cos\chi + y\sin\chi)\,d\rho \\ &\quad + (x\rho\sin\chi - y\rho\cos\chi)\,d\chi, \end{aligned}$$

which shows that

$$\begin{aligned} p_\rho &\equiv -x\cos\chi - y\sin\chi, \\ p_\chi &\equiv x\rho\sin\chi - y\rho\cos\chi, \end{aligned} \tag{12}$$

together with ρ and χ are canonical coordinates, which guarantees that Eqs. (4) are applicable to them. From Eqs. (12) we obtain

$$x^2 + y^2 = p_\rho^2 + \frac{p_\chi^2}{\rho^2}, \tag{13}$$

and substituting Eqs. (11) and (13) into Eq. (10), with $H = E$, we arrive at

$$p_\rho^2 + \frac{p_\chi^2}{\rho^2} = \left(\frac{2mk}{\rho^2 - 2mE}\right)^2. \tag{14}$$

Hence, Eqs. (14) and (7) are the same equation if we identify

$$a = 2mk, \qquad b = -2mE,$$

which demonstrates the equivalence of the two problems. The hodograph, which is the curve described by the particle in momentum space, is therefore a circle of radius R whose center is at a distance d from the origin with [see Eqs. (9)]

$$R = \frac{mk}{|L|}, \tag{15a}$$

$$d = \sqrt{R^2 + 2mE}. \tag{15b}$$

From Eq. (15b) we see that, if $E > 0$, the origin lies outside the circle, if $E = 0$, the origin is on the circle and if $E < 0$, the origin is inside the circle.

4. Conjugate points in the Kepler problem

The Kepler problem possesses a constant of the motion, known as the Laplace-Runge-Lenz vector [1, 3, 4], given by

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - \frac{mk}{r}\mathbf{r}.$$

From $\mathbf{A}$ and $\mathbf{L}$ one can obtain the Hamilton vector, $\mathbf{h}$, which is also a constant of the motion, as [3]

$$\mathbf{h} = \frac{\mathbf{L} \times \mathbf{A}}{L^2},$$

hence

$$\mathbf{h} = \mathbf{p} + \frac{mk}{L^2}\left[(\mathbf{p}\cdot\mathbf{r})\frac{\mathbf{r}}{r} - r\mathbf{p}\right]. \qquad (16)$$

The vector inside the square brackets in Eq. (16) has constant magnitude, in fact

$$\left|(\mathbf{p}\cdot\mathbf{r})\frac{\mathbf{r}}{r} - r\mathbf{p}\right|^2 = |\mathbf{r} \times \mathbf{p}|^2 = L^2,$$

therefore,

$$|\mathbf{p} - \mathbf{h}| = \frac{mk}{|L|} \qquad (17)$$

and by means of a straightforward computation, from Eq. (16) it follows that

$$h^2 = 2mE + \frac{m^2k^2}{L^2}. \qquad (18)$$

Thus, Eqs. (17) and (18) give an alternative proof of the fact that the hodograph is a circle (or an arc of a circle) of radius $mk/|L|$ (*cf.* Eq. (15a)) centered at the end point of $\mathbf{h}$ [note that Eq. (18) is equivalent to Eq. (15b)].

Now we want to show that given a momentum, $\mathbf{p}$, there exists another point in momentum space, $\mathbf{p}'$, the conjugate of $\mathbf{p}$, such that, for a fixed negative value of E, every hodograph passing through $\mathbf{p}$, passes through $\mathbf{p}'$. Using the fact that the Hamilton vector is a constant of the motion, from Eq. (16) we have

$$\mathbf{p} = \mathbf{h} + \frac{mk}{|L|}\hat{\mathbf{a}} \qquad (19)$$

and

$$\mathbf{p}' = \mathbf{h} + \frac{mk}{|L|}\hat{\mathbf{b}}, \qquad (20)$$

where $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ are unit vectors. Owing to the spherical symmetry of the problem, we can assume that $\mathbf{p}'$ is antiparallel to $\mathbf{p}$, then, from Eqs. (18)–(20) we have

$$\begin{aligned}
\mathbf{p}\cdot\mathbf{p}' &= h^2 + \frac{mk}{|L|}\mathbf{h}\cdot(\hat{\mathbf{a}}+\hat{\mathbf{b}}) + \frac{m^2k^2}{L^2}\hat{\mathbf{a}}\cdot\hat{\mathbf{b}} \\
&= 2mE + \frac{mk}{|L|}\mathbf{h}\cdot(\hat{\mathbf{a}}+\hat{\mathbf{b}}) + \frac{m^2k^2}{L^2}(1+\hat{\mathbf{a}}\cdot\hat{\mathbf{b}}) \\
&= 2mE + \frac{mk}{|L|}\mathbf{h}\cdot(\hat{\mathbf{a}}+\hat{\mathbf{b}}) + \frac{m^2k^2}{2L^2}(\hat{\mathbf{a}}+\hat{\mathbf{b}})^2 \\
&= 2mE + \frac{mk}{2|L|}(\hat{\mathbf{a}}+\hat{\mathbf{b}})\cdot\left[2\mathbf{h} + \frac{mk}{|L|}(\hat{\mathbf{a}}+\hat{\mathbf{b}})\right] \\
&= 2mE + \frac{mk}{2|L|}(\hat{\mathbf{a}}+\hat{\mathbf{b}})\cdot(\mathbf{p}+\mathbf{p}'),
\end{aligned}$$

the product $(\hat{\mathbf{a}}+\hat{\mathbf{b}})\cdot(\mathbf{p}+\mathbf{p}')$ vanishes because $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ point along the directions going from the center of the hodograph to $\mathbf{p}$ and $\mathbf{p}'$, which lie on the circle, hence $(\hat{\mathbf{a}}+\hat{\mathbf{b}})$ is orthogonal to the (common) direction of $\mathbf{p}$ and $\mathbf{p}'$. Thus, we find that

$$\mathbf{p}\cdot\mathbf{p}' = 2mE. \qquad (21)$$

Since the right-hand side of Eq. (21) only involves the value of the energy, this equation implies that, for a fixed energy, every hodograph passing through $\mathbf{p}$, passes through the conjugate point $\mathbf{p}'$. Since $\mathbf{p}$ and $\mathbf{p}'$ are antiparallel, Eq. (21) implies that E must be negative. Correspondingly, there exist conjugate points in the Maxwell fish-eye only when $b > 0$ (recalling that $b = -2mE$).

An alternative proof of the existence of conjugate points in the Kepler problem when $E < 0$, can be given without restricting the motion to the xy-plane by means of the stereographic projection, which relates the momentum space with the unit sphere in four dimensions (see, *e.g.*, Refs. 4 and 10) according to

$$\mathbf{p} = (p_x, p_y, p_z) = p_0\frac{(u_x, u_y, u_z)}{1 - u_w} \qquad (22)$$

and

$$\begin{aligned}
\hat{\mathbf{u}} &= (u_x, u_y, u_z, u_w) \\
&= \frac{(2p_0p_x, 2p_0p_y, 2p_0p_z, p^2 - p_0^2)}{p^2 + p_0^2},
\end{aligned} \qquad (23)$$

where $p_0 \equiv \sqrt{-2mE}$ and $\hat{\mathbf{u}}$ is a unit vector. By means of this mapping, the equivalence of the Kepler problem with the free motion of a particle on the unit sphere in four dimensions can be established. The orbits of the latter system are great circles, which, projected onto the momentum space are circles, showing that in this case ($E < 0$) the hodograph is a circle [4]. Clearly, every great circle passing through a point $\hat{\mathbf{u}}$ of the unit sphere in four dimensions also passes through the point $\hat{\mathbf{u}}' = -\hat{\mathbf{u}}$, the antipode of $\hat{\mathbf{u}}$. Therefore, if the points $\mathbf{p}$ and $\mathbf{p}'$ in momentum space correspond to $\hat{\mathbf{u}}$ and $\hat{\mathbf{u}}'$, respectively, then by combining Eqs. (22) and (23), we find that

$$\mathbf{p}' = p_0 \frac{\left(-\frac{2p_0p_x}{p^2+p_0^2}, -\frac{2p_0p_y}{p^2+p_0^2}, -\frac{2p_0p_z}{p^2+p_0^2}\right)}{1-\frac{p_0^2-p^2}{p^2+p_0^2}} = -\frac{p_0^2}{p^2}\mathbf{p},$$

hence,

$$p' = \frac{p_0^2}{p} = -\frac{2mE}{p},$$

which coincides with the expression obtained above [Eq. (21)].

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