The three-dimensional Kepler problem and the four-dimensional isotropic harmonic oscillator

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Making use of the fact that certain submanifolds of the phase space of the four-dimensional isotropic harmonic oscillator can be identified with the phase space of the Kepler problem with an interaction with the field of a magnetic monopole and a centrifugal potential if the energy is negative, the solution of this problem in classical and quantum mechanics is obtained. Similar results for the case where the energy is equal to zero are also given.

Keywords: Kepler problem; Kustaanheimo-Stiefel transformation; magnetic monopole

Usando el hecho de que ciertas subvariedades del espacio fase del oscilador armónico isótropo en cuatro dimensiones se pueden identificar con el espacio fase del problema de Kepler con una interacción con el campo de un monopolo magnético y un potencial centrífugo si la energía es negativa, se obtiene la solución de este problema en la mecánica clásica y en la mecánica cuántica. Se presentan resultados similares para el caso en el que la energía es igual a cero.

Descriptores: Problema de Kepler; transformación de Kustaanheimo-Stiefel; monopolo magnético

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1. Introduction

The Kepler problem, in classical or quantum mechanics and in any number of dimensions, is known to be related to other simple or interesting problems (see, *e.g.*, Refs. 1–5). In particular, by means of the complex mapping $w = z^2$, the Kepler problem in two dimensions can be related to the two-dimensional isotropic harmonic oscillator (see, *e.g.*, Refs. 2, 3, 6–9) and, similarly, making use of the Kustaanheimo-Stiefel transformation, the Kepler problem in three dimensions can be shown to be equivalent to the four-dimensional isotropic harmonic oscillator (FIHO) with a constraint (see *e.g.*, Refs. 10 and 11, and the references cited therein). It turns out that, in the same way, the FIHO can be related to the Kepler problem with a magnetic monopole field and a centrifugal potential (proportional to the square of the magnetic charge). This latter problem (sometimes called the MIC-Kepler problem) was studied by McIntosh and Cisneros [12], who were interested in spherically symmetric systems in the presence of a magnetic monopole field. Making use of vectorial methods, they showed that by adding to the potential $1/r$ and the field of a magnetic monopole, a repulsive centrifugal potential of the appropriate strength, the orbits are plane, as well as confined to the surface of a cone. The corresponding Schrödinger equation was also solved by separation of variables in spherical coordinates, finding that, if the magnetic charge of the monopole does not vanish, the ground state is degenerate. In Ref. 5, it was shown that the MIC-Kepler problem can be related to a "conformal" Kepler problem in four dimensions, which was introduced to associate the FIHO to the three-dimensional Kepler problem (*cf.* also Ref. 13).

In this paper, the relationship between the FIHO and the Kepler problem in three dimensions is reviewed, showing that the solution of the Kepler problem with negative or zero energy, with or without the monopole field and the centrifugal potential, can be easily obtained from the solution of the FIHO (in classical or quantum mechanics) and that a dynamical symmetry group for the Kepler problem can be derived from one of the FIHO (see also Ref. 5). In Sect. 2, the solution of the Kepler problem in classical mechanics with negative energy, including the monopole field and the centrifugal potential, is derived from the much simpler solution of the FIHO. We find the analogs of some well-known properties of the solution of the ordinary Kepler problem, some of which were already given in Ref. 12. We also show that the constants of motion that generate the SU(4) dynamical symmetry of the FIHO give rise to the angular momentum and the Hermann-Bernoulli-Laplace-Runge-Lenz vector (usually known as the Runge-Lenz vector), or their analogs when the magnetic charge is not zero (*cf.* also Ref. 5). In Sect. 3, the solution of the Kepler problem in quantum mechanics with negative energy, in the presence of the magnetic monopole is derived from the solution of the quantum FIHO. The solution so obtained turns out to be separable in parabolic coordinates, which, when the magnetic charge is zero, agrees with the well-known fact that the hydrogen atom admits separable solutions in parabolic coordinates. Section 4 contains similar results to those given in Sects. 2 and 3 for the case where the energy of the Kepler problem is equal to zero.

2. The four-dimensional isotropic harmonic oscillator and the Kustaanheimo-Stiefel transformation

Making use of the complex four-component vector

$$\psi \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \equiv \begin{pmatrix} ip_1 + M\omega u_1 \\ ip_2 + M\omega u_2 \\ ip_3 + M\omega u_3 \\ ip_4 + M\omega u_4 \end{pmatrix}, \tag{1}$$

the Hamiltonian function of the four-dimensional isotropic harmonic oscillator (FIHO)

$$H = \frac{1}{2M}(p_1^2 + p_2^2 + p_3^2 + p_4^2) + \frac{M\omega^2}{2}(u_1^2 + u_2^2 + u_3^2 + u_4^2), \tag{2}$$

can be expressed as

$$H = \frac{1}{2M}\psi^\dagger \psi, \tag{3}$$

and the Poisson brackets among the components of ψ and their complex conjugates are

$$\{\psi_i, \psi_j\} = 0 = \{\overline{\psi_i}, \overline{\psi_j}\},$$
$$\{\psi_i, \overline{\psi_j}\} = -2iM\omega\delta_{ij} \qquad i,j = 1,2,3,4. \tag{4}$$

Let α be a constant (complex) 4×4 matrix, the function $f = \psi^\dagger \alpha \psi$ is real if and only if α is Hermitian, $\alpha^\dagger = \alpha$. By virtue of Eq. (4), the Poisson bracket of the function $f = \psi^\dagger \alpha \psi$ with $g = \psi^\dagger \beta \psi$, where β is another 4×4 matrix, is given by

$$\{f, g\} = -2iM\omega\psi^\dagger[\alpha, \beta]\psi. \tag{5}$$

Since the Hamiltonian (3) is given by $H = (1/2M)\psi^\dagger I \psi$, where I is the 4×4 unit matrix and any matrix commutes with I, Eq. (5) implies that any function $f = \psi^\dagger \alpha \psi$ is a constant of the motion:

$$\{f, H\} = -i\omega\psi^\dagger[\alpha, I]\psi = 0. \tag{6}$$

Thus, if α is any Hermitian constant 4×4 matrix, $\psi^\dagger \alpha \psi$ is a real-valued constant of the motion.

As is well known, any constant of the motion is the generating function of a one-parameter group of canonical transformations that leave the Hamiltonian invariant. In order to find explicitly the canonical transformations generated by the functions of the form $\psi^\dagger \alpha \psi$, for α Hermitian, we recall that any function of the coordinates and momenta, G, is the generating function of a one-parameter group of transformations, parameterized by a variable s, in such a way that the rate of change of an arbitrary function f under the transformations generated by G is given by

$$\frac{df}{ds} = \{f, G\}. \tag{7}$$

Thus, from Eqs. (4) and (7), it follows that under the transformations generated by $\psi^\dagger \alpha \psi$,

$$\frac{d\psi}{ds} = -2iM\omega\alpha\psi, \tag{8}$$

hence

$$\psi(s) = \exp(-2iM\omega s\alpha)\psi(0). \tag{9}$$

If we further assume that α is traceless, $U \equiv \exp(-2iM\omega s\alpha)$ is a unitary matrix with determinant $+1$, *i.e.*, U belongs to the group SU(4). Thus, SU(4) is a dynamical symmetry group of the FIHO. If the trace of α is not restricted, then U is an element of U(4), which is also a dynamical symmetry group of the FIHO; SU(4) is associated with the quadratic constants of the motion apart from the Hamiltonian itself. (Recalling that H corresponds to $(1/2M)I$, Eq. (9) with $\alpha = (1/2M)I$ gives the time evolution of the FIHO, with the parameter s being the time.)

Now we introduce the four variables x, y, z, w, defined by

$$x = 2(u_1u_3 + u_2u_4), \qquad y = 2(u_1u_4 - u_2u_3),$$
$$z = u_1^2 + u_2^2 - u_3^2 - u_4^2, \quad w = \arctan\frac{u_2}{u_1}, \tag{10}$$

then if p_x, p_y, p_z, p_w denote the momenta conjugate to these coordinates, by differentiating Eqs. (10) one finds that

$$p_x dx + p_y dy + p_z dz + p_w dw = p_1 du_1 + p_2 du_2 + p_3 du_3 + p_4 du_4$$

with

$$p_1 = 2(u_3p_x + u_4p_y + u_1p_z) - \frac{u_2p_w}{u_1^2 + u_2^2},$$
$$p_2 = 2(u_4p_x - u_3p_y + u_2p_z) + \frac{u_1p_w}{u_1^2 + u_2^2},$$
$$p_3 = 2(u_1p_x - u_2p_y - u_3p_z),$$
$$p_4 = 2(u_2p_x + u_1p_y - u_4p_z), \tag{11}$$

where p_1, p_2, p_3, p_4 are the momenta conjugate to u_1, u_2, u_3, u_4. The Kustaanheimo-Stiefel transformation is usually defined as the relation between the u_i and x, y, z given above (see, *e.g.*, Refs. 10 and 11); the extra coordinate, w, can be defined in many other ways. It may be noticed that the replacement of the coordinate w by $w + \xi(x, y, z)$ leaves p_w unchanged while $(p_x, p_y, p_z) \mapsto (p_x, p_y, p_z) + p_w\nabla\xi$, which corresponds to the effect of a *gauge transformation* [see Eq. (34), below]. By means of straightforward computations, from Eqs. (10) and (11), one finds that

$$r^2 \equiv x^2 + y^2 + z^2 = (u_1^2 + u_2^2 + u_3^2 + u_4^2)^2, \tag{12}$$

$$p_1^2 + p_2^2 + p_3^2 + p_4^2 = 4r(p_x^2 + p_y^2 + p_z^2) + \frac{2p_w^2}{r+z} + \frac{4p_w}{r+z}(yp_x - xp_y) \tag{13}$$

and

$$K \equiv u_1 p_2 - u_2 p_1 + u_3 p_4 - u_4 p_3 = p_w. \tag{14}$$

Clearly, K is a constant of the motion, being the sum of two components of the angular momentum; alternatively, making use of Eq. (1), it may be noticed that $K = (1/2M\omega)\psi^\dagger \gamma \psi$, where γ is the 4×4 Hermitian matrix [see Eq. (6)]

$$\gamma = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} \tag{15}$$

and σ_2 is one of the Pauli matrices.

2.1. The Kepler problem with negative energy

Substituting Eqs. (12)-(14) into the Hamiltonian of the FIHO [Eq. (2)] one obtains

$$H = \frac{2r(p_x^2 + p_y^2 + p_z^2)}{M} + \frac{M\omega^2 r}{2} + \frac{K^2}{M(r+z)} + \frac{2K}{M(r+z)}(yp_x - xp_y), \tag{16}$$

therefore, if $E > 0$, by defining

$$h \equiv \frac{H - E}{4r} - \frac{M\omega^2}{8}, \tag{17}$$

we find that

$$dh = \frac{1}{4r} dH - \frac{(H-E)}{4r^2} dr. \tag{18}$$

Hence,

$$h = \frac{p_x^2 + p_y^2 + p_z^2}{2M} - \frac{E}{4r} + \frac{K^2}{4Mr(r+z)} + \frac{K}{2Mr(r+z)}(yp_x - xp_y) \tag{19}$$

and, restricted to the (six-dimensional) submanifold $H = E$, $K = \text{const.}$, from Eq. (17) we have

$$h = -\frac{M\omega^2}{8}, \tag{20}$$

and, according to Eq. (18),

$$dh = \frac{1}{4r} dH, \tag{21}$$

which means that the "time parameter", τ, conjugate to h, is related to the time, t, of the FIHO by

$$d\tau = 4r\, dt = 4(u_1^2 + u_2^2 + u_3^2 + u_4^2)\, dt \tag{22}$$

since, by virtue of Hamilton's equations we have, *e.g.*,

$$\frac{dx}{d\tau} = \frac{\partial h}{\partial p_x} = \frac{1}{4r}\frac{\partial H}{\partial p_x} = \frac{1}{4r}\frac{dx}{dt},$$

$$\frac{dp_x}{d\tau} = -\frac{\partial h}{\partial x} = -\frac{1}{4r}\frac{\partial H}{\partial x} = \frac{1}{4r}\frac{dp_x}{dt}.$$

(Note that the difference between the transformations for the coordinates and the velocities given in Ref. 11 follows from the difference between the time parameters used in the FIHO and the Kepler problem; the "explanation" given in Ref. 11, asserting that the coordinate transformation equations should not be used to determine the velocity transformation equations, makes no sense.)

If $K = 0$, Eq. (19) coincides with the Hamiltonian of the (three-dimensional) Kepler problem,

$$\frac{p_x^2 + p_y^2 + p_z^2}{2M} - \frac{k}{r}, \tag{23}$$

with a negative energy,

$$\varepsilon = -\frac{M\omega^2}{8} \tag{24}$$

[Eq. (20)], if we make the identification

$$k \equiv \frac{E}{4}. \tag{25}$$

Thus, on the submanifold of the phase space defined by $H = E$, $K = 0$, the dynamics of the FIHO reproduces that of the Kepler problem with negative energy. Since H and K are constants of the motion, any orbit starting at a point of the submanifold $H = E$, $K = \text{const.}$, remains on that submanifold.

The properties of the orbits for the Kepler problem with negative energy can be easily derived using the fact that the FIHO is equivalent to four independent one-dimensional harmonic oscillators and the orbits in the configuration space are ellipses centered at the origin. In order to simplify the analysis, we shall consider an orbit of the FIHO lying on the plane $u_2 = u_4 = 0$ in such a way that the axes of the orbit coincide with the u_1 and u_3 axes. Thus, assuming that

$$u_1 = a_1 \cos\omega t, \qquad u_3 = a_3 \sin\omega t, \tag{26}$$

with $a_1 \leq a_3$, we have $p_1 = -M\omega a_1 \sin\omega t$, $p_3 = M\omega a_3 \cos\omega t$, $p_2 = p_4 = 0$, hence $K = 0$ and the Hamiltonian H [Eq. (2)] has the constant value

$$E = \frac{M\omega^2}{2}(a_1^2 + a_3^2). \tag{27}$$

Substituting $u_2 = 0 = u_4$ into Eqs. (10) we obtain $x = 2u_1 u_3$, $z = u_1^2 - u_3^2$ (which is essentially the relationship between cartesian and parabolic coordinates on the plane) and $y = 0 = w$. Then, from Eqs. (26) we obtain

$$x = a_1 a_3 \sin 2\omega t,$$

$$z = -\frac{a_3^2 - a_1^2}{2} + \frac{a_1^2 + a_3^2}{2}\cos 2\omega t, \tag{28}$$

which are parametric equations of an ellipse with one focus at the origin, of eccentricity $e = (a_3^2 - a_1^2)/(a_1^2 + a_3^2)$, semiminor axis $b = a_1 a_3$ and semimajor axis $a = (a_1^2 + a_3^2)/2$, or, making use of Eqs. (24), (25), and (27) one finds that

$$a = \frac{E}{M\omega^2} = -\frac{k}{2\varepsilon}, \tag{29}$$

which is the well-known fact that the energy depends only on the length of the major axis of the orbit. By suitably choosing the constant of integration, Eqs. (22) and (26) yield

$$\tau = \frac{a_1^2 + a_3^2}{\omega}(2\omega t - e\sin 2\omega t), \tag{30}$$

which relates the time parameter of the Kepler problem, τ, with the eccentric anomaly $2\omega t$ (see, *e.g.*, Ref. 14).

The relationship between the coordinates $u_1, u_2, u_3, u_4, p_1, p_2, p_3, p_4$, and $x, y, z, w, p_x, p_y, p_z, p_w$, given by Eqs. (10) and (11) is two to one; the points $(u_1, u_2, u_3, u_4, p_1, p_2, p_3, p_4)$ and $(-u_1, -u_2, -u_3, -u_4, \ -p_1, -p_2, -p_3, -p_4)$ have the same coordinates $(x, y, z, w, p_x, p_y, p_z, p_w)$. Among other things, this fact implies that a complete cycle of the FIHO on the submanifold $H = E$, $K = 0$, corresponds to two complete cycles of the Kepler problem, hence, according to Eq. (22), the period in the Kepler problem is given by

$$T_{\text{Kepler}} = 2\int_0^{2\pi/\omega} (u_1^2 + u_2^2 + u_3^2 + u_4^2)\, dt. \tag{31}$$

On the other hand, the solution of the equations of motion of the FIHO is given by $u_i = a_i\cos(\omega t + \phi_i)$, where the a_i and the ϕ_i are constants. Then the momenta are given by $p_i = M du_i/dt = -M\omega a_i \sin(\omega t + \phi_i)$, hence, $E = (M\omega^2/2)(a_1^2 + a_2^2 + a_3^2 + a_4^2)$. Since the mean value of $\cos^2$ over a complete period is $1/2$, using Eqs. (31), (25) and (24) we find

$$T_{\text{Kepler}} = 2\frac{2\pi}{\omega}\frac{1}{2}\sum_{i=1}^{4} a_i^2 = \frac{4\pi E}{M\omega^3} = \frac{2\pi k M^{1/2}}{(-2\varepsilon)^{3/2}}, \tag{32}$$

which, taking into account Eq. (29), amounts to the relation

$$T_{\text{Kepler}}^2 = \frac{4\pi^2 M}{k} a^3, \tag{33}$$

which corresponds to Kepler's third law.

2.2. The MIC-Kepler problem

When K is a constant different from zero, one can express Eq. (19) in the form

$$h = \frac{1}{2M}\left(\mathbf{p} - \frac{q}{c}\mathbf{A}\right)^2 - \frac{E}{4r} + \frac{K^2}{8Mr^2}, \tag{34}$$

where $\mathbf{p} \equiv (p_x, p_y, p_z)$, c is the speed of light, q is a constant interpretable as an electric charge and

$$\mathbf{A} \equiv \frac{c}{q}\frac{K}{2}\frac{(-y\hat{\imath} + x\hat{\jmath})}{r(r+z)}, \tag{35}$$

is a vector potential for the magnetic field of a monopole at the origin of magnetic charge

$$g = \frac{c}{q}\frac{K}{2}. \tag{36}$$

Then, Eq. (34) can be interpreted as the Hamiltonian of a (nonrelativistic) particle of mass M and electric charge q in the field of a magnetic monopole superimposed to the central potential

$$V(r) = -\frac{k}{r} + \left(\frac{qg}{c}\right)^2 \frac{1}{2Mr^2}. \tag{37}$$

As in the case of the Kepler problem ($K = 0$), the orbits determined by the Hamiltonian (34) can be obtained from the solution of the equations of motion of the FIHO. By suitably choosing the coordinate axes, we can assume that $u_1 = a_1\cos\omega t$, $u_2 = a_2\sin\omega t$, $u_3 = \mu a_2\sin\omega t$, $u_4 = \mu a_1\cos\omega t$, where a_1, a_2 and μ are arbitrary constants. Then one finds that

$$\begin{aligned} E &= \frac{M\omega^2}{2}(1+\mu^2)(a_1^2 + a_2^2), \\ K &= M\omega(1-\mu^2)a_1 a_2, \end{aligned} \tag{38}$$

and substituting the expressions given above into Eqs. (10) we obtain

$$\begin{aligned} x &= 2\mu a_1 a_2 \sin(2\omega t), \\ y &= \mu[(a_1^2 + a_2^2)\cos(2\omega t) + a_1^2 - a_2^2], \\ z &= \frac{1-\mu^2}{2}[(a_1^2 - a_2^2)\cos(2\omega t) + a_1^2 + a_2^2], \end{aligned} \tag{39}$$

which are parametric equations of the intersection of the cone

$$x^2 + y^2 = \left(\frac{2\mu}{1-\mu^2}\right)^2 z^2 \tag{40}$$

and the plane

$$(1-\mu^2)(a_1^2 - a_2^2)y - 2\mu(a_1^2 + a_2^2)z + 4\mu(1-\mu^2)a_1^2 a_2^2 = 0 \tag{41}$$

thus, also in the case where K is different from zero, the orbit is an ellipse, but the plane of this ellipse does not contain the origin. It is easy to see that Eq. (32) also holds in the present case and making use of Eqs. (36), and (38)-(41) one finds that the semimajor axis of the orbit (39) is given by

$$a = \frac{\sqrt{k^2 + \frac{2\varepsilon}{M}\left(\frac{qg}{c}\right)^2}}{-2\varepsilon} \tag{42}$$

[*cf.* Eq. (29)] thus, again, the value of the Hamiltonian depends only on the semimajor axis of the ellipse

$$\varepsilon = -\frac{\sqrt{(2Mka)^2 + (qg/c)^4} - (qg/c)^2}{4Ma^2}. \tag{43}$$

2.3. Constants of the motion

The transformations (9) generated by a 4×4 Hermitian matrix α leave the hypersurface $H = E$ invariant, but only those Hermitian matrices α commuting with γ [Eq. (15)] leave the submanifold $H = E$, $K = \text{const.}$, invariant. It is easy to see that all the Hermitian 4×4 matrices that commute with γ are linear combinations with real coefficients of I, γ and the six Hermitian traceless matrices

$$\lambda_1 \equiv \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \quad \lambda_2 \equiv \begin{pmatrix} 0 & iI_2 \\ -iI_2 & 0 \end{pmatrix}, \quad \lambda_3 \equiv \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix},$$

$$\alpha_1 \equiv \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \quad \alpha_2 \equiv \begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix}, \quad \alpha_3 \equiv \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \tag{44}$$

where I_2 is the unit 2×2 matrix. A straightforward computation shows that the matrices (44) satisfy the commutation relations of a basis of the Lie algebra of the rotation group in four dimensions SO(4),

$$[\lambda_i, \lambda_j] = -2i\varepsilon_{ijk}\lambda_k,$$
$$[\alpha_i, \alpha_j] = -2i\varepsilon_{ijk}\lambda_k,$$
$$[\lambda_i, \alpha_j] = -2i\varepsilon_{ijk}\alpha_k. \tag{45}$$

Thus, the functions

$$L_i \equiv -\frac{1}{4M\omega}\psi^\dagger\lambda_i\psi, \qquad R_i \equiv -\frac{1}{8}\psi^\dagger\alpha_i\psi \tag{46}$$

are constants of the motion [Eq. (6)] which have a vanishing Poisson bracket with K [Eq. (5)]; therefore, the functions (46) generate canonical transformations that leave the submanifold $H = E$, $K = \text{const.}$, invariant and are constants of the motion for the Kepler problem, when $K = 0$, or for the Hamiltonian (34), when $K \neq 0$. From Eqs. (5), (45) and (46) one readily finds that

$$\{L_i, L_j\} = \varepsilon_{ijk}L_k,$$
$$\{L_i, R_j\} = \varepsilon_{ijk}R_k,$$
$$\{R_i, R_j\} = (-2M\varepsilon)\varepsilon_{ijk}L_k, \tag{47}$$

where we have made use of Eq. (24) (*cf.* also Ref. 5).

Making use of Eqs. (1), (10), (20), (25), (44) and (46) one finds that on the submanifold $H = E$, $K = 0$, the functions L_i are the components of the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ and the R_i are the components of the Hermann-Bernoulli-Laplace-Runge-Lenz (HBLRL) vector

$$\mathbf{R} = \mathbf{p} \times \mathbf{L} - \frac{Mk\mathbf{r}}{r}. \tag{48}$$

In an analogous manner, making use of the relation $\mathbf{p} = M\mathbf{v} + (q/c)\mathbf{A}$, one finds that, when $K \neq 0$, the conserved quantities L_i and R_i are the components of the vectors

$$\mathbf{L} = \mathbf{r} \times M\mathbf{v} - \frac{qg}{c}\frac{\mathbf{r}}{r} \tag{49}$$

and

$$\mathbf{R} = M\mathbf{v} \times \left[\mathbf{r} \times M\mathbf{v} - \frac{qg}{c}\frac{\mathbf{r}}{r}\right] - \frac{Mk\mathbf{r}}{r}. \tag{50}$$

(Note that $\mathbf{v}$ amounts to $d\mathbf{r}/d\tau$.)

As is well known, in the case of the Kepler problem, the orbit is plane and orthogonal to the angular momentum vector $\mathbf{L}$. The HBLRL vector $\mathbf{R}$ lies in the plane of the orbit and points to the point of the orbit closest to the origin. When $K \neq 0$ (*i.e.*, $g \neq 0$), the orbit is also plane and is the intersection of a plane normal to

$$\mathbf{N} = Mk\mathbf{L} - \frac{qg}{c}\mathbf{R}, \tag{51}$$

with $\mathbf{L}$ and $\mathbf{R}$ given by Eqs. (49) and (50), respectively, and a cone whose axis is the vector $\mathbf{L}$. The half-angle of the cone, θ, is given by

$$\cos\theta = -\frac{qg}{c|\mathbf{L}|} \tag{52}$$

and the center of the orbit is at the point $\mathbf{R}/(2M\varepsilon)$. [These properties can be derived by considering the orbit (39).]

The vector $\mathbf{h} \equiv (\mathbf{L} \times \mathbf{R})/L^2$ is also a constant of the motion, analogous to the Hamilton vector of the ordinary Kepler problem (see, *e.g.*, Ref. 10). From Eqs. (49) and (50) one finds that

$$\mathbf{h} = M\mathbf{v} + \frac{qg}{c}\frac{M}{L^2}\frac{\mathbf{r}\cdot\mathbf{v}}{r}\mathbf{L} - \frac{M^2k}{L^2}\left(r\mathbf{v} - \frac{\mathbf{r}\cdot\mathbf{v}}{r}\mathbf{r}\right),$$

therefore, taking into account the equality $\mathbf{v}\cdot\mathbf{L} = -(qg/c)(\mathbf{r}\cdot\mathbf{v}/r)$, which follows from Eq. (49), the component of the velocity orthogonal to $\mathbf{L}$, given by $\mathbf{v}_\perp \equiv \mathbf{v} - (\mathbf{v}\cdot\mathbf{L}/L^2)\mathbf{L}$, is

$$\mathbf{v}_\perp = \frac{\mathbf{h}}{M} + \frac{Mk}{L^2}\left(r\mathbf{v} - \frac{\mathbf{r}\cdot\mathbf{v}}{r}\mathbf{r}\right),$$

hence

$$|M\mathbf{v}_\perp - \mathbf{h}| = \frac{Mk}{L^2}\sqrt{L^2 - (qg/c)^2}, \tag{53}$$

which means that the vector $M\mathbf{v}_\perp$ describes a circle. Thus, the hodograph is the ellipse determined by the intersection of the cylinder (53) and the plane orthogonal to $\mathbf{N}$ passing through the origin. Only if $\mathbf{N}$ and $\mathbf{L}$ are parallel, the hodograph is a circle (which happens when $g = 0$ or when $\mathbf{L}$ and $\mathbf{R}$ are parallel). Finally, it may be noticed that

$$\mathbf{L}\cdot\mathbf{R} = Mk\frac{qg}{c}. \tag{54}$$

3. Solution of the Schrödinger equation with negative energy

The (time-independent) Schrödinger equation for the FIHO can be written as

$$-\frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial u_1^2} + \frac{\partial^2}{\partial u_2^2} + \frac{\partial^2}{\partial u_3^2} + \frac{\partial^2}{\partial u_4^2}\right)\Psi + \frac{M\omega^2}{2}(u_1^2 + u_2^2 + u_3^2 + u_4^2)\Psi = E\Psi \tag{55}$$

or, in terms of the coordinates x, y, z and w defined by Eqs. (10),

$$-\frac{\hbar^2}{2M}\left[\nabla^2\Psi + \frac{1}{2r(r+z)}\frac{\partial^2\Psi}{\partial w^2} + \frac{1}{r(r+z)}\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)\frac{\partial\Psi}{\partial w}\right] + \frac{M\omega^2}{8}\Psi = \frac{E}{4r}\Psi, \tag{56}$$

where $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$. Since w is an ignorable coordinate, we can look for separable solutions of Eq. (56) of the form

$$\Psi(x,y,z,w) = \psi(x,y,z)e^{iKw/\hbar} \tag{57}$$

where K is a constant, ψ satisfies the equation

$$\frac{1}{2M}\left(\frac{\hbar}{i}\nabla - \frac{q}{c}\mathbf{A}\right)^2\psi - \frac{E}{4r}\psi + \frac{K^2}{8Mr^2}\psi = -\frac{M\omega^2}{8}\psi \tag{58}$$

[*cf.* Eq. (34)] and $\mathbf{A}$ is the vector potential of a magnetic monopole given by Eq. (35). Equation (58) is the Schrödinger equation for a charged particle in the field of a magnetic monopole of charge $g = (cK/2q)$ [Eq. (36)] and the central field (37). Taking into account that $-i\hbar\partial/\partial w$ amounts to $-i\hbar(u_1\partial/\partial u_2 - u_2\partial/\partial u_1 + u_3\partial/\partial u_4 - u_4\partial/\partial u_3)$, *i.e.*, is the sum of two components of the FIHO angular momentum, its eigenvalues, K, must be of the form $N\hbar$, where N is an integer and, therefore, from Eq. (36), one obtains the Dirac quantization condition

$$\frac{qg}{c} = \frac{N\hbar}{2}. \tag{59}$$

When $K = 0$, Eq. (58) reduces to the Schrödinger equation for the hydrogen atom.

It is convenient to consider the FIHO as two two-dimensional isotropic harmonic oscillators of the same frequency, one with coordinates u_1, u_2 and the other with coordinates u_3, u_4; then, the condition

$$\frac{\hbar}{i}\frac{\partial\Psi}{\partial w} = K\Psi \tag{60}$$

[Eq. (57)] means that the sum of the angular momenta of these two two-dimensional oscillators is equal to K. On the other hand, the energy eigenvalues of a two-dimensional isotropic harmonic oscillator are of the form $(2s+1)\hbar\omega$, with $s = 0, \frac{1}{2}, 1, \ldots$, while the eigenvalues of its angular momentum are given by $2m\hbar$, with $m = -s, -s+1, \ldots, s$ (see, *e.g.*, Ref. 15); hence m can be an integer if and only if s is an integer. The common (unnormalized) eigenfunctions of the angular momentum and of the Hamiltonian of the two-dimensional isotropic harmonic oscillator expressed in polar coordinates are of the form

$$\psi_{sm}(\rho,\theta) = e^{-M\omega\rho^2/\hbar}\,\rho^{2|m|}\,e^{2im\theta}\,L^{2|m|}_{s-|m|}\left(\frac{M\omega\rho^2}{\hbar}\right), \tag{61}$$

where L^k_n denotes the associated Laguerre polynomials (the subscript n corresponds to the degree of the polynomial L^k_n). Thus, the energy eigenvalues of the FIHO can be expressed as

$$E = 2(s_1 + s_2 + 1)\hbar\omega, \tag{62}$$

where s_1 and s_2 are two independent quantum numbers that can take the values $0, \frac{1}{2}, 1, \ldots$, and the constraint (60) amounts to

$$2(m_1 + m_2)\hbar = K, \tag{63}$$

where m_1 and m_2 can take the values $m_1 = -s_1, -s_1+1, \ldots, s_1$ and $m_2 = -s_2, -s_2+1, \ldots, s_2$. Since the points (u_1, u_2, u_3, u_4) and $(-u_1, -u_2, -u_3, -u_4)$ have the same coordinates x, y, z (and w) [see Eqs. (10)], the single-valued solutions of Eq. (58) correspond to the even solutions of Eq. (55); hence, in order to have single-valued solutions of Eq. (58), m_1 and m_2 must be both integers or half-integers, in such a way that $m_1 + m_2$ will always be an integer, and, therefore s_1 and s_2 are both integers or half-integers and $s_1 + s_2$ is an integer greater than, or equal to, zero. Letting $n \equiv s_1 + s_2 + 1$, and making use again of the identifications (24) and (25), from Eq. (62) we obtain

$$\varepsilon = -\frac{Mk^2}{2\hbar^2 n^2}, \tag{64}$$

with $n = 1, 2, \ldots$.

In the case where $K = 0$ (hydrogen atom without magnetic monopole), Eq. (63) gives $m_2 = -m_1$, which does not impose further restrictions on n and Eq. (64) is just the well-known expression for the energy levels of the hydrogen atom. Since the wave functions of the FIHO can be obtained by multiplying the wave functions of the two equivalent two-dimensional isotropic harmonic oscillators, the wave functions of the hydrogen atom can be easily obtained using Eq. (61). To that end, we write $u_1 + iu_2 = \rho_1 e^{i\theta_1}$ and $u_3 + iu_4 = \rho_2 e^{i\theta_2}$; then, from Eqs. (10) we find that

$$\begin{aligned} x &= 2\rho_1\rho_2\cos(\theta_2 - \theta_1), \\ y &= 2\rho_1\rho_2\sin(\theta_2 - \theta_1), \\ z &= \rho_1^2 - \rho_2^2, \end{aligned} \tag{65}$$

which coincides with the relationship between the cartesian coordinates x, y, z and the parabolic coordinates ρ_1, ρ_2, $\theta_2 - \theta_1$, and taking into account that $m_2 = -m_1$ from Eq. (61) we obtain

$$\psi_{s_1 s_2 m_2} = e^{-\kappa(\rho_1^2+\rho_2^2)/2}(\rho_1\rho_2)^{2|m_2|}e^{2im_2(\theta_2-\theta_1)} L^{2|m_2|}_{s_1-|m_2|}(\kappa\rho_1^2)\,L^{2|m_2|}_{s_2-|m_2|}(\kappa\rho_2^2), \tag{66}$$

with $\kappa \equiv M\omega/\hbar = 2Mk/(\hbar^2 n)$ [see Eqs. (24) and (64)], which agrees with the solution of the hydrogen atom in parabolic coordinates. (See, *e.g.*, Ref. 16; note that the parabolic coordinates used there, ζ, η and ϕ, correspond to $2\rho_2^2$, $2\rho_1^2$ and $\theta_2-\theta_1$, respectively.)

When K is different from zero, condition (63) restricts the values of n appearing in Eq. (64). For instance, the lowest admissible nonzero value of $|K|$ is $2\hbar$ (*i.e.*, $|qg|=\hbar c$). Taking $K=2\hbar$, Eq. (63) implies that $m_1+m_2=1$, and the smallest value of s_1+s_2 compatible with this constraint is 1; in fact there are three possible combinations of (s_1, s_2, m_1, m_2) such that $m_1+m_2=1$ and $s_1+s_2=1$ [namely: (1,0,1,0), (0,1,0,1) and $(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$], which means that the lowest energy level corresponds to $n=2$ and is threefold degenerate. In a similar manner, one finds that if $K=N\hbar$, where $|N|$ is an even integer, the lowest value of n is $|N|/2+1$ and the degeneracy of the corresponding energy level is $|N|+1$.

The solutions of Eq. (58), with the monopole field present, can be obtained again making use of Eqs. (57) and (61), taking into account that, according to Eqs. (10), $w=\theta_1$, and, now, $m_1=(N/2)-m_2$ [Eq. (63)], thus

$$\psi_{s_1s_2m_2}=e^{-\kappa(\rho_1^2+\rho_2^2)/2}\,\rho_1^{|N-2m_2|}\,\rho_2^{|2m_2|}\,e^{2im_2(\theta_2-\theta_1)}\,L_{s_1-|N/2-m_2|}^{|N-2m_2|}(\kappa\rho_1^2)\,L_{s_2-|m_2|}^{|2m_2|}(\kappa\rho_2^2), \quad (67)$$

with $\kappa=2Mk/(\hbar^2 n)$. The solutions of the Schrödinger equation (58) were obtained directly by separation of variables in spherical coordinates in Ref. 12. Thus, as in the case of the hydrogen atom, Eq. (58) is separable in spherical and parabolic coordinates.

4. Solution of the MIC-Kepler problem with zero energy

The results of the preceding two sections can be extended to the cases where ε is positive or equal to zero, which, according to Eq. (24), correspond to $\omega^2<0$ or $\omega^2=0$, respectively. In the latter case, Eq. (2) is the Hamiltonian of a free particle in four dimensions and the orbits in configuration space are straight lines. By suitably choosing the coordinate axes, we can assume that

$$u_1=b_1t, \qquad u_2=b_2,$$
$$u_3=\mu b_2, \qquad u_4=\mu b_1 t, \quad (68)$$

where b_1, b_2 and μ are arbitrary constants, then substituting into Eq. (2), with $\omega=0$, and Eq. (14) one finds that the values of H and K are

$$E=\frac{M}{2}(1+\mu^2)b_1^2,$$
$$K=M(\mu^2-1)b_1b_2 \quad (69)$$

and from Eqs. (10) it follows that

$$x=4\mu b_1b_2t,$$
$$y=2\mu(b_1^2t^2-b_2^2),$$
$$z=(1-\mu^2)(b_1^2t^2+b_2^2), \quad (70)$$

which are parametric equations of the parabola given by the intersection of the plane

$$(\mu^2-1)y+2\mu z+4\mu(\mu^2-1)b_2^2=0 \quad (71)$$

and the cone

$$x^2+y^2=\left(\frac{2\mu}{1-\mu^2}\right)^2 z^2. \quad (72)$$

Thus, as in the cases where ε is different from zero, the orbit lies on a cone whose axis is **L**, is plane and only if $K=0$ (*i.e.*, $\mu=1$) the origin lies on the plane of the orbit. The vectors **L** and **R** given by Eqs. (49) and (50) are conserved and the vector **N** defined by Eq. (51) is normal to the plane of the orbit.

In a similar way, if we set $\omega^2=0$ in Eq. (55), we obtain the time-independent Schrödinger equation for a free particle in four dimensions, which corresponds to the Schrödinger equation for a charged particle in the field of a magnetic monopole and the central field (37) with zero energy [see Eq. (58)]. Proceeding as in the previous section, we shall make use of the fact that the separable eigenfunctions of the Hamiltonian of a free particle in two dimensions, in polar coordinates, are of the form

$$\psi_{Em}(\rho,\theta)=J_{2m}(\sqrt{2ME}\,\rho/\hbar)e^{i2m\theta}, \quad (73)$$

where J_{2m} is a Bessel function and m is an integer or half-integer [*cf.* Eq. (61)]. Hence, Eq. (55) with $\omega=0$ admits solutions of the form $\psi_{E_1m_1}(\rho_1,\theta_1)\psi_{E_2m_2}(\rho_2,\theta_2)$, where $E_1+E_2=E$, m_1 and m_2 are both integers or half-integers and $u_1+iu_2=\rho_1e^{i\theta_1}$, $u_3+iu_4=\rho_2e^{i\theta_2}$. Therefore, taking into account Eqs. (57) and (63) we conclude that

$$\psi_{\kappa_2m_2}=J_{N-2m_2}(\kappa_1\rho_1)J_{2m_2}(\kappa_2\rho_2)e^{2im_2(\theta_2-\theta_1)} \quad (74)$$

where $|N|$ is an even integer ($N=K/\hbar$), $\kappa_1^2+\kappa_2^2=8Mk/\hbar^2$ [see Eq. (25)], is a solution of the Schrödinger equation (58) with zero energy.

5. Concluding remarks

As we have shown, certain submanifolds of the FIHO phase space correspond to the phase spaces of MIC-Kepler problems with different values of the parameters appearing in the potentials (35) and (37). Another remarkable fact is that a similar relationship also holds regarding the Schrödinger equation. The fact that the interaction with a magnetic monopole, which involves the gauge-dependent vector potential, can be derived from a Hamiltonian with a potential energy that depends on the coordinates only [see Eq. (2)], allows us to avoid the difficulties related to the dependence of the canonical momenta on the choice of gauge (*cf.* Ref. 12).

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