Symmetry of the Kepler problem in classical mechanics

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It is shown that the Kepler problem can be transformed into the problem of a free particle moving in the sphere, the plane, or a hyperboloid in Minkowski space, depending on the value of the energy.

Keywords: Kepler problem; dynamical symmetry

Se muestra que el problema de Kepler puede transformarse en el problema de una partícula libre moviéndose en la esfera, el plano, o un hiperboloide en el espacio de Minkowski, dependiendo del valor de la energía.

Descriptores: Problema de Kepler; simetría dinámica

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1. Introduction

It is well known that the Kepler problem possesses a hidden symmetry, which is associated with the existence of a conserved vector—the Hermann-Bernoulli-Laplace-Runge-Lenz (HBLRL) vector—that lies on the plane of the orbit and points to the perihelion. The conservation of the angular momentum and of the HBLRL vector permits to obtain the orbit without the explicit integration of the equations of motion. (In fact, these results hold for the Kepler problem in any number of dimensions [1].) Depending on the value of the energy, by suitably normalizing the HBLRL vector, one finds that the Poisson brackets between its components and those of the angular momentum coincide with the commutation relations for the Lie algebra of the rotation group, the Euclidean group or the Lorentz group.

In the case of the quantum Kepler problem for bound states, Fock [2] showed that, by means of the stereographic projection, the Schrödinger equation in momentum space can be transformed into an integral equation on a sphere, in such a way that the invariance under the rotation group becomes manifest (see also, *e.g.*, Refs. 3–5). When the energy is positive or zero, the Schrödinger equation in momentum space can be transformed into an integral equation on a hyperboloid or a hyperplane, respectively, that displays the invariance of the problem under the Lorentz group or the Euclidean group (see, *e.g.*, Refs. 2 and 6). In this paper it is shown in an elementary manner that a similar result applies to the Kepler problem in classical mechanics, giving canonical transformations that relate the Kepler problem with the problem of a free particle on a sphere, a hyperplane or a hyperboloid. Some earlier treatments of this correspondence can be found, *e.g.*, in Refs. 7–9. In order to present the procedure in a simple way, we consider in some detail the two-dimensional Kepler problem and for the three-dimensional Kepler problem we only discuss the case where the energy is equal to zero.

2. The Kepler problem in two dimensions

The Hamiltonian corresponding to the Kepler problem in two dimensions, expressed in terms of cartesian coordinates, is given by

$$H = \frac{1}{2M}(p_x^2 + p_y^2) - \frac{k}{\sqrt{x^2+y^2}}, \tag{1}$$

where k is a positive constant. Therefore, the hypersurface in phase space $H = E$ corresponds to

$$x^2 + y^2 = \left(\frac{2Mk}{p^2 - 2ME}\right)^2, \tag{2}$$

where $p \equiv \sqrt{p_x^2 + p_y^2}$. As we shall show now, depending on whether E is positive, negative or zero, one can find a canonical transformation that takes Eq. (2) into $h = \text{const.}$, where h is the Hamiltonian of a free particle in a maximally symmetric two-dimensional space.

2.1. $E < 0$

Following Refs. 2 and 3, making use of the stereographic projection, we replace the vector $\mathbf{p} = (p_x, p_y)$ by a unit vector $\mathbf{n} = (n_x, n_y, n_z)$ according to

$$\mathbf{p} = (p_x, p_y) = p_0 \frac{(n_x, n_y)}{1 - n_z}, \tag{3}$$

where

$$p_0 \equiv \sqrt{-2ME}, \tag{4}$$

hence,

$$\mathbf{n} = (n_x, n_y, n_z) = \frac{(2p_0 p_x, 2p_0 p_y, p^2 - p_0^2)}{p^2 + p_0^2}. \tag{5}$$

Under this correspondence, the plane is mapped into the unit sphere and making use of the spherical coordinates θ, ϕ, of $\mathbf{n}$, from Eq. (3) we find that

$$\mathbf{p} = \frac{p_0}{1-\cos\theta}(\sin\theta\cos\phi, \sin\theta\sin\phi) = p_0 \cot\left(\frac{\theta}{2}\right)(\cos\phi, \sin\phi), \tag{6}$$

therefore,

$$p = p_0 \cot\left(\frac{\theta}{2}\right) \tag{7}$$

and

$$\begin{aligned} p_x\,dx + p_y\,dy &= p_0 \cot\left(\frac{\theta}{2}\right)(\cos\phi\,dx + \sin\phi\,dy) \\ &= d\left[p_0 \cot\left(\frac{\theta}{2}\right)(x\cos\phi + y\sin\phi)\right] \\ &\quad + \frac{1}{2}p_0\csc^2\left(\frac{\theta}{2}\right)(x\cos\phi + y\sin\phi)\,d\theta \\ &\quad + p_0\cot\left(\frac{\theta}{2}\right)(x\sin\phi - y\cos\phi)\,d\phi, \end{aligned}$$

which means that θ, ϕ and

$$\begin{aligned} p_\theta &\equiv \tfrac{1}{2}p_0\csc^2\left(\frac{\theta}{2}\right)(x\cos\phi + y\sin\phi), \\ p_\phi &\equiv p_0\cot\left(\frac{\theta}{2}\right)(x\sin\phi - y\cos\phi) \end{aligned} \tag{8}$$

are canonical coordinates. From Eq. (8) and (7) it follows that

$$\begin{aligned} x^2 + y^2 &= \frac{4\sin^4\left(\frac{\theta}{2}\right)}{p_0^2}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right) \\ &= \left(\frac{2p_0}{p^2+p_0^2}\right)^2\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right), \end{aligned}$$

therefore, taking into account Eq. (4), one finds that Eq. (2) amounts to

$$\frac{1}{2}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right) = \frac{1}{2}\left(\frac{Mk}{p_0}\right)^2. \tag{9}$$

The left-hand side of the last equation is recognized as the Hamiltonian of a free particle (of unit mass) moving on the unit sphere; which is manifestly invariant under the rotation group SO(3). This invariance is related to the fact that the functions

$$\begin{aligned} l_x &\equiv -\sin\phi\,p_\theta - \cot\theta\cos\phi\,p_\phi, \\ l_y &\equiv \cos\phi\,p_\theta - \cot\theta\sin\phi\,p_\phi, \\ l_z &\equiv p_\phi, \end{aligned} \tag{10}$$

which generate rotations about the n_x-, n_y- and n_z-axis, respectively, have vanishing Poisson brackets with the Hamiltonian

$$h \equiv \frac{1}{2}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right). \tag{11}$$

The auxiliary Hamiltonian (11) is conjugate to a fictitious "time", τ, which is related to the true time, t, by

$$\frac{d\tau}{dt} = \frac{p_0^2}{M^2k\sqrt{x^2+y^2}} = \frac{p_0^2(p^2+p_0^2)}{2M^3k^2}. \tag{12}$$

Hence, the functions (10) are also constants of the motion for the Hamiltonian (1). (Alternatively, $\{h, f\} = 0$ if and only if f generates canonical transformations that leave the hypersurface $h = \text{const.}$ invariant; therefore, since $h = \text{const.}$ is equivalent to $H = \text{const.}$, $\{h, f\} = 0$ is equivalent to $\{H, f\} = 0$.) From Eqs. (6) and (8) one finds that $p_\theta = \csc\theta(xp_x + yp_y)$ and $p_\phi = xp_y - yp_x$; hence, using Eqs. (6), (7) and (2), it follows that the constants of the motion (10), expressed in terms of the original coordinates, are

$$l_x = \frac{A_y}{p_0}, \qquad l_y = -\frac{A_x}{p_0}, \qquad l_z = xp_y - yp_x, \tag{13}$$

where

$$\begin{aligned} A_x &= p_y(xp_y - yp_x) - \frac{Mkx}{\sqrt{x^2+y^2}}, \\ A_y &= -p_x(xp_y - yp_x) - \frac{Mky}{\sqrt{x^2+y^2}}, \end{aligned} \tag{14}$$

are the nonvanishing components of the Hermann-Bernoulli-Laplace-Runge-Lenz (HBLRL) vector $\mathbf{A} \equiv \mathbf{p}\times\mathbf{L} - Mk\mathbf{r}/r$ with $\mathbf{L} = \mathbf{r}\times\mathbf{p}$. From Eq. (13) and the well-known Poisson bracket relations $\{l_x, l_y\} = l_z$, $\{l_y, l_z\} = l_x$, $\{l_z, l_x\} = l_y$, which can be derived from Eq. (10), one can easily obtain the Poisson brackets between A_x, A_y and l_z.

The orbits on the unit sphere can be easily obtained by noting that Eq. (10) leads to $l_x\sin\theta\cos\phi + l_y\sin\theta\sin\phi + l_z\cos\theta = 0$, or, equivalently, $(l_x, l_y, l_z)\cdot\mathbf{n} = 0$, which is the equation of a great circle, as expected. Then making use of Eq. (5) one finds that $2l_xp_0p_x + 2l_yp_0p_y + l_z(p_x^2 + p_y^2 - p_0^2) = 0$. On the other hand, form Eqs. (10) and (9), $l_x^2 + l_y^2 + l_z^2 = p_\theta^2 + p_\phi^2/\sin^2\theta = (Mk/p_0)^2$; hence, assuming $l_z \neq 0$, we have

$$\left(p_x + \frac{l_xp_0}{l_z}\right)^2 + \left(p_y + \frac{l_yp_0}{l_z}\right)^2 = \left(\frac{Mk}{l_z}\right)^2, \tag{15}$$

which is the equation of a circle of radius $Mk/|l_z|$ whose center is at a distance

$$\sqrt{\left(\frac{l_xp_0}{l_z}\right)^2 + \left(\frac{l_yp_0}{l_z}\right)^2} = \frac{Mk}{|l_z|}\sqrt{1 + \frac{2El_z^2}{Mk^2}} = \frac{|\mathbf{A}|}{|l_z|} \tag{16}$$

from the origin. The orbit (15) is the so-called hodograph and the fact that this curve is a circle follows immediately from the fact that the orbits of the Hamiltonian (11) are great circles and that the stereographic projection maps circles onto circles.

As is well known, the orbit in the configuration space is easily obtained making use of the conservation of the HBLRL vector (14). Indeed, if ψ is the angle between $\mathbf{A}$ and the position vector $\mathbf{r}$, then $r|\mathbf{A}|\cos\psi = \mathbf{r}\cdot\mathbf{A} = \mathbf{r}\cdot(\mathbf{p}\times\mathbf{L}) - Mkr = l_z^2 - Mkr$; hence

$$r = \frac{l_z^2}{Mk(1+\varepsilon\cos\psi)}, \tag{17}$$

where $\varepsilon \equiv |\mathbf{A}|/Mk$ is the eccentricity of the orbit.

2.2. $E = 0$

In this case the vector $\mathbf{p}$ will be replaced by a two-component dimensionless vector $\mathbf{u} = (u_x, u_y)$ according to

$$\mathbf{p} = (p_x, p_y) = \rho_0 \frac{(u_x, u_y)}{u_x^2 + u_y^2} = \rho_0 \frac{\mathbf{u}}{u^2}, \tag{18}$$

where ρ_0 is a constant with dimensions of momentum. Under the inversion (18), the plane is mapped into itself. Then,

$$\mathbf{u} = \rho_0 \frac{\mathbf{p}}{p^2}, \qquad p = \frac{\rho_0}{u} \tag{19}$$

and

$$p_x dx + p_y dy = \rho_0 \frac{u_x dx + u_y dy}{u_x^2 + u_y^2} = d\left[\rho_0 \frac{u_x x + u_y y}{u_x^2 + u_y^2}\right] + \rho_0 \frac{[x(u_x^2 - u_y^2) + 2yu_xu_y]du_x + [y(u_y^2 - u_x^2) + 2xu_xu_y]du_y}{(u_x^2 + u_y^2)^2}$$

hence, u_x, u_y and

$$P_1 \equiv \rho_0 \frac{x(u_x^2 - u_y^2) + 2yu_xu_y}{(u_x^2 + u_y^2)^2},$$
$$P_2 \equiv \rho_0 \frac{y(u_y^2 - u_x^2) + 2xu_xu_y}{(u_x^2 + u_y^2)^2} \tag{20}$$

are canonical coordinates. From these last expressions one finds that $P_1^2 + P_2^2 = p^4(x^2+y^2)/\rho_0^2$; thus, Eq. (2) with $E = 0$ takes the form

$$\frac{1}{2}(P_1^2 + P_2^2) = \frac{1}{2}\left(\frac{2Mk}{\rho_0}\right)^2. \tag{21}$$

The left-hand side of Eq. (21) is the Hamiltonian of a free particle (of unit mass) in the plane, which is manifestly invariant under the Euclidean group SE(2).

Clearly, the functions P_1, P_2 and $u_xP_2 - u_yP_1$ have vanishing Poisson brackets with the auxiliary Hamiltonian

$$h = \frac{1}{2}(P_1^2 + P_2^2). \tag{22}$$

The functions P_1, P_2 and $u_xP_2 - u_yP_1$ generate the action of SE(2) on the phase space alluded to above and their Poisson brackets with H also vanish [see the discussion after Eq. (12)]. From Eqs. (19), (20) and Eq. (2) with $E = 0$ it follows that

$$P_1 = -\frac{2A_x}{\rho_0},$$
$$P_2 = -\frac{2A_y}{\rho_0},$$
$$u_xP_2 - u_yP_1 = xp_y - yp_x, \tag{23}$$

with A_x and A_y defined by Eq. (14).

Using the fact that P_1, P_2 and $u_xP_2 - u_yP_1$ are constants of the motion, one finds that the equation of the orbit is $u_xP_2 - u_yP_1 = l_z$, where l_z, P_1 and P_2 are constants, which corresponds to a straight line, as expected. Then, from Eq. (19) it follows that, in terms of the original variables, the orbit is given by $P_2p_x - P_1p_y = (l_z/\rho_0)(p_x^2 + p_y^2)$ or, taking into account Eq. (21),

$$\left(p_x - \frac{\rho_0 P_2}{2l_z}\right)^2 + \left(p_y + \frac{\rho_0 P_1}{2l_z}\right)^2 = \left(\frac{Mk}{l_z}\right)^2 \tag{24}$$

[*cf.* Eq. (15)], which is the equation of a circle of radius $Mk/|l_z|$ passing through the origin.

The orbit in configuration space is also given by Eq. (17). Owing to Eqs. (21) and (23), $|\mathbf{A}| = Mk$, therefore the eccentricity is equal to 1.

It is known that, making use of parabolic coordinates, the two-dimensional Kepler problem with energy E can be related to a two-dimensional isotropic harmonic oscillator of angular frequency $\omega = \sqrt{-2E/M}$ (see, *e.g.*, Refs. 5 and 10); therefore, by means of this relationship, the two-dimensional Kepler problem with zero energy corresponds to the problem of a free particle in the plane (see, *e.g.*, Ref. 11). However, by contrast with the canonical transformation given by Eqs. (19) and (20), this procedure cannot be applied to other dimensions.

2.3. $E > 0$

Now the vector $\mathbf{p}$ will be replaced by a vector $\mathbf{n} = (n_x, n_y, n_z)$ satisfying the condition [2, 6]

$$n_x^2 + n_y^2 - n_z^2 = -1 \tag{25}$$

by means of

$$\mathbf{p} = (p_x, p_y) = p_0 \frac{(n_x, n_y)}{1 - n_z} \tag{26}$$

where, in the present case,

$$p_0 \equiv \sqrt{2ME}. \tag{27}$$

Therefore,

$$\mathbf{n} = (n_x, n_y, n_z) = \frac{(-2p_0 p_x, -2p_0 p_y, p^2 + p_0^2)}{p^2 - p_0^2}. \qquad (28)$$

Since $p > p_0$ [see Eq. (1)], $n_z > 1$. Under the correspondence given by Eqs. (26) and (28), the region $p > p_0$ of the plane is mapped into the hyperboloid (25) with $n_z > 1$. Thus, we can write $\mathbf{n} = (\sinh\theta\cos\phi, \sinh\theta\sin\phi, \cosh\theta)$ and from Eq. (26) we obtain

$$\mathbf{p} = -p_0 \coth\left(\frac{\theta}{2}\right)(\cos\phi, \sin\phi), \qquad (29)$$

therefore

$$p = p_0 \coth\left(\frac{\theta}{2}\right) \qquad (30)$$

and

$$\begin{aligned} p_x dx + p_y dy &= -p_0 \coth\left(\frac{\theta}{2}\right)(\cos\phi\, dx + \sin\phi\, dy) \\ &= d[-p_0 \coth\left(\frac{\theta}{2}\right)(x\cos\phi + y\sin\phi)] \\ &\quad - \frac{1}{2} p_0 \operatorname{csch}^2\left(\frac{\theta}{2}\right)(x\cos\phi + y\sin\phi)\, d\theta \\ &\quad - p_0 \coth\left(\frac{\theta}{2}\right)(x\sin\phi - y\cos\phi)\, d\phi. \end{aligned}$$

Hence, θ, ϕ and

$$\begin{aligned} p_\theta &\equiv -\tfrac{1}{2} p_0 \operatorname{csch}^2\left(\tfrac{\theta}{2}\right)(x\cos\phi + y\sin\phi), \\ p_\phi &\equiv -p_0 \coth\left(\frac{\theta}{2}\right)(x\sin\phi - y\cos\phi) \end{aligned} \qquad (31)$$

are canonical coordinates and from Eqs. (31) and (30) we obtain

$$\begin{aligned} x^2 + y^2 &= \frac{4\sinh^4\left(\frac{\theta}{2}\right)}{p_0^2}\left(p_\theta^2 + \frac{p_\phi^2}{\sinh^2\theta}\right) \\ &= \left(\frac{2p_0}{p^2 - p_0^2}\right)^2\left(p_\theta^2 + \frac{p_\phi^2}{\sinh^2\theta}\right) \end{aligned}$$

which implies that Eq. (2) is equivalent to

$$\frac{1}{2}\left(p_\theta^2 + \frac{p_\phi^2}{\sinh^2\theta}\right) = \frac{1}{2}\left(\frac{Mk}{p_0}\right)^2 \qquad (32)$$

[*cf.* Eq. (9)]. The left-hand side of Eq. (32) is the Hamiltonian of a free particle of unit mass on the (maximally symmetric) two-dimensional Riemannian manifold defined by Eq. (25), for $n_z > 1$, with the metric induced by the Lorentzian metric $(dn_x)^2 + (dn_y)^2 - (dn_z)^2$ and, hence, is invariant under $SO^{\uparrow}(2,1)$.

It can be readily seen that the functions

$$\begin{aligned} l_x &\equiv \sin\phi\, p_\theta + \coth\theta\cos\phi\, p_\phi, \\ l_y &\equiv -\cos\phi\, p_\theta + \coth\theta\sin\phi\, p_\phi, \\ l_z &\equiv p_\phi, \end{aligned} \qquad (33)$$

have vanishing Poisson brackets with the Hamiltonian

$$h \equiv \frac{1}{2}\left(p_\theta^2 + \frac{p_\phi^2}{\sinh^2\theta}\right). \qquad (34)$$

l_x and l_y generate Lorentz transformations on the $n_y n_z$- and $n_x n_z$-plane, respectively, and l_z generates rotations on the $n_x n_y$-plane. These functions satisfy the Poisson bracket relations $\{l_x, l_y\} = -l_z$, $\{l_y, l_z\} = l_x$, $\{l_z, l_x\} = l_y$, and are constants of the motion. Making use of Eqs. (29) and (31) one finds that $p_\theta = (xp_x + yp_y)/\sinh\theta$, $p_\phi = xp_y - yp_x$, which, together with Eqs. (33), (14) and (2) imply that

$$l_x = \frac{A_y}{p_0}, \qquad l_y = -\frac{A_x}{p_0}, \qquad l_z = xp_y - yp_x. \qquad (35)$$

Again, the orbit is easily obtained using the constants of the motion. From Eq. (33) we have $l_x \sinh\theta\cos\phi + l_y \sinh\theta\sin\phi - l_z\cosh\theta = 0$, or, $l_x n_x + l_y n_y - l_z n_z = 0$, which, owing to Eq. (28), amounts to $2l_x p_0 p_x + 2l_y p_0 p_y + l_z(p_x^2 + p_y^2 + p_0^2) = 0$. Noting that $l_x^2 + l_y^2 - l_z^2 = p_\theta^2 + p_\phi^2/\sinh^2\theta = (Mk/p_0)^2$ and assuming $l_z \neq 0$, we can also write

$$\left(p_x + \frac{l_x p_0}{l_z}\right)^2 + \left(p_y + \frac{l_y p_0}{l_z}\right)^2 = \left(\frac{Mk}{l_z}\right)^2 \qquad (36)$$

[*cf.* Eq. (15)], which is the equation of a circle of radius $Mk/|l_z|$ whose center is at a distance from the origin given by Eq. (16); however, in the present case, the hodograph is not the complete circle (36) but only the arc contained in the region $p > p_0$.

Finally, from Eq. (35) we find that $|\mathbf{A}| = p_0\sqrt{l_x^2 + l_y^2} = Mk\sqrt{1 + (2El_z^2)/(Mk^2)}$ [*cf.* Eq. (16)], therefore, the eccentricity of the orbit (17) is greater than 1.

3. The zero-energy Kepler problem in three dimensions

The procedure shown in the preceding section can be applied to the three-dimensional Kepler problem (and, in fact, to the Kepler problem in any number of dimensions). As an illustration, we shall consider here only the case where the energy is equal to zero. Starting from the Hamiltonian

$$H = \frac{1}{2M}(p_x^2 + p_y^2 + p_z^2) - \frac{k}{\sqrt{x^2 + y^2 + z^2}}, \qquad (37)$$

one finds that the condition $H = 0$ is equivalent to

$$x^2 + y^2 + z^2 = \left(\frac{2Mk}{p^2}\right)^2, \qquad (38)$$

where $p = \sqrt{p_x^2 + p_y^2 + p_z^2}$.

Introducing the vector $\mathbf{u} = (u_x, u_y, u_z)$ by

$$\mathbf{u} \equiv \rho_0 \frac{\mathbf{p}}{p^2}, \qquad \text{or} \qquad \mathbf{p} = \rho_0 \frac{\mathbf{u}}{u^2}, \tag{39}$$

where ρ_0 is a constant with dimensions of momentum and $\mathbf{p} = (p_x, p_y, p_z)$, making use of the notation $\mathbf{r} = (x, y, z)$ we obtain

$$\begin{aligned} p_x dx + p_y dy + p_z dz = \mathbf{p} \cdot d\mathbf{r} &= \rho_0 \frac{\mathbf{u} \cdot d\mathbf{r}}{u^2} \\ &= d\left[\rho_0 \frac{\mathbf{u} \cdot \mathbf{r}}{u^2}\right] + \mathbf{P} \cdot d\mathbf{u}, \end{aligned}$$

where

$$\mathbf{P} = (P_1, P_2, P_3) \equiv \frac{\rho_0}{u^4}(2(\mathbf{r} \cdot \mathbf{u})\mathbf{u} - u^2 \mathbf{r}). \tag{40}$$

Thus, the new variables $\mathbf{P}$, $\mathbf{u}$ are related with $\mathbf{p}$, $\mathbf{r}$ by a canonical transformation. From Eq. (40) one easily sees that $\mathbf{P}^2 = \rho_0^2 r^2 / u^4$; therefore, making use of Eq. (39) we find that Eq. (38) amounts to

$$\frac{1}{2}(P_1^2 + P_2^2 + P_3^2) = \frac{1}{2}\left(\frac{2Mk}{\rho_0}\right)^2 \tag{41}$$

[*cf.* Eq. (21)]. Clearly, the left-hand side of Eq. (41) is the Hamiltonian of a free particle in three-dimensional space, which is invariant under the Euclidean group SE(3). This invariance is related to the conservation of P_1, P_2, P_3 and of the components of the angular momentum $\mathbf{L} = \mathbf{u} \times \mathbf{P}$.

Using Eqs. (39) and (40) one finds that

$$\mathbf{L} = \mathbf{u} \times \frac{\rho_0}{u^4}[2(\mathbf{r} \cdot \mathbf{u})\mathbf{u} - u^2 \mathbf{r}] = \frac{\rho_0}{u^2} \mathbf{r} \times \mathbf{u} = \mathbf{r} \times \mathbf{p}, \tag{42}$$

which generate the "obvious" (rotational) symmetry of (37), and that the remaining constants of the motion amount to

$$\begin{aligned} \mathbf{P} &= \frac{1}{\rho_0}[2(\mathbf{r} \cdot \mathbf{p})\mathbf{p} - p^2 \mathbf{r}] \\ &= -\frac{2}{\rho_0}\left[\mathbf{p} \times (\mathbf{r} \times \mathbf{p}) - \frac{1}{2} p^2 \mathbf{r}\right] \\ &= -\frac{2}{\rho_0}\left(\mathbf{p} \times \mathbf{L} - \frac{Mk\mathbf{r}}{r}\right) = -\frac{2\mathbf{A}}{\rho_0}, \end{aligned} \tag{43}$$

where we have made use of the fact that $E = 0$ and $\mathbf{A}$ is the HBLRL vector [*cf.* Eq. (23)]. Thus, the translational invariance of (41) corresponds to the "hidden symmetry" of the Hamiltonian (37) when $E = 0$.

4. Concluding remarks

We have shown that by means of suitable coordinate transformations in phase space, the Kepler problem can be related to the problem of a free particle in a homogeneous space. These transformations can be regarded as the analogs in classical mechanics of Fock's transformations. A slight difference between the classical and the quantum Kepler problem with positive energy comes from the fact that in the latter case both sheets of the hyperboloid (25) have to be taken into account; the sheet with $n_z < -1$ corresponds to the classically forbidden region.

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